6 Given $\vec{A} = 2\hat{i} - \hat{j}$

$\vec{B} = 2\hat{j} + 3\hat{k}$

$\vec{C} = \hat{i} + \hat{j} + \hat{k}$ Find

a $\vec{A} \cdot (\vec{B} \times \vec{C})$ and $(\vec{A} \times \vec{B}) \cdot C$

b $\vec{A} \times (\vec{B} \times \vec{C})$ and $(\vec{A} \times \vec{B}) \times \vec{C}$

a $\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i}(2-3) + \hat{j}(3-0) + \hat{k}(0-2)$

$= -\hat{i} + 3\hat{j} - 2\hat{k}$

$\therefore \vec{A} \cdot (\vec{B} \times \vec{C}) = (2\hat{i} - \hat{j}) \cdot (-\hat{i} + 3\hat{j} - 2\hat{k})$

$= -2 - 3 = -5$

$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 0 \\ 0 & 2 & 3 \end{vmatrix} = \hat{i}(-3-0) + \hat{j}(0-6) + \hat{k}(4-0)$

$= -3\hat{i} - 6\hat{j} + 4\hat{k}$

$(\vec{A} \times \vec{B}) \cdot \vec{C} = (-3\hat{i} - 6\hat{j} + 4\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k})$

$= -3 - 6 + 4 = -5$

Not $\Rightarrow$ $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot C$

b $\vec{B} \times \vec{C} = -\hat{i} + 3\hat{j} - 2\hat{k}$

$\bar{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 0 \\ -1 & 3 & -2 \end{vmatrix}$

$= \hat{i}(2-0) + \hat{j}(0+4) + \hat{k}(6-1)$

$= 2\hat{i} - \hat{j} + 5\hat{k}$

(7) A force $\vec{F_1} = \hat{i} + \hat{j}$ is applied to abody at point P_1 where the Vector $OP_1 = r_1 = 2\hat{i} + \hat{j}$. A second Force $F_2 = \hat{j} - \hat{k}$ is applied at the point $r_2 = \hat{i} + \hat{j} + \hat{k}$ Find the total moment $(\vec{N})$, the magnitude of $\vec{N}$ and the direction Cosines of the resultant axis of rotation?

* $\vec{N_1} = \vec{r_1} \times \vec{F_1} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i}(0) + \hat{j}(0) + \hat{k}(2-1)$
$= \hat{k}$

* $\vec{N_2} = \vec{r_2} \times \vec{F_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{i}(-1-1) + \hat{j}(0+1) + \hat{k}(1-0)$
$= -2\hat{i} + \hat{j} + \hat{k}$

* $\vec{N}_{total} = \vec{N_1} + \vec{N_2} = -2\hat{i} + \hat{j} + 2\hat{k}$

* Magnitude of $\vec{N}$

$$N = |\vec{N}| = \sqrt{(-2)^2 + (1)^2 + (2)^2} = \sqrt{9} = 3$$

* the direction Cosine

$$n = \frac{\vec{N}}{N}$$ وحدة المتجه العمودي تعطي جيوب تمام الزوايا

$$= \frac{-2\hat{i} + \hat{j} + 2\hat{k}}{3} = \frac{-2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$= \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$$

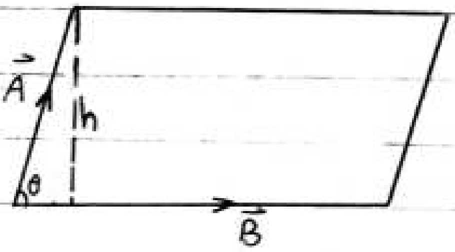
$\cos\alpha \downarrow \frac{-2}{3}$ $\quad \cos\beta \downarrow \frac{1}{3}$ $\quad \cos\gamma \downarrow \frac{2}{3}$

تلامس

(9) Two vectors $\vec{A}$ and $\vec{B}$ represent concurrent sides of a parallelogram (متوازي الأضلاع). Prove that the area of parallelogram is $|\vec{A} \times \vec{B}|$.

