# Chapter Two : Continuity and Derived Topological Spaces

**Definition : Continuous & discontinuous Functions**

Let $(X, \tau)$ and $(Y, \tau')$ be two topological spaces and $f : (X, \tau) \to (Y, \tau')$. The function f is called **continuous** if the inverse image for any open set in Y is an open set in X. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is continuous} \Leftrightarrow f^{-1}(V) \in \tau \quad \forall V \in \tau'$$

and the function f is called **discontinuous** if there exist an open set in Y, but inverse image is not open in X. i.e.,

$$f \text{ is discontinuous} \Leftrightarrow \exists V \in \tau' \wedge f^{-1}(V) \notin \tau$$



**Example :** Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi, \{b\}\}$

**(1)** Define $f : (X, \tau) \to (Y, \tau')$ ; $f(1) = b, f(2) = f(3) = a$. Is f continuous??

The open sets in Y are Y, $\phi$, $\{b\}$. Now take the inverse image of this sets.

$Y \in \tau' \Rightarrow f^{-1}(Y) = X \in \tau$ the set of all element in X its image in Y

$\phi \in \tau' \Rightarrow f^{-1}(\phi) = \phi \in \tau$ the set of all element in $\phi$ its image in $\phi$

$\{b\} \in \tau' \Rightarrow f^{-1}(\{b\}) = \{x \in X ; f(x) = b\} = \{1\} \in \tau$

the set of all element in X its image in $\{b\}$

Therefore, the inverse image of every element in $\tau'$ is element in $\tau$, hence f is continuous.

**(2)** Define $g : (X, \tau) \to (Y, \tau')$ ; $g(1) = a, g(2) = g(3) = b$. Is g continuous??

$Y \in \tau' \Rightarrow g^{-1}(Y) = X \in \tau$

$\phi \in \tau' \Rightarrow g^{-1}(\phi) = \phi \in \tau$

$\{b\} \in \tau' \Rightarrow g^{-1}(\{b\}) = \{2, 3\} \notin \tau$

Therefore, f is discontinuous.

**(3)** Define $h : (X, \tau) \to (Y, \tau')$ ; $h(1) = h(2) = h(3) = a$. Is h continuous??

$Y \in \tau' \Rightarrow h^{-1}(Y) = X \in \tau$

$\phi \in \tau' \Rightarrow h^{-1}(\phi) = \phi \in \tau$

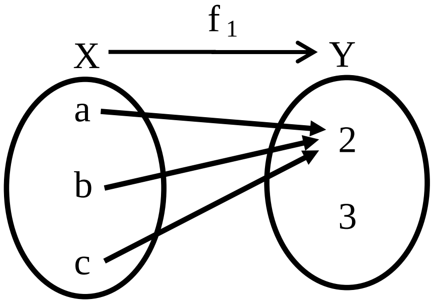
$\{b\} \in \tau' \Rightarrow h^{-1}(\{b\}) = \phi \in \tau$ since there is no element its image is b

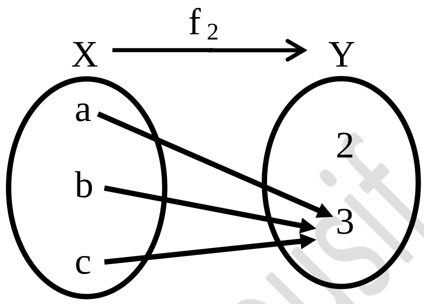
Therefore, f is continuous.

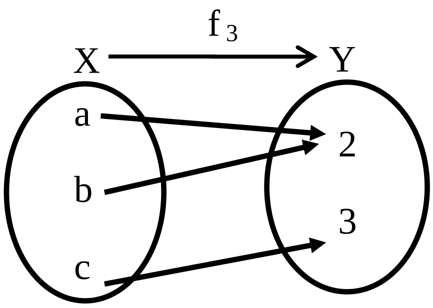
**Remark :** Always the inverse image of Y is X and the inverse image of $\phi$ is $\phi$.

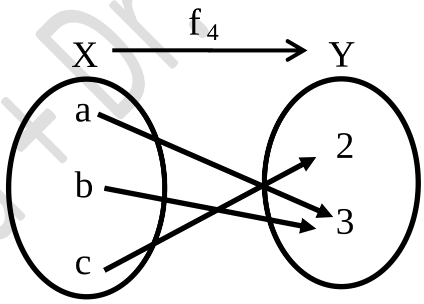
**Example :** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}\}$, $Y = \{2, 3\}$ and $\tau' = \{Y, \phi, \{2\}\}$. Find all continuous function define from $(X, \tau)$ to $(Y, \tau')$.

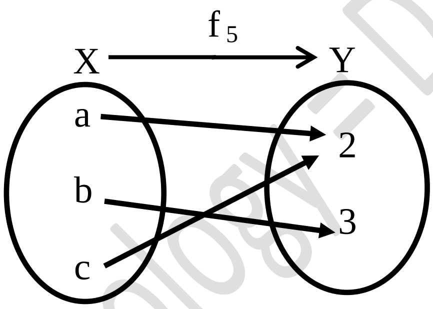
**Solution :** There are $2^3 = 8$ from difference functions from X to Y which are some of them continuous and some others of them discontinuous . Now we introduce the figure for all functions from X to Y and discuses there continuous.

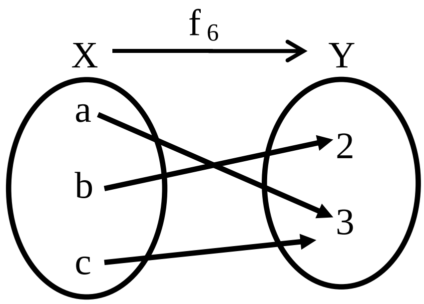


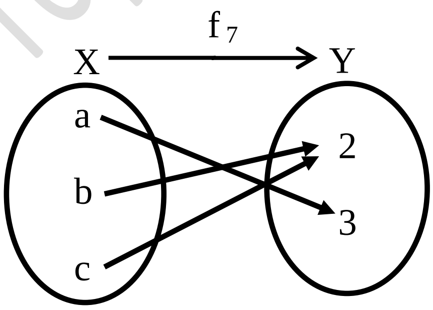



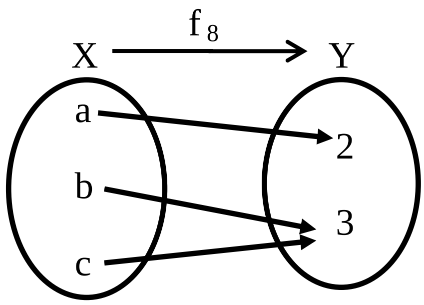



From remark above $f_i^{-1}(Y) = X$ and $f_i^{-1}(\phi) = \phi$, i = 1, 2, 3, 4, 5, 6, 7, 8.
$f_1$ is continuous, since $f_1^{-1}(\{2\}) = \{a, b, c\} = X \in \tau$.
$f_2$ is continuous, since $f_2^{-1}(\{2\}) = \phi \in \tau$.
$f_3$ is discontinuous, since $f_3^{-1}(\{2\}) = \{a, b\} \notin \tau$.
$f_4$ is discontinuous, since $f_4^{-1}(\{2\}) = \{c\} \notin \tau$.
$f_5$ is discontinuous, since $f_5^{-1}(\{2\}) = \{a, c\} \notin \tau$.
$f_6$ is discontinuous, since $f_6^{-1}(\{2\}) = \{b\} \notin \tau$.
$f_7$ is continuous, since $f_7^{-1}(\{2\}) = \{b, c\} \in \tau$.
$f_8$ is discontinuous, since $f_8^{-1}(\{2\}) = \{a\} \notin \tau$.
Therefore, the continuous functions in this example are $f_1$, $f_2$, $f_7$ only.

**Remark :** There are special cases of continuous functions.

**[1]** Every constant function from a space $(X, \tau)$ to a space $(Y, \tau')$ is continuous. i.e.,

$$f : (X, \tau) \to (Y, \tau')\ ;\ f(x) = c \quad \forall\, x \in X \text{ and } c = \text{constant in } Y.$$

To show that f is continuous.
Let $V \in \tau' \Rightarrow V$ is open in Y, then

$$f^{-1}(V) = \begin{cases} X & \text{if } \ c \in V \\ \emptyset & \text{if } \ c \notin V \end{cases}$$

$\Rightarrow X, \phi \in \tau \Rightarrow$ f is continuous.

**[2]** If $\tau' = I$, then the function $f : (X, \tau) \to (Y, I)$ is continuous for any set Y and any topological space $(X, \tau)$. i.e., $I = \{Y, \phi\}$ and $f^{-1}(Y) = X \in \tau$, $f^{-1}(\phi) = \phi \in \tau$.
Special case : $f : (X, I) \to (Y, I)$ is continues
And the function $f : (X, I) \to (Y, \tau')$ ; $\tau' \neq I$ is discontinuous in general for example :

$$f : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u) \quad ; \quad f(x) = x$$

f is discontinuous since $(0, 1) \in \tau_u$ and $f^{-1}((0, 1)) = (0, 1) \notin I = \{\mathbb{R}, \phi\}$.

**[3]** If $\tau = D$, then the function $f : (X, D) \to (Y, \tau')$ is continuous for any set X and any topological space $(Y, \tau')$ and for any function f since, if $V \in \tau'$, then $f^{-1}(V) \subseteq X$ this means $f^{-1}(V) \in IP(X)$, but $D = IP(X)$ and this implies $f^{-1}(V) \in D$. Therefore f is continuous.
Special case : $f : (X, D) \to (Y, D)$ is continuous
And the function $f : (X, \tau) \to (Y, D)$ ; $\tau \neq D$ is discontinuous in general for example :

$$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, D) \quad ; \quad f(x) = x$$

f is discontinuous since $\{1\} \in D$ and $f^{-1}(\{1\}) = \{1\} \notin \tau_u$.

Notes that the function $f : (X, D) \to (Y, I)$ is continuous always for any set X and any set Y since its add the remark [2] and [3] such that $\tau = D$ and $\tau' = I$.

**[4]** Every identity function from a spaces to the same space is continuous. i.e.,

$$f : (X, \tau) \to (X, \tau) \quad ; \quad f(x) = x \quad \forall\, x \in X$$

is continuous function since $f^{-1}(V) = V$ for any open set V in $(X, \tau)$ and this implies $f^{-1}(V)$ is open in $(X, \tau)$

notes that the identity function from a space to another space may be continuous and its clear in example in remark [2] and [3] above.

**<u>Theorem :</u>** If $f : (X, \tau) \to (Y, \tau')$ and $g : (Y, \tau') \to (Z, \tau'')$ are both continuous functions, then the composition $gof : (X, \tau) \to (Z, \tau'')$ is continuous.

**<u>Proof :</u>**

Let $W \in \tau'' \Rightarrow g^{-1}(W) \in \tau'$ (since g is continuous)

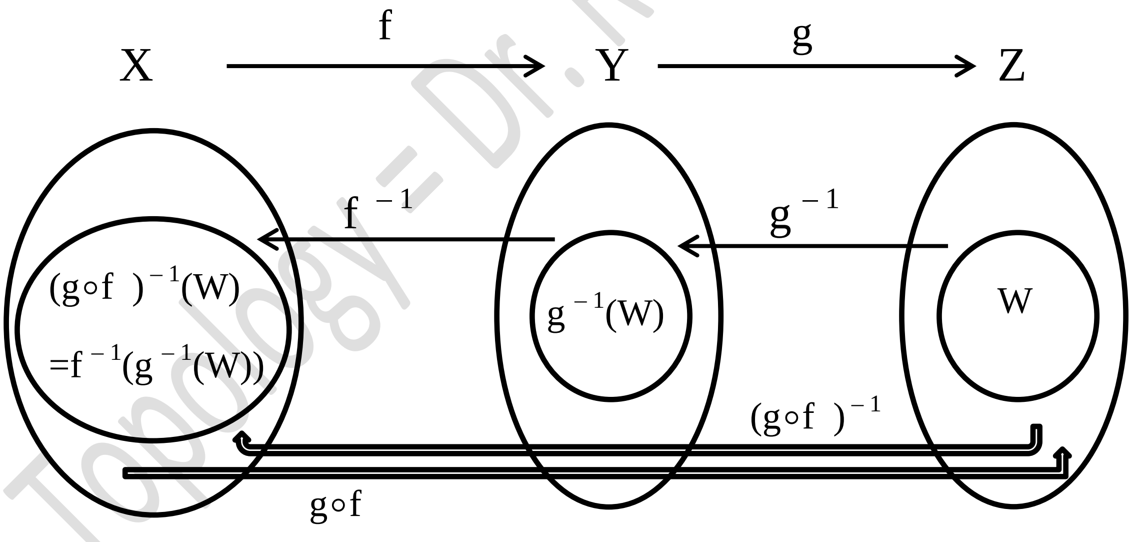
notes that $g^{-1}(w) \subseteq Y$

$\Rightarrow f^{-1}(g^{-1}(W)) \in \tau$ (since f is continuous)

$\Rightarrow (f^{-1}og^{-1})(W) \in \tau$ (by composition of function)

$\Rightarrow (gof)^{-1}(W) \in \tau$ (since $(gof)^{-1} = f^{-1}og^{-1}$ )

$\therefore$ gof is continuous. The figure below clear this theorem.




**<u>Remake :</u>** The composition of finite number of continuous functions is continuous. i.e., the composition of three or five or hundred continuous functions is continuous. For example if f , g, h, k are continuous, then kohogof is continuous …. etc.

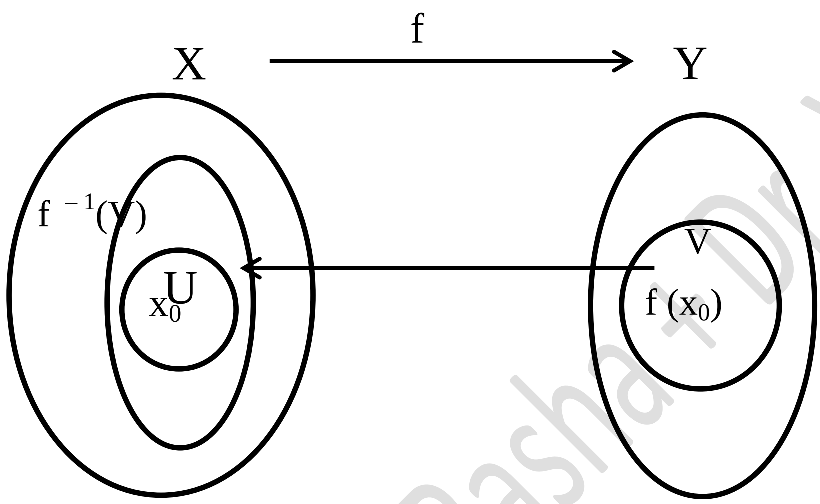
Now we introduce the definition of continuous function at a point :

**Definition : Continuous at a Point**

Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces and $f : (X, \tau) \to (Y, \tau')$. the function f is called **continuous at a point** $x_0 \in X$ if the inverse image for any open nbd for $f(x_0)$ in Y contains an open nbd for $x_0$ in X. i.e.,

$$f \text{ is continuous at } x_0 \in X \Leftrightarrow \forall\ V \in \tau' ;\ f(x_0) \in V\ \exists\ U \in \tau ;\ x_0 \in U \wedge U \subseteq f^{-1}(V)$$

The following figure clear this definition :




Such that V is an open nbd for $f(x_0)$ in Y and $f^{-1}(V)$ is inverse image for V and U is an open nbd for $x_0$ in X contains in $f^{-1}(V)$.

**Remark :** If f is continuous function. Then its continuous at every point in the domain. Also, if f is continuous at every point in the domain, then its continuous.

**Notes that**, if f is continuous at a point in the domain, then its discontinuous in general and the following example show that :

**Example :** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, $Y = \{1, 2\}$ and $\tau' = \{Y, \phi, \{1\}\}$. Define f as follow :

$$f : (X, \tau) \to (Y, \tau') \ ; \quad f(a) = f(b) = 2,\ f(c) = 1$$

notes that f is discontinuous since $\{1\} \in \tau'$, but $f^{-1}(\{1\}) = \{c\} \notin \tau$.

On the other hand, thought that f is discontinuous in general, but its continuous at a point a as follow :

$f(a) = 2$ and the open nbd of 2 is Y only and $f^{-1}(Y) = X$ and X is an open nbd

There are several characterizations of continuous functions and, hence, that any one of them may be used to show continuity of a function. These are given in the next theorem :

**Theorem :** Let $f : (X, \tau) \to (Y, \tau')$ be a function. Then f is continuous iff satisfy one of the following properties :

**(1)** $f^{-1}(F) \in \mathcal{F} \quad \forall \quad F \in \mathcal{F}'$ ; $\mathcal{F}$ family of closed sets in X and $\mathcal{F}'$ family of closed sets in Y i.e., The inverse image of every closed set in Y is closed in X.

**(2)** $f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$ ; $\beta'$ is a basis for $\tau'$.

i.e., The inverse image of every element in any basis for $\tau'$ is open set in X.

**(3)** $f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathcal{S}'$ ; $\mathcal{S}'$ is a subbasis for $\tau'$.

i.e., The inverse image of every element in any subbasis for $\tau'$ is open set in X.

**(4)** $f^{-1}(N_y) \in \tau \quad \forall \quad y \in Y \quad \forall \quad N_y \in \eta_y$ ; $\eta_y$ is a family of open nbd for a point y in Y.

i.e., The inverse image of every open nbd for any element Y is open set in X.

**(5)** $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ ; $B \subseteq Y$.

**(6)** $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ ; $B \subseteq Y$.

**Proof :**

(1) To prove, f is continuous $\Leftrightarrow X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$

($\Rightarrow$) Suppose that f is cont. , to prove $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$

Let F closed set in Y $\Rightarrow Y - F$ open set in Y (def. of closed set)

$\Rightarrow f^{-1}(Y - F) \in \tau$ (since f is continuous)

But, $f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F)$

$= X - f^{-1}(F)$ (since $f^{-1}(Y) = X$)

$\because \quad f^{-1}(Y - F) \in \tau \Rightarrow X - f^{-1}(F) \in \tau$

($\Leftarrow$) Suppose that $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\therefore$ Y – V closed set in Y since V open

$\Rightarrow f^{-1}(Y - V)$ closed set in X (by hypothesis )

$\Rightarrow X - f^{-1}(Y - V) \in \tau$

But, $X - f^{-1}(Y - V) = X - [f^{-1}(Y) - f^{-1}(V)] = X - [X - f^{-1}(V)] = f^{-1}(V)$

$\Rightarrow f^{-1}(V) \in \tau$

$\therefore$ f is continuous.