$$\sin\theta = \frac{h}{A}$$

$$h = A\sin\theta \quad \text{—} (1)$$



$$\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$$

$$\vec{A} \times \vec{B} = B(A\sin\theta)\,\hat{n}$$

$$\vec{A} \times \vec{B} = Bh\,\hat{n}$$

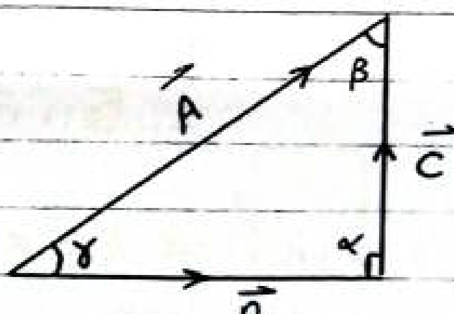
$$|\vec{A} \times \vec{B}| = Bh = \text{the area of parallelogram}$$

مثلث

(11) Prove the trigonometric law of sines using vector methods?

$$\vec{A} = \vec{B} + \vec{C}$$

$$\vec{A} \times \vec{B} = \cancel{\vec{B} \times \vec{B}} + \vec{C} \times \vec{B}$$

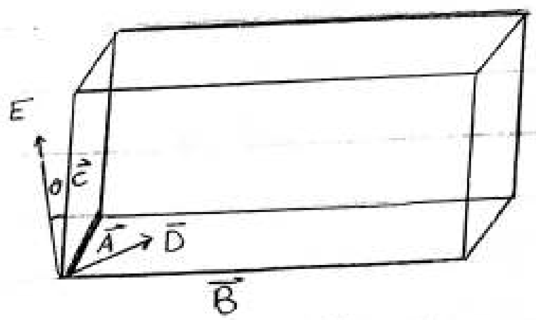


$$A\cancel{B}\sin\gamma\,\hat{n} = C\cancel{B}\sin\alpha\,\hat{n}$$

$$\frac{A}{\sin\alpha} = \frac{C}{\sin\gamma}$$

(12) Three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ represent concurrent sides of a parallelepiped (متوازي المستطيلات). Show that the volume of the parallelepiped is

$$|\vec{A} \cdot (\vec{B} \times \vec{C})| .$$



$$\vec{A} \times \vec{B} = \vec{D}$$

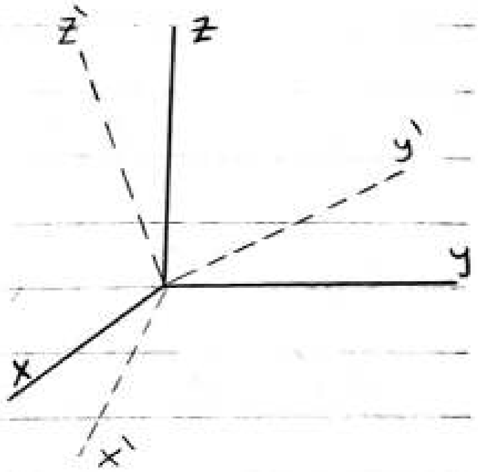
$$\cos\theta = \frac{E}{C} \Rightarrow E = C\cos\theta$$

$$\text{Volume} = D\,C\cos\theta$$
$$= \vec{D} \cdot \vec{C}$$
$$= (\vec{A} \times B) \cdot \vec{C}$$

* The Change of Coordinate System

Consider the Vector $\vec{A}$ expressed relative to the $(\hat{i}\,\hat{j}\,\hat{k})$

$$\vec{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z$$



$$\vec{A} \cdot \hat{i}' = A_x' = (\hat{i} \cdot \hat{i}') A_x + (\hat{j} \cdot \hat{i}') A_y + (\hat{k} \cdot \hat{i}') A_z \quad \text{—— (1)}$$

$$\vec{A} \cdot \hat{j}' = A_y' = (\hat{i} \cdot \hat{j}') A_x + (\hat{j} \cdot \hat{j}') A_y + (\hat{k} \cdot \hat{j}') A_z \quad \text{—— (2)}$$

$$\vec{A} \cdot \hat{k}' = A_z' = (\hat{i} \cdot \hat{k}') A_x + (\hat{j} \cdot \hat{k}') A_y + (\hat{k} \cdot \hat{k}') A_z \quad \text{—— (3)}$$