(2) To prove, f is continuous $\Leftrightarrow f^{-1}(B') \in \tau \ \forall \ B' \in \beta'$ ; $\beta'$ is a basis for $\tau'$.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$

Let $\beta'$ be a base of $\tau'$ and $B' \in \beta'$

$\Rightarrow B' \in \tau'$ (since $\beta' \subseteq \tau'$)

$\Rightarrow f^{-1}(B') \in \tau$ (since f is continuous)

$\Rightarrow f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$

($\Leftarrow$) Suppose that $f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow V = \bigcup_i B_i' \quad ; \quad B_i' \in \beta'$ (def. of basis)

$\Rightarrow f^{-1}(V) = f^{-1}(\bigcup_i B_i') = \bigcup_i f^{-1}(B_i')$

$\Rightarrow \bigcup_i f^{-1}(B_i') \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $f^{-1}(V) = f^{-1}(\bigcup_i B_i') = \bigcup_i f^{-1}(B_i')$ )

$\therefore$ f is continuous

(3) To prove, f is continuous $\Leftrightarrow f^{-1}(S') \in \tau \ \forall \ S' \in \mathcal{S}'$ ; $\mathcal{S}'$ is a subbasis for $\tau'$.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathcal{S}'$

Let $\mathcal{S}'$ be a subbase of $\tau'$ and $S' \in \mathcal{S}'$

$\Rightarrow S' \in \tau'$ (since $\mathcal{S}' \subseteq \tau'$)

$\Rightarrow f^{-1}(S') \in \tau$ (since f is continuous)

$\Rightarrow f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathcal{S}'$

($\Leftarrow$) Suppose that $f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathcal{S}'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow V = \bigcup_i (\bigcap_{j=1}^{n} S_j')$ (def. of basis and subbasis)

$\Rightarrow f^{-1}(V) = f^{-1}(\bigcup_i (\bigcap_{j=1}^{n} S_j'))$

$= \bigcup_i f^{-1}(\bigcap_{j=1}^{n} S_j')$ (inverse image distribution on union)

$= \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j'))$ (inverse image distribution on intersection)

$\because f^{-1}(S_j') \in \tau \quad \Rightarrow \quad f^{-1}(S_j')$ open in X

$\Rightarrow \bigcap_{j=1}^{n} f^{-1}(S_j') \in \tau$ (by second condition of def. of top.)

$\Rightarrow \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j')) \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $f^{-1}(V) = \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j'))$ )

$\therefore$ f is continuous,

(4) To prove, f is continuous $\Leftrightarrow f^{-1}(N_y) \in \tau \quad \forall \quad y \in Y \quad \forall \quad N_y \in \eta_y$

($\Rightarrow$) Suppose that f is continuous, to prove $f^{-1}(N_y) \in \tau \quad \forall \quad y \in Y \quad \forall \quad N_y \in \eta_y$

Let $N_y \in \eta_y$

$\because$ $\eta_y$ is open nbd system for y, then $\eta_y$ is a family of open set, therefore

$\Rightarrow N_y \in \tau'$

$\Rightarrow f^{-1}(N_y) \in \tau$ (since f is continuous)

($\Leftarrow$) Suppose that $f^{-1}(N_y) \in \tau \ \forall \ y \in Y \ \forall \ N_y \in \eta_y$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow V = \bigcup_{y \in V} N_y$ i.e., V = union of a family of open sets for every point in V by using the fifth condition of def. of o.n.s

$(U \in \tau \Leftrightarrow \exists N_y \in \eta_y ; N_y \subseteq U \ \forall \ y \in U)$

$\therefore f^{-1}(V) = f^{-1}(\bigcup_{y \in V} N_y) = \bigcup_{y \in V} f^{-1}(N_y)$ (inverse image distribution on union)

$\because f^{-1}(N_y) \in \tau$ (by hypothesis )

$\Rightarrow \bigcup_{y \in V} f^{-1}(N_y) \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $(f^{-1}(V) = \bigcup_{y \in V} f^{-1}(N_y))$ )

$\therefore$ f is continuous.

(5) To prove, f is continuous $\Leftrightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ ; $B \subseteq Y$.

($\Rightarrow$) Suppose that f is continuous , to prove $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ ; $B \subseteq Y$.

Let $B \subseteq Y \Rightarrow B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$

$\because \overline{B}$ closed set in Y by (1) $f^{-1}(\overline{B})$ closed set in X

$\Rightarrow \bigcap\{F \subseteq X : F^c \in \tau \wedge f^{-1}(B) \subseteq F\} \subseteq f^{-1}(\overline{B})$

since $\overline{f^{-1}(B)}$ is intersection of all closed sets that contain $f^{-1}(B)$ and $f^{-1}(\overline{B})$ is one of the closed set that contain $f^{-1}(B)$, then

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ (since $\bigcap\{F : F^c \in \tau \wedge f^{-1}(B) \subseteq F\} = \overline{f^{-1}(B)}$)

($\Leftarrow$) Suppose that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ ; $B \subseteq Y$. , to prove f is continuous.

We will use part (1) in this theorem

Let F closed set in Y, to prove $f^{-1}(F)$ closed in X i.e., $\overline{f^{-1}(F)} = f^{-1}(F)$.

$f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ (since $A \subseteq \overline{A}$ ) ---------------(1)

$\because$ F closed $\Rightarrow F = \overline{F} \Rightarrow f^{-1}(F) = f^{-1}(\overline{F})$

By hypnoses, $\overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) = f^{-1}(F)$

$\Rightarrow \overline{f^{-1}(F)} \subseteq f^{-1}(F)$ ---------------(2)

From (1) and (2) we have $\overline{f^{-1}(F)} = f^{-1}(F)$

$\therefore f^{-1}(F)$ closed in X.

$\therefore$ f is continuous.

(6) To prove, f is continuous $\Leftrightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ ; $B \subseteq Y$.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ ; $B \subseteq Y$

Let $B \subseteq Y \Rightarrow B^o \subseteq B \Rightarrow f^{-1}(B^o) \subseteq f^{-1}(B)$

$\because$ $B^o$ is open in Y $\Rightarrow f^{-1}(B^o)$ is open in X (since f is continuous)
$f^{-1}(B^o) \subseteq \bigcup\{O \subseteq X ; O \in \tau , O \subseteq f^{-1}(B)\}$
since $(f^{-1}(B))^o$ is union of all open sets that contain in $f^{-1}(B)$ and $f^{-1}(B^o)$ is one of the open set that contain in $f^{-1}(B)$, then

$\Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ (since $\bigcup\{O \subseteq X ; O \in \tau , O \subseteq f^{-1}(B)\} = (f^{-1}(B))^o$ )
($\Leftarrow$) Suppose that $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ ; $B \subseteq Y$, to prove f is continuous.
Let V open set in Y i.e., $V \in \tau'$ to prove $f^{-1}(V)$ open in X i.e., $f^{-1}(V) = (f^{-1}(V))^o$.
$\because$ $V^o \subseteq V \Rightarrow ((f^{-1}(V))^o \subseteq f^{-1}(V)$ ---------------(1)
$\because$ V open $\Rightarrow V = V^o \Rightarrow f^{-1}(V) = f^{-1}(V^o)$
By hypnoses, $f^{-1}(V) = f^{-1}(V^o) \subseteq (f^{-1}(V))^o$
$\Rightarrow f^{-1}(V) \subseteq (f^{-1}(V))^o$ ---------------(2)
From (1) and (2) we have $f^{-1}(V) = (f^{-1}(V))^o$
$\therefore$ $f^{-1}(V)$ open in X
$\therefore$ f is continuous.

**Remark :** The six characterizations in the previous theorem for definition of continuity is not unique, but there are another characterizations for example :

$$f \text{ is continuous } \Leftrightarrow f(\overline{A}) \subseteq \overline{f(A)} \;\; ; \;\; A \subseteq X.$$
$$f \text{ is continuous } \Leftrightarrow (f(A))^o \subseteq f(A^o) \;\; ; \;\; A \subseteq X.$$

**Remarks :**

**[1]** Notes that, if $f : (X, \tau) \rightarrow (Y, \tau')$ is continuous function, then it's not necessary that the direct image of open set in X is open set in Y. i.e.,

$$U \in \tau \not\Rightarrow f(U) \in \tau' \text{ (in general not true)}$$
$$V \in \tau' \Rightarrow f^{-1}(V) \in \tau \text{ (is true)}$$

This two statements is difference and the following example show this :

**Example :** Let $f : (\mathbb{R}, \tau_u) \rightarrow (\mathbb{R}, I)$ be a function, then f is continuous (see, page 37). Now we will show that the direct image of open set is not open in general :
$f : (\mathbb{R}, \tau_u) \rightarrow (\mathbb{R}, I)$ ; $f(x) = x \;\; \forall \; x \in \mathbb{R}$
Let $U = (0, 1)$ open set in $(\mathbb{R}, \tau_u)$ and $f(U) = U$
$\Rightarrow f(U) = U$ is not open in $(\mathbb{R}, I)$ since $U = (0, 1) \notin I = \{\mathbb{R}, \phi\}$
So, we show that $U \in \tau \wedge f(U) \notin \tau'$.

**[2]** Notes that, if $f : (X, \tau) \rightarrow (Y, \tau')$ is continuous function, then it's not necessary that the direct image of closed set in X is closed set in Y. i.e.,

$F^c \in \tau \not\Rightarrow (f(F))^c \in \tau'$ (in general not true)

$F^c \in \tau' \Rightarrow (f^{-1}(F))^c \in \tau$ (is true)

This two statements is difference and the following example show this :

**Example :** In the previous example :

$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$ ; $f(x) = x \quad \forall\ x \in \mathbb{R}$, f is continuous

Let $F = [0, 1]$ closed set in $(\mathbb{R}, \tau_u)$

$\Rightarrow f(F) = F$ is not closed in $(\mathbb{R}, I)$ since $F^c = [0, 1]^c \notin I = \{\mathbb{R}, \phi\}$

So, we show that $[0, 1]^c \in \tau_u \wedge (f([0, 1])^c = [0, 1]^c \notin I$.

Now we will introduce a new definitions for functions satisfy the condition [1] and [2] in the previous remark as follows :

**Definition : Open & Closed Functions**

Let $f : (X, \tau) \to (Y, \tau')$ be a function.

**(1)** The function f is called **open** if the direct image for any open set in X is open set in Y. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is open function } \Leftrightarrow \forall\ U \in \tau \Rightarrow f(U) \in \tau'$$

$$f : (X, \tau) \to (Y, \tau') \text{ is not open function } \Leftrightarrow \exists\ U \in \tau \wedge f(U) \notin \tau'$$

**(2)** The function f is called **closed** if the direct image for any closed set in X is closed set in Y. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is closed function } \Leftrightarrow \forall\ F^c \in \tau \Rightarrow (f(F))^c \in \tau'$$

$$f : (X, \tau) \to (Y, \tau') \text{ is not closed function } \Leftrightarrow \exists\ F^c \in \tau \wedge (f(F))^c \notin \tau'$$

**Remark :** There are no relation between the concepts continuous, open, closed functions and the following table show that :

| f continuous function | f open function | f closed function |
|---|---|---|
| T | T | T |
| T | T | F |
| T | F | T |
| F | T | T |
| T | F | F |
| F | T | F |
| F | F | T |
| F | F | F |

Such that T = True (i.e., the function is satisfy) and F = False (i.e., the function is not satisfy). Also, there are eight probability may be taken the function for example ( T F T) means that the function f is continuous, not open , closed. Therefore we will introduce an eight examples satisfy this probability :

**Example (1) :** (T T T) means the function f is continuous, open , closed.
Define the identity function $f : (X, \tau) \to (X, \tau)$ ; $f(x) = x \quad \forall\ x \in X$.
f is continuous since $\forall\ V \in \tau$ in rang X $\Rightarrow$ $f^{-1}(V) = V \in \tau$ in domain X.
f is open since $\forall\ U \in \tau$ open in domain X $\Rightarrow$ $f(U) = U$ is open in rang X.
f is closed since $\forall$ F closed in domain X $\Rightarrow$ $f(F) = F$ is closed in rang X.

**Example (2) :** (T T F) means the function f is continuous, open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}\}$
Define the constant function $f : (X, \tau) \to (Y, \tau')$ ; $f(1) = f(2) = f(3) = a$
f is continuous since it is constant.
f is open since : $X \in \tau \Rightarrow f(X) = \{a\} \in \tau'$, $\phi \in \tau \Rightarrow f(\phi) = \phi \in \tau'$ and $\{1\} \in \tau \Rightarrow f(\{1\}) = \{a\} \in \tau'$ (i.e., $\forall\ U \in \tau \Rightarrow f(U) \in \tau'$ ).
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$)
But, $f(\{2, 3\}) = \{a\} \notin \mathcal{F}'$ since $\{a\}^c = \{b, c\} \notin \tau'$

**Example (3) :** (T F T) means the function f is continuous, not open , closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a, b\}\}$
Define the constant function $f : (X, \tau) \to (Y, \tau')$ ; $f(1) = f(2) = f(3) = c$
f is continuous since it is constant.
f is not open since $\exists$ open set $\{1\} \in \tau$ ,but $f(\{1\}) = \{c\} \notin \tau'$.
f is closed since : The family of closed sets in X is $\mathcal{F} = \{X, \phi, \{2, 3\}\}$ and the family of closed set in Y is $\mathcal{F}' = \{Y, \phi, \{c\}\}$, then
$f(X) = \{c\}$, $f(\phi) = \phi$, $f(\{2, 3\}) = \{c\}$ (i.e., $\forall\ F \in \mathcal{F} \Rightarrow f(F) \in \mathcal{F}'$).

**Example (4) :** (F T T) means the function f is not continuous, open , closed.
Define the function $f : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$ ; $f(x) = x \quad \forall\ x \in \mathbb{R}$
f is not continuous since $\exists$ (0, 1) open in $(\mathbb{R}, \tau_u)$, but $f^{-1}((0, 1)) = (0, 1)$ not open in $(\mathbb{R}, I)$.
f is open and closed since the only open and closed sets in $(\mathbb{R}, I)$ are $\mathbb{R}$, $\phi$ and $f(\mathbb{R}) = \mathbb{R}$ and $f(\phi) = \phi$ (i.e., the direct image of open (rep., closed) set is open (rep., closed) set )

**Example (5) :** (T F F) means the function f is continuous, not open , not closed.
Define the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$ ; $f(x) = x \quad \forall x \in \mathbb{R}$
f is continuous since the rang is $(\mathbb{R}, I)$ (see, page 37).
f is not open since $\exists$ open set $(0, 1)$ in $(\mathbb{R}, \tau_u)$, but $f((0, 1)) = (0, 1)$ is not open in $(\mathbb{R}, I)$.
f is not closed since $\exists$ closed set $\{0\}$ in $(\mathbb{R}, \tau_u)$, but $f(\{0\}) = \{0\}$ is not closed in $(\mathbb{R}, I)$.

**Example (6) :** (F T F) means the function f is not continuous, open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}, \{a, b\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$ ; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{a\} \in \tau'$ ,but $f^{-1}(\{a\}) = \{1, 2\} \notin \tau$.
f is open since : $X \in \tau \Rightarrow f(X) = \{a, b\} \in \tau'$, $\phi \in \tau \Rightarrow f(\phi) = \phi \in \tau'$ and $\{1\} \in \tau \Rightarrow f(\{1\}) = \{a\} \in \tau'$ (i.e., $\forall\ U \in \tau \Rightarrow f(U) \in \tau'$ ).
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$), but, $f(\{2, 3\}) = \{a, b\} \notin \mathcal{F}'$ since $\{a, b\}^c = \{c\} \notin \tau'$.

**Example (7) :** (F F T) means the function f is not continuous, not open , closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{c\}, \{b, c\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$ ; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{b, c\} \in \tau'$ ,but $f^{-1}(\{b, c\}) = \{3\} \notin \tau$.
f is not open since $\exists$ open set $\{1\} \in \tau$ ,but $f(\{1\}) = \{a\} \notin \tau'$.
f is closed since : The family of closed sets in X is $\mathcal{F} = \{X, \phi, \{2, 3\}\}$ and the family of closed set in Y is $\mathcal{F}' = \{Y, \phi, \{a, b\}, \{a\}\}$, then
$f(X) = \{a, b\}$, $f(\phi) = \phi$, $f(\{2, 3\}) = \{a, b\}$ (i.e., $\forall\ F \in \mathcal{F} \Rightarrow f(F) \in \mathcal{F}'$).

**Example (8) :** (F F F) means the function f is not continuous, not open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$ ; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{a\} \in \tau'$ ,but $f^{-1}(\{a\}) = \{1, 2\} \notin \tau$.
f is not open since $\exists$ open set $X \in \tau$ ,but $f(X) = \{a, b\} \notin \tau'$.
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$), but, $f(\{2, 3\}) = \{a, b\} \notin \mathcal{F}'$ since $\{a, b\}^c = \{c\} \notin \tau'$.

**Remark :** The open function is closed and the closed function is open if the function is bijective (injective and surjective). i.e.,

$$f \text{ is bijective function } \Rightarrow (f \text{ open} \Leftrightarrow f \text{ closed})$$

$$f \text{ is bijective function } \Rightarrow (f \text{ not open} \Leftrightarrow f \text{ not closed})$$

This means if we wanted to get a function is open not closed or closed not open must be define a function not bijection (not injective or not surjective) since if we define a bijective function then it's either open and closed or not open and not closed.

**Remark :** We talking about the function $f : (X, \tau) \to (Y, \tau')$ is continuous or discontinuous. Now we will question if the function f is bijective and continuous this implies that $f^{-1}$ is continuous (i.e., if $f^{-1}$ exists function and f is continuous, then that implies to $f^{-1}$ is continuous ?? or conversaly). The answer of this question is no since we can find a continuous function but your inverse is not continuous and the following example show that :

**Example :** Define the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$ ; $f(x) = x \quad \forall x \in \mathbb{R}$

Notes that f is continuous since the rang is $(\mathbb{R}, I)$ (see, page 37).

Notes that f is bijective , then $f^{-1}$ is exists function and

$f^{-1} : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$, but this function not continuous since

$\exists$ open set (0, 1) in $(\mathbb{R}, \tau_u)$, but $(f^{-1})^{-1}((0, 1)) = (0, 1)$ is not open in $(\mathbb{R}, I)$.