These equations are conveniently expressed in matrix

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} (\hat{i} \cdot \hat{i}') & (\hat{j} \cdot \hat{i}') & (\hat{k} \cdot \hat{i}') \\ (\hat{i} \cdot \hat{j}') & (\hat{j} \cdot \hat{j}') & (\hat{k} \cdot \hat{j}') \\ (\hat{i} \cdot \hat{k}') & (\hat{j} \cdot \hat{k}') & (\hat{k} \cdot \hat{k}') \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Coefficients of transformation

$$x' = R \quad x$$

Relative to a new $(\grave{i}\ \grave{j}\ \grave{k})$ having a different orientation from that of $\hat{i}\,\hat{j}\,\hat{k}$, the same vector $\vec{A}$ is expressed as:.

$$\vec{A} = \grave{i}\,\grave{A}_x + \grave{j}\,\grave{A}_y + \grave{k}\,\grave{A}_z$$

$$\vec{A'} \cdot \hat{i} = A_x$$

$$A_x = \vec{A'} \cdot \hat{i} = (\grave{i} \cdot \hat{i})\,\grave{A}_x + (\grave{j} \cdot \hat{i})\,\grave{A}_y + (\grave{k} \cdot \hat{i})\,\grave{A}_z \quad \text{——} \textcircled{1}$$

$$A_y = \vec{A'} \cdot \hat{j} = (\grave{i} \cdot \hat{j})\,\grave{A}_x + (\grave{j} \cdot \hat{j})\,\grave{A}_y + (\grave{k} \cdot \hat{j})\,\grave{A}_z \quad \text{——} \textcircled{2}$$

$$A_z = \vec{A'} \cdot \hat{k} = (\grave{i} \cdot \hat{k})\,\grave{A}_x + (\grave{j} \cdot \hat{k})\,\grave{A}_y + (\grave{k} \cdot \hat{k})\,\grave{A}_z \quad \text{——} \textcircled{3}$$

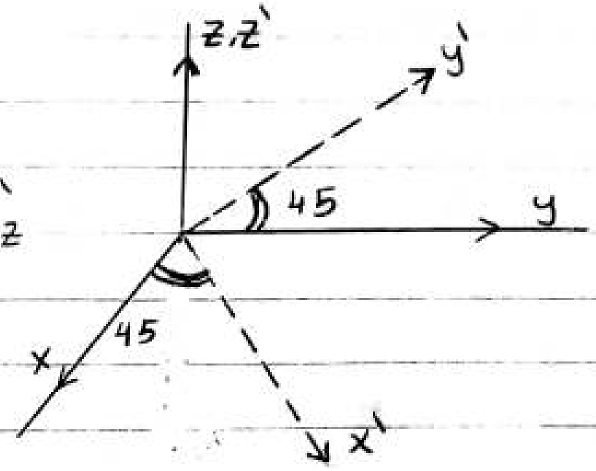
These equations are conveniently expressed in Matrix

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} (\grave{i} \cdot \hat{i}) & (\grave{j} \cdot \hat{i}) & (\grave{k} \cdot \hat{i}) \\ (\grave{i} \cdot \hat{j}) & (\grave{j} \cdot \hat{j}) & (\grave{k} \cdot \hat{j}) \\ (\grave{i} \cdot \hat{k}) & (\grave{j} \cdot \hat{k}) & (\grave{k} \cdot \hat{k}) \end{bmatrix} \begin{bmatrix} \grave{A}_x \\ \grave{A}_y \\ \grave{A}_z \end{bmatrix}$$

نجد R^T . بجعل الصف عمود في مصفوفة التحويل R

ex)

Express the Vector $\vec{A} = 3\hat{i} + 2\hat{j} + \hat{k}$ in terms of the triad $\hat{i}', \hat{j}', \hat{k}'$ where the $x'y'$ axes are rotated (45°) around the (z) axis, the z and z' axes Coinciding ?