Now the question : Is there are functions is continuous and there inverse is continuous two ??. The answer is yes and the following definition introduce this functions :

**Definition : Homeomorphism Functions**

Let $f : (X, \tau) \to (Y, \tau')$ be a function. The function f is called **homeomorphism** if its injective, surjective, continuous and $f^{-1}$ continuous. i.e.,

$f : (X, \tau) \to (Y, \tau')$ is homeomorphism $\Leftrightarrow$ f 1-1, onto, continuous and $f^{-1}$ continuous

$f : (X, \tau) \to (Y, \tau')$ is not homeomorphism $\Leftrightarrow$

$$f \text{ not 1-1} \vee f \text{ not onto} \vee f \text{ not continuous} \vee f^{-1} \text{ not continuous}$$

**Remark :** Clear that every homeomorphism function is continuous, but the converse is not true for example :

$$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I) \; ; \; f(x) = x \quad \forall \; x \in X$$

The function f is1-1, onto, continuous, but $f^{-1}$ is not continuous. Therefore f is not homeomorphism.

**Remark :** If $f : (X, \tau) \rightarrow (Y, \tau')$ is homeomorphism function, this means : $(f^{-1})^{-1}(U) \in \tau' \quad \forall \ U \in \tau$ (def of continuity), but $(f^{-1})^{-1}(U) = f(U)$ (since f bijective), so we can said

$$f^{-1} \text{ is continuous } \Leftrightarrow f(U) \in \tau' \quad \forall \ U \in \tau$$

but this is the definition of open function, so if $f^{-1}$ is continuous this means f is open and vise versa with property that f is bijective. i.e.,

$$f^{-1} \text{ is continuous } \Leftrightarrow f \text{ is open}$$

if f is bijective (by previous remark, p. 47, f is open $\Leftrightarrow$ f is closed), so that

$$f^{-1} \text{ is continuous } \Leftrightarrow f \text{ is open } \Leftrightarrow f \text{ is closed}$$

i.e., the three concepts are equivalent and we can replace the definition of homeomorphism as follow :

f is homeomorphism $\Leftrightarrow$ f is 1-1, onto, continuous and open.

f is homeomorphism $\Leftrightarrow$ f is 1-1, onto, continuous and closed.

such that we replace the statement $f^{-1}$ is continuous in definition of homeomorphism by either f open function or f closed function.

**Remarks :**

**[1]** If f is homeomorphism function, then $f^{-1}$ is also homeomorphism function.
since f is 1-1 and onto, then $f^{-1}$ is 1-1 and onto
since f is homeomorphism, then $f^{-1}$ is continuous, also, $f = (f^{-1})^{-1}$ is continuous
Therefore, $f^{-1}$ is homeomorphism function.

**[2]** If $f : (X, \tau) \rightarrow (Y, \tau')$ and $g : (Y, \tau') \rightarrow (Z, \tau'')$ are both homeomorphism functions, then the composition $gof : (X, \tau) \rightarrow (Z, \tau'')$ is homeomorphism.
since f and g are 1-1 and onto, then gof is 1-1 and onto
since f and g are continuous, then gof is continuous (by previous theorem)
since f and g are homeomorphism, then $f^{-1}$ and $g^{-1}$ are continuous also $f^{-1}og^{-1}$ is continuous (by previous theorem)
but, $f^{-1}og^{-1} = (gof)^{-1}$ is continuous.
Therefore, gof is homeomorphism function.

**Definition : Homeomorphic Topologies**

We called two topological spaces $(X, \tau)$ and $(Y, \tau')$ are **homeomorphic** if there exists a homeomorphism function from $(X, \tau)$ to $(Y, \tau')$ and denoted by $(X, \tau) \cong (Y, \tau')$ or $(Y, \tau') \cong (X, \tau)$. i.e.,

$$(X, \tau) \cong (Y, \tau') \Leftrightarrow \exists \text{ homeomorphism function } f : (X, \tau) \to (Y, \tau')$$

**Theorem :** The relation $\cong$ is an equivalent relation on the family of topological spaces.

**Proof :** We must prove the relation $\cong$ is reflexive, symmetric and transitive.

**(1)** To prove $\cong$ is reflexive. i.e., $(X, \tau) \cong (X, \tau)$ ??

Define the identity function $f : (X, \tau) \to (X, \tau)$ ; $f(x) = x \quad \forall x \in X$

Clear that f is 1-1, onto, continuous and $f = f^{-1}$ so that $f^{-1}$ is continuous

Therefore, $\cong$ is reflexive.

**(2)** To prove $\cong$ is symmetric. i.e., if $(X, \tau) \cong (Y, \tau') \Rightarrow (Y, \tau') \cong (X, \tau)$ ??

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists$ homo. funct. $f : (X, \tau) \to (Y, \tau')$

by remark [1] above, we have $f^{-1}$ is homo. funct. and

$f^{-1} : (Y, \tau') \to (X, \tau) \Rightarrow (Y, \tau') \cong (X, \tau)$

Therefore, $\cong$ is symmetric.

**(3)** To prove $\cong$ is transitive. i.e., if $(X, \tau) \cong (Y, \tau') \cong (Z, \tau'') \Rightarrow (X, \tau) \cong (Z, \tau'')$ ??

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists$ homo. funct. $f : (X, \tau) \to (Y, \tau')$ and

$\because (Y, \tau') \cong (Z, \tau'') \Rightarrow \exists$ homo. funct. $g : (Y, \tau') \to (Z, \tau'')$ and

by remark [2] above, we have gof is homo. funct. and

$gof : (X, \tau) \to (Z, \tau'') \Rightarrow (X, \tau) \cong (Z, \tau'')$

Therefore, $\cong$ is transitive.

**Theorem :**

**(1)** The bijective function $f : (X, \tau) \to (Y, \tau')$ is homeomorphism iff $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$ ; $B \subseteq Y$.

**(2)** The bijective function $f : (X, \tau) \to (Y, \tau')$ is homeomorphism iff $f^{-1}(B^o) = (f^{-1}(B))^o$ ; $B \subseteq Y$.

**Proof :**

(1) ($\Leftarrow$) Suppose that $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and $f^{-1}$ is cont.

$\because \overline{f^{-1}(B)} = f^{-1}(\overline{B}) \Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ ; $B \subseteq Y$)

and $\because$ $\overline{f^{-1}(B)} = f^{-1}(\overline{B}) \Rightarrow f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)} \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow f(\overline{B}) \subseteq \overline{f(B)}$ ; $B \subseteq X$ and f replace by $f^{-1}$)

$\therefore$ f is Home.

($\Rightarrow$) Suppose that f is Home., to prove $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$

$\because$ f is home. $\Rightarrow$ f is cont. and $f^{-1}$ is cont.

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ and $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$

$\Rightarrow \overline{f^{-1}(B)} = f^{-1}(\overline{B})$

(2) ($\Leftarrow$) Suppose that $f^{-1}(B^o) = (f^{-1}(B))^o$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and $f^{-1}$ is cont.

$\because f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ ; $B \subseteq Y$)

and $\because$ $f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow (f^{-1}(B))^o \subseteq f^{-1}(B^o) \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow (f(B))^o \subseteq f(B^o)$ ; $B \subseteq X$ and f replace by $f^{-1}$)

$\therefore$ f is Home.

($\Leftarrow$) Suppose that f is Home., to prove $f^{-1}(B^o) = (f^{-1}(B))^o$

$\because$ f is home. $\Rightarrow$ f is cont. and $f^{-1}$ is cont.

$\Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ and $(f^{-1}(B))^o \subseteq f^{-1}(B^o)$

$\Rightarrow f^{-1}(B^o) = (f^{-1}(B))^o$

**Definition :** **Topological Property**

A property "P" of a topological space $(X, \tau)$ is called a **topological property** iff every topological space $(Y, \tau')$ homeomorphic to $(X, \tau)$ also has the same property. i.e., if $(X, \tau) \cong (Y, \tau')$ and $(X, \tau)$ has a property "P", then $(Y, \tau')$ has the same property and vise versa.

## Subspace or Induced space

**Definition :** Let $(X, \tau)$ be a topological space and $W \subseteq X$. Define the family $\tau_W$ as a family of subset of W as follow :

$$\tau_W = \{W \cap U \;:\; U \in \tau\}$$

Notes that the elements of $\tau_W$ are intersection W with every open set in X.

**Theorem :** Let $(X, \tau)$ be a topological space and $W \subseteq X$. Then $\tau_W = \{W \cap U \;:\; U \in \tau\}$ is a topology on W.

**Proof :** We will satisfy the three conditions in the definition of topological space.

**(1)** To prove, $W \in \tau_W$ and $\phi \in \tau_W$ .

$\because$ $X \in \tau \wedge W \subseteq X \Rightarrow W = W \cap X \Rightarrow W \in \tau_W$ (def. of $\tau_W$)

$\because$ $\phi \in \tau \wedge \phi \subseteq X \Rightarrow \phi = W \cap \phi \Rightarrow \phi \in \tau_W$ (def. of $\tau_W$)

**(2)** Let $V_1, V_2 \in \tau_W$ , to prove $V_1 \cap V_2 \in \tau_W$

$\because$ $V_1 \in \tau_W \Rightarrow \exists\ U_1 \in \tau$ ; $V_1 = W \cap U_1$ (def. of $\tau_W$)

$\because$ $V_2 \in \tau_W \Rightarrow \exists\ U_2 \in \tau$ ; $V_2 = W \cap U_2$ (def. of $\tau_W$)

$\Rightarrow V_1 \cap V_2 = (W \cap U_1) \cap (W \cap U_2)$

$\Rightarrow V_1 \cap V_2 = W \cap \underbrace{(U_1 \cap U_2)}_{\in\ \tau}$ (since $\cap$ distribution on $\cap$)

$\Rightarrow V_1 \cap V_2 \in \tau_W$ (def. of $\tau_W$)

**(3)** Let $V_\alpha \in \tau_W$ ; $\alpha \in \Lambda$, to prove $\bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_W$

$\because$ $V_\alpha \in \tau_W \Rightarrow \exists\ U_\alpha \in \tau$ ; $V_\alpha = W \cap U_\alpha$ ; $\alpha \in \Lambda$ (def. of $\tau_W$)

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha = \bigcup_{\alpha\in\Lambda}(W \cap U_\alpha)$

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha = W \cap \underbrace{(\bigcup_{\alpha\in\Lambda} U_\alpha)}_{\in\ \tau}$

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_W$ (def. of $\tau_W$)

Therefore, $\tau_W$ is a topology on W.

**Definition : Subspace (or Induced) Topology**

Let $(X, \tau)$ be a topological space and $W \subseteq X$. Then the topology $\tau_W$ is called the **subspace (or induced) topology** for W and the pair $(W, \tau_W)$ is called **subspace** of $(X, \tau)$.

**Example :** Let X = {a, b, c}, $\tau$ = {X, $\phi$, {a}, {a, c}}, W = {a, b}, Z = {b} and K = {a, c}. Find $\tau_W$ , $\tau_Z$ , $\tau_K$ .

**Solution :**

$\tau_W = \{W \cap U\ :\ U \in \tau\}$

$\tau_W = \{W \cap X, W \cap \phi, W \cap \{a\}, W \cap \{a, c\}\}$

$= \{W, \phi, \{a\}\}$

By similar way we compute $\tau_Z$ , $\tau_K$ .

$\tau_Z = \{Z \cap U\ :\ U \in \tau\}$

$\tau_Z = \{Z \cap X, Z \cap \phi, Z \cap \{a\}, Z \cap \{a, c\}\}$

$= \{Z, \phi\} = I_Z$ = **indiscrete topology on Z**

$\tau_K = \{K \cap U\ :\ U \in \tau\}$

$\tau_K = \{K \cap X, K \cap \phi, K \cap \{a\}, K \cap \{a, c\}\} = \{K, \phi, \{a\}\}$.

**Remarks :**

**[1]** Notes that there is an open set in the subspace but it's not open in the space. In the previous example the set W = {a, b} is open in the subspace (W, $\tau_W$) but it is not open in the space (X, τ) i.e., W ∉ τ, so we have :

$$V \in \tau_w \not\Rightarrow V \in \tau$$

In other word $\tau_W \not\subset \tau$ (in general).

**[2]** If W ∈ τ, then $\tau_W \subseteq \tau$.

**[3]** Notes that, in the previous example $\tau_Z = I_Z = \{Z, \phi\}$, but $\tau \neq I_X = \{X, \phi\}$.

**[4]** There are some example $\tau_W = D_W$ = **discrete topology on W**, but $\tau \neq D_X$ i.e.,

$$(\tau_w = D_W \not\Rightarrow \tau = D_X)$$

**For example :** Let X = {a, b, c, d}, τ = {X, ϕ, {a, b}, {c, d}} and W = {a, c}, then $\tau_W = \{W, \phi, \{a\}, \{c\}\} = D_W$

**[5]** If $\tau_X = D_X$ , then $\tau_W = D_W$ for all W ⊆ X. (i.e., $\tau_X = D_X \Rightarrow \tau_W = D_W$)

**To prove this property** it's enough to prove that every singleton subset of W is open set in W.

Let y ∈ W ⇒ {y} ⊆ W, to prove $\{y\} \in \tau_W$ ??

∵ {y} ⊆ W ⊆ X ⇒ {y} ⊆ X ⇒ $\{y\} \in D_X$

since {y} = W ⋂ {y} ⇒ $\{y\} \in \tau_W$

$$\Rightarrow \tau_W = D_W.$$

**[6]** If $\tau = I_X = \{X, \phi\}$, then $\tau_W = I_W = \{W, \phi\}$. (i.e., $\tau = I_X \Rightarrow \tau_W = I_W$)

**To prove this property**

$\tau_W = \{W \cap U : U \in \tau\} = \{W \cap X, W \cap \phi\} = \{W, \phi\}$.

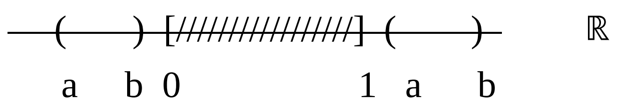
**Example :** In the usual topological space (ℝ, $\tau_u$). Find the induced topology for the following sets : W = [0, 1] , H = ℕ, M = ℚ, K = [2, 3).

**Solution :** The open sets in (ℝ, $\tau_u$) is the union of family of open interval and the family of open interval is a basis for topology $\tau_u$ . So we will use the open interval to compute the basis for induce topology for given set as follow :

**W = [0, 1] ??**

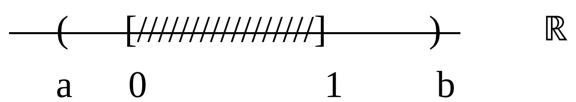
if, a, b ≤ 0 ∨ a, b ≥ 1

then, [0, 1] ⋂ (a, b) = ϕ



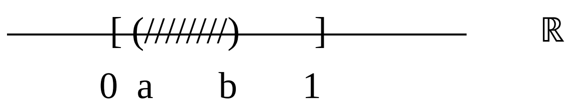

if, b > 1 ∧ a < 0

then, [0, 1] ⋂ (a, b) = [0, 1]



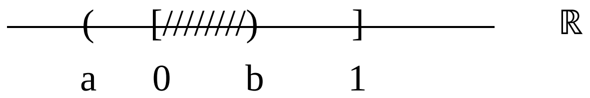

if, b ≤ 1 ∧ a ≥ 0

then, [0, 1] ⋂ (a, b) = (a, b)



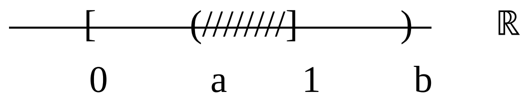
if, $a < 0 \;\wedge\; 0 < b \leq 1$

then, $[0, 1] \bigcap (a, b) = [0, b)$




if, $0 \leq a < 1 \;\wedge\; b > 1$

then, $[0, 1] \bigcap (a, b) = (a, 1]$




From the probability above the basis for induce topology $\tau_W$ is

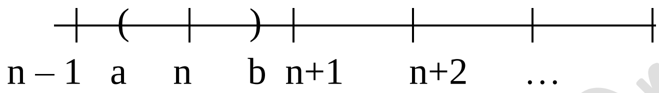
$$\beta_W = \{[0, 1], \phi, (a, b), [0, b), (a, 1]\}$$

Notes that the elements in this family is infinite since a , b $\in \mathbb{R}$.

**H = ℕ ??**

The induce topology for H = ℕ is discrete topology $D_{\mathbb{N}}$ since :

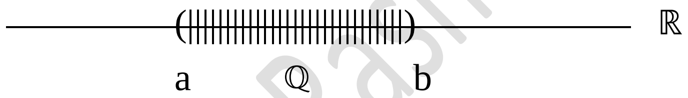
Let (a, b) open interval in $(\mathbb{R}, \tau_u)$ ; a, b $\in \mathbb{R}$ and n $\in \mathbb{N}$ ; $n - 1 < a < n$ and $n < b < n+1$.




This means that every singleton set (i.e., $\mathbb{N} \bigcap (a, b) = \{n\}$) from ℕ is open in ℕ (i.e., $\{n\} \in \tau_{\mathbb{N}}$ ). Therefore $\tau_{\mathbb{N}} = D$.

**M = ℚ ??**

We will intersect ℚ with every open interval (a, b) in $(\mathbb{R}, \tau_u)$ ; a, b $\in \mathbb{R}$.