$$\vec{A} = \hat{i}' A_x' + \hat{j}' A_y' + \hat{k}' A_z'$$

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} (\hat{i}\cdot\hat{i}') & (\hat{j}\cdot\hat{i}') & (\hat{k}\cdot\hat{i}') \\ (\hat{i}\cdot\hat{j}') & (\hat{j}\cdot\hat{j}') & (\hat{k}\cdot\hat{j}') \\ (\hat{i}\cdot\hat{k}') & (\hat{j}\cdot\hat{k}') & (\hat{k}\cdot\hat{k}') \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45 & \cos 45 & \cos 90 \\ \cos(90+45) & \cos 45 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Not $\Rightarrow \cos(90+45) = -\sin 45$

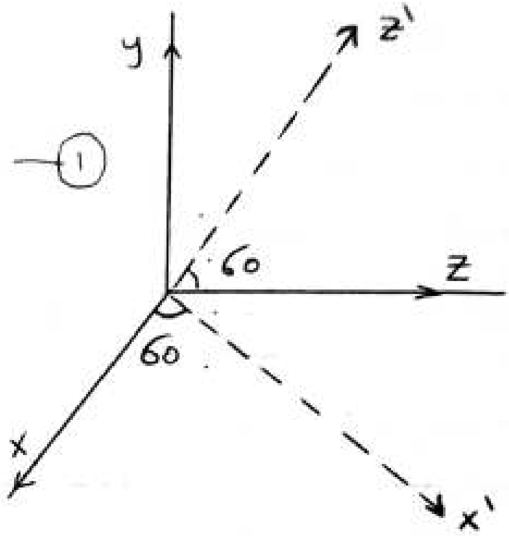
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 \\ -\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 \\ 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

$$\vec{A} = \frac{5}{\sqrt{2}}\hat{i}' - \frac{1}{\sqrt{2}}\hat{j}' + \hat{k}'$$

13) Express the Vector $\hat{i}+\hat{j}$ interms of the triad $\hat{i}'\ \hat{j}'\ \hat{k}'$ where the $x'\ z'$ axes are rotated about the (y) axis (which coincides with the (y') axis through an angle of (60°)

$$\vec{A} = = \hat{i}' A'_x + \hat{j}' A'_y + \hat{k}' A'_z \quad \text{—} (1)$$



$$\begin{bmatrix} A'_x \\ A'_y \\ A'_z \end{bmatrix} = \begin{bmatrix} (\hat{i}, \hat{i}') & (\hat{j}, \hat{i}') & (\hat{k}, \hat{i}') \\ (\hat{i}, \hat{j}') & (\hat{j}, \hat{j}') & (\hat{k}, \hat{j}') \\ (\hat{i}, \hat{k}') & (\hat{j}, \hat{k}') & (\hat{k}, \hat{k}') \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos 60 & \cos 90 & \cos 30 \\ \cos 90 & \cos 0 & \cos 90 \\ \underset{\displaystyle -\sin 60}{\underset{\downarrow}{\cos(90+60)}} & \cos 90 & \cos 60 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{-\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+0+0 \\ 0+1+0 \\ \frac{-\sqrt{3}}{2}+0+0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{-\sqrt{3}}{2} \end{bmatrix}$$

$$\therefore \vec{A} = \frac{1}{2}\hat{i}' + \hat{j}' - \frac{3}{\sqrt{2}}\hat{k}' \Rightarrow |\vec{A}| = \sqrt{(\tfrac{1}{2})^2 + 1 + (\tfrac{\sqrt{3}}{2})^2}$$

$$= \sqrt{\tfrac{1}{4} + 1 + \tfrac{3}{4}} = \sqrt{2}$$

$$(\hat{i}, \hat{j}, \hat{k}) \Rightarrow |\vec{A}| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

∴ مقدار المتجه لا يتغير سواء في الاحداثيات $\hat{i}\ \hat{j}\ \hat{k}$ أو $\hat{i}'\ \hat{j}'\ \hat{k}'$