$\mathbb{Q} \bigcap (a, b)$ = the rational numbers in (a, b) and the basis for induce topology $\tau_{\mathbb{Q}}$ is

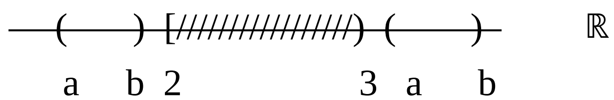
$$\beta_{\mathbb{Q}} = \{\mathbb{Q} \bigcap (a, b) \;;\; a, b \in \mathbb{R}\}.$$

**K = [2, 3) ??**

To compute the induce topology $\tau_K$ ; K = [2, 3) is similar of compute the induce topology $\tau_W$ ; W = [0, 1] above by replace [0,1] by [2, 3) as follow :

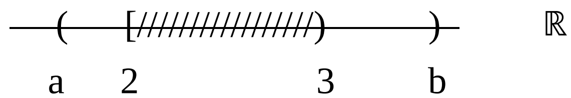
if $a, b \leq 2 \;\vee\; a, b \geq 3$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = \phi$



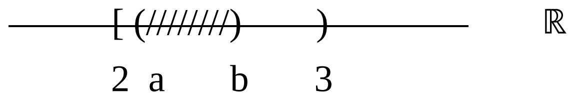

if $a < 2 \;\wedge\; b \geq 3$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = [2, 3)$



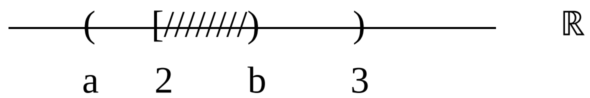

if $a \geq 2 \;\wedge\; b \leq 3$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = (a, b)$



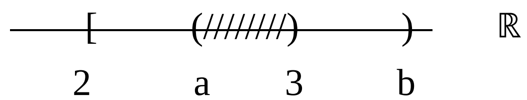

if $a < 2 \;\wedge\; 2 < b \leq 3$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = [2, b)$




if $2 \leq a < 3 \;\wedge\; b > 3$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = (a, 3)$




From the probability above the basis for induce topology $\tau_K$ is

$$\beta_K = \{[2, 3), \phi, (a, b), [2, b), (a, 3)\}.$$

We can compute the induce topology for the intervals [c, d], [c, d), (c, d], (c, d) by similar way by taken this probality and replace W = [0, 1] or K = [2, 3) by [c, d], [c, d), (c, d], (c, d).

**Theorem :** Let $(X, \tau)$ be a topological space and $(W, \tau_W)$ be a subspace topology of X. If $W \in \tau$, then $\tau_W$ is subfamily of $\tau$. i.e.,
If W open set in X, then every open set in W is open in X.

**Proof :** We must prove the following statement $\tau_W \subseteq \tau$ (i.e., if $V \in \tau_W \Rightarrow V \in \tau$ )

Let $V \in \tau_W \Rightarrow \exists U \in \tau ; V = W \cap U$ (def. of $\tau_W$)

$\because$ $W \in \tau$ (by hypothesis) $\wedge$ $U \in \tau$

$\Rightarrow W \cap U \in \tau$ (def. of Top.)

$\Rightarrow V \in \tau$ (since $V = W \cap U$)

The following example clear this theorem :

**Example :** Let X = {a, b, c, d}, $\tau$ = {X, $\phi$, {a}, {a, b}, {a, b, c}} and W = {a, b, c}. Find $\tau_W$ .

**Solution :**

Clear $W \in \tau$. To compute $\tau_W$ :

$\tau_W = \{W \cap U : U \in \tau\}$

$\tau_W = \{W \cap X, W \cap \phi, W \cap \{a\}, W \cap \{a, b\}, W \cap \{a, b, c\}\}$

$= \{W, \phi, \{a\}, \{a, b\}\}$

Notes that $\tau_W$ is subfamily of $\tau$ (i.e., $\tau_W \subseteq \tau$).

**Remark :** From definition of induce topology $\tau_W$ , notes that :

$$V \in \tau_W \Leftrightarrow \exists U \in \tau ; V = W \cap U$$

The question now what about the close set, the previous statement satisfy or not ??
The answer **yes** such that :

$$A \in (\tau_W)^c \Leftrightarrow \exists F \in \tau^c ; A = W \cap F$$

By other statement :

$$A \in \mathcal{F}_W \Leftrightarrow \exists F \in \mathcal{F} ; A = W \cap F$$

Such that $\mathcal{F}$ is the family of closed sets in X and $\mathcal{F}_W$ is the family of closed sets in W.

**Example :** Let X = {a, b, c}, $\tau$ = {X, $\phi$, {a}, {b}, {a, b}} and W = {b, c}.

Then $\tau_W = \{W, \phi, \{b\}\}$, $\mathcal{F} = \{X, \phi, \{b, c\}, \{a, c\}, \{c\}\}$ and $\mathcal{F}_W = \{W^c, \phi^c, \{b\}^c\}$ such that :

$\mathcal{F}_W = \{W^c, \phi^c, \{b\}^c\} = \{\phi, W, \{c\}\}$

$= \{W \cap X, W \cap \phi, W \cap \{b, c\}, W \cap \{a, c\}, W \cap \{c\}\}$

**Theorem :** If K is a subspace from W and W is a subspace from X, then K is a subspace from X.

**Proof :** Let $(X, \tau)$ be a topological space and $W \subseteq X$ , $K \subseteq W$, to prove K is a subspace from X, must prove :

**(1)** $K \subseteq X$

**(2)** if $A \in (\tau_W)_K \Rightarrow \exists\, U \in \tau$ ; $A = K \cap U$

Now,

(1) Since $K \subseteq W \subseteq X \Rightarrow K \subseteq X$.

(2) Let $A \in (\tau_W)_K \Rightarrow \exists\, V \in \tau_W$ ; $A = K \cap V$

(def. of induce top. and K is a sub space from W)

$\because$ $V \in \tau_W \Rightarrow \exists\, U \in \tau$ ; $V = W \cap U$

(def. of induce top. and W is a sub space from X)

$\because$ $A = K \cap V \Rightarrow A = K \cap (W \cap U)$ (since $V = W \cap U$)

$\Rightarrow A = (K \cap W) \cap U$ ($\cap$ associative)

$\Rightarrow A = K \cap U$ (since $K \subseteq W$ and $K = K \cap W$)

**Definition : Restriction Function**

Let $f : X \to Y$ be a function and let $A \subseteq X$. We say the function $g : A \to Y$ such that $g(a) = f(a)$ for all $a \in A$ is the **restriction function** on the set A and denoted by $g = f|_A$ .

**Example :** Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 1$ be a function.

Notes that the domain of f is $\mathbb{R}$, take $\mathbb{N} \subseteq \mathbb{R}$ and

$f|_{\mathbb{N}} : \mathbb{N} \to \mathbb{R}$ such that $(f|_{\mathbb{N}})(x) = x + 1$.

$f|_{\mathbb{N}}$ is restriction function on the set $\mathbb{N}$.

We will use this definition to introduce the following theorem :

**Theorem :** Let $f : (X, \tau) \to (Y, \tau')$ be a continuous function and W be a subspace topology from X. Then $f|_W$ is continuous.

**Proof :** To prove, $f|_W : (W, \tau_W) \to (Y, \tau')$ is continuous ??
i.e., to prove if $V \in \tau' \Rightarrow (f|_W)^{-1}(V) \in \tau_W$
Let $V \in \tau' \Rightarrow f^{-1}(V) \in \tau$ (since f is continuous)
$\Rightarrow W \cap f^{-1}(V) \in \tau_W$ (By def. of $\tau_W$)
$\Rightarrow (f|_W)^{-1}(V) \in \tau_W$ (since $W \cap f^{-1}(V) = (f|_W)^{-1}(V)$ )
$\therefore f|_W$ is cont.

**Remark :** From previous theorem we can get on an infinite number from continuous functions thought out know one continuous function for example :
$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ such that $f(x) = x + 2$ is continuous function.
$\therefore$ every functions of follows is continues function :
$f|_{\mathbb{N}}$, $f|_{\mathbb{Q}}$, $f|_{[2, 3]}$, $f|_{(0,\infty)}$ … etc.

We can get another of an infinite number from new continuous functions by theorem blow :

**Theorem :** Let $(X, \tau)$ be a topological space and $(W, \tau_W)$ be a subspace of X. Then the inclusion function $i : (W, \tau_W) \to (X, \tau)$ such that $i(x) = x$ for all $x \in W$ is continuous.
**Proof :** To prove if $V \in \tau \Rightarrow i^{-1}(V) \in \tau_W$
Let $V \in \tau \Rightarrow i^{-1}(V) = W \cap V$ (since $W \subseteq X$ and def. of $i$)
$\Rightarrow i^{-1}(V) \in \tau_W$ (since $W \cap V \in \tau_W$ and by def. of $\tau_W$)
$\therefore i$ is cont.

Let $(X, \tau)$ be a topological space and $(W, \tau_W)$ be a subspace topology of X and $A \subseteq W \subseteq X$. We can compute $\overline{A}$, $A^o$, $A^b$ in $(X, \tau)$ and from the other hand we can compute $\overline{A}$, $A^o$, $A^b$ in $(W, \tau_W)$. The question what relation between ($\overline{A}$ in X and $\overline{A}$ in W), ($A^o$ in X and $A^o$ in W) and ($A^b$ in X and $A^b$ in W) ?? The theorem blow answer on this questions :

**Theorem :** Let $(X, \tau)$ be a topological space and $(W, \tau_W)$ be a subspace topology of X and $A \subseteq W \subseteq X$, then
**(1)** $W \cap \overline{A} = \overline{A}$ in W ; $\overline{A}$ is closure of A in X
**(2)** $W \cap A^o \subseteq A^o$ in W.
**(3)** $W \cap A^b \supseteq A^b$ in W.
**Proof :**

(1) To prove, $W \cap \overline{A} = \overline{A}$ in W

we must prove, $W \cap \overline{A} \subseteq \overline{A}$ in W and $W \cap \overline{A} \supseteq \overline{A}$ in W

$\because A \subseteq W \subseteq X \Rightarrow \overline{A} \in \mathcal{F}$ (by previous theorem $\overline{A}$ is closed in X)

$\Rightarrow \overline{A} \cap W \in \mathcal{F}_W$ ($\overline{A}$ is closed in W)

Now,

$A \subseteq W \wedge A \subseteq \overline{A} \Rightarrow A \subseteq W \cap \overline{A}$

Notes that $W \cap \overline{A}$ is closed set in W and containing A, but $\overline{A}$ in W is the smallest closed set in W contain A, so we get

$\Rightarrow \overline{A}$ in $W \subseteq W \cap \overline{A}$ ---------------(1)

and,

$\because W \cap \overline{A} \in \mathcal{F}_W \Rightarrow \exists F \in \mathcal{F} : \overline{A}$ in $W = W \cap F$

$\Rightarrow A \subseteq F$

$\Rightarrow \overline{A} \subseteq \overline{F} = F \Rightarrow \overline{A} \subseteq F$ ($\overline{F} = F$ since F is closed)

$\Rightarrow W \cap \overline{A} \subseteq W \cap F = \overline{A}$ in W

$\Rightarrow W \cap \overline{A} \subseteq \overline{A}$ in W ----------------(2)

From (1) and (2) we have, $W \cap \overline{A} = \overline{A}$ in W.

(2) To prove, $W \cap A^o \subseteq A^o$ in W

$A^o \in \tau \Rightarrow W \cap A^o \in \tau_W$

$\Rightarrow W \cap A^o \subseteq A^o \subseteq A \Rightarrow W \cap A^o \subseteq A$

$\Rightarrow W \cap A^o \subseteq A^o$ in W (since $W \cap A^o$ open in W contain in A)

i.e., $A^o$ in W must contain all open set in W contain in A.

(3) To prove, $A^b$ in $W \subseteq A^b \cap W$.

Let $x \in A^b$ in $W \Rightarrow \forall V \in \tau_W, x \in V ; V \cap A \neq \phi \wedge V \cap A^c \neq \phi$

(By def. of boundary point)

$\because V \in \tau_W \Rightarrow \exists U \in \tau ; V = W \cap U$ (def. of $\tau_W$)

$\Rightarrow \forall U \in \tau, x \in U, U \cap A \neq \phi \wedge U \cap A^c \neq \phi$

$\Rightarrow x \in A^b \Rightarrow x \in A^b \cap W$ (since $x \in V \subseteq W$ )

$\therefore A^b$ in $W \subseteq A^b \cap W$

**Remark :** The equality of properties (2) and (3) in the previous theorem is not true in general and the following example clear that :

**Example :** Let X = {a, b, c, d}, $\tau$ = {X, $\phi$, {a}, {a, b}, {a, b, c}} and W = {a, c, d}.

Then $\tau_W$ = {W, $\phi$, {a}, {a, c}}, A = {a, c}

A = {a, c} = $A^o$ in W since $A \in \tau_W$ and $A^o$ = {a}

Since {a} is largest open set in X containing in A $\Rightarrow$

$A^o \cap W = \{a\} \cap \{a, c, d\} = \{a\} \Rightarrow A^o$ in $W \neq A^o \cap W$ since $\{a, c\} \neq \{a\}$
On the other hand to compute $A^b$, $A^b$ in W, we will compute $A^x$, $A^x$ in W such that
$A^x$ in $W = \phi$ , then
$A^b$ in $W = W - (A^o$ in $W \cup A^x$ in $W) = W - \{a, c\} = \{d\}$
$\therefore$ $A^b$ in $W = \{d\}$
$A^x = \phi \Rightarrow A^b = X - (A^o \cup A^x) = X - \{a\} = \{b, c, d\}$
$\Rightarrow A^b \cap W = \{c, d\}$
$\therefore$ $A^b$ in $W \neq A^b \cap W$
To check property (1) in the previous theorem we compute $\bar{A}$ and $\bar{A}$ in W as follow :
$\bar{A}$ in $W = W$ and $\bar{A} = X \Rightarrow \bar{A} \cap W = X \cap W = W$
$\therefore$ $\bar{A}$ in $W = \bar{A} \cap W$

## ***Product Space***

**Definition : Cartesian Product**
Let X and Y be any two sets. The **Cartesian product** , or simply **product** of X by Y is denoted by X×Y and denoted as :

$$X \times Y = \{(x, y) ; x \in X \wedge y \in Y\}$$

**Definition : Product Space**
Let $(X, \tau)$ and $(Y, \tau')$ be two topological spaces. We say the topology has a base $\beta$ ;

$$\beta = \{ U \times V ; U \in \tau \wedge V \in \tau'\}$$

Is the **Product Topology** on the set X×Y and denoted by $\tau_{X\times Y}$ and called the spaces $(X\times Y, \tau_{X\times Y})$ is the **Product Space** of X by Y.

**Remark :** Notes that $\beta$ in general not topology since it's not satisfy the third condition of topology, but since $\beta$ is a base for topology, so we can get the three condition of topology and the following example show that :

**Example :** Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi, \{b\}\}$.
Compute $\tau_{X\times Y}$ .
**Solution :**
$X\times Y = \{(x, y) ; x \in X \wedge y \in Y\}$
$= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
$\beta = \{ U\times V ; U \in \tau \wedge V \in \tau'\}$
$= \{X\times Y, X\times\phi, X\times\{b\}, \phi\times Y, \phi\times\phi, \phi\times\{b\}, \{1\}\times Y, \{1\}\times\phi, \{1\}\times\{b\}\}$

$= \{X\times Y, \phi, X\times\{b\}, \{1\}\times Y, \{1\}\times\{b\}\}$

Since $A\times\phi = \phi$ and $\phi\times A = \phi$ for any set A.

$\therefore\ \beta = \{X\times Y, \phi, \{(1, b), (2, b), (3, b)\}, \{(1, a), (1, b)\}, \{(1, b)\}\}$

Notes that β is not topology since

$\{(1, b), (2, b), (3, b)\} \bigcup \{(1, a), (1, b)\}\} \notin \beta$

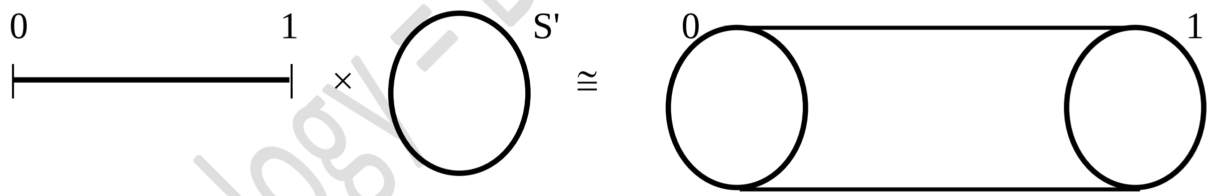
The elements of $\tau_{X\times Y}$ is elements of β and add all possible union of elements of β ;

$\therefore\ \tau_{X\times Y} = \{X\times Y, \phi, \{(1, b), (2, b), (3, b)\}, \{(1, a), (1, b)\}, \{(1, b)\}, \{(1, b), (2, b), (3, b), (1, a)\}\}$.

**Remark :** We can compute the product space $(X\times X, \tau_{X\times X})$ depending on $(X, \tau_X)$ only, also we can compute $(Y\times Y, \tau_{Y\times Y})$, $(Y\times X, \tau_{Y\times X})$, $(X\times Y\times Z, \tau_{X\times Y\times Z})$, … etc., there are an infinite number from product spaces which can computing from one space known or more than one space. In general $X\times Y \neq Y\times Y$.

From known product spaces which we use always is usual space $\mathbb{R}^n$ ; $n \in \mathbb{N}$ and the most common one is $\mathbb{R}^2$ which represent the plane and its product space follow from product $(\mathbb{R}, \tau_u)$ by self.

**Example :** Let X = [0, 1] be a subspace of $(\mathbb{R}, \tau_u)$ and take $S' = \{(x, y) \in \mathbb{R}^2; x^2+y^2 =1\}$ be a subspace of $(\mathbb{R}^2, \tau_u)$ such that S' geometry represented as a circle in plane its center the original point (0, 0). Then $[0, 1] \times S'$ is a cylinder as follow :




**Remarks :** Let $(X, \tau)$ and $(Y, \tau')$ be any two topological spaces.

**[1]** If $\tau = I_X$ and $\tau' = I_Y$ , then $\tau_{X\times Y} = I_{X\times Y}$ , i.e.,

If $\tau = \{X, \phi\}$ and $\tau' = \{Y, \phi\}$, then $\tau_{X\times Y} = \{X\times Y, \phi\}$ and β is :

$$\beta = \{X\times Y, X\times\phi, \phi\times Y, \phi\times\phi\} = \{X\times Y, \phi\} = I_{X\times Y}$$

**[2]** If $\tau = I_X$ and $\tau' \neq I_Y$ or $\tau \neq I_X$ and $\tau' = I_Y$, then $\tau_{X\times Y} \neq I_{X\times Y}$ , for example :

If X = {1, 2}, $\tau = \{X, \phi, \{1\}\}$, Y = {a, b} and $\tau' = \{Y, \phi\}$, then

$$\beta = \{X\times Y, \phi, \{1\}\times Y\} = \{X\times Y, \phi, \{(1, a), (1, b)\}\} = \tau_{X\times Y} \neq I_{X\times Y}$$

**[3]** If $\tau = D_X$ and $\tau' = D_Y$ , then $\tau_{X\times Y} = D_{X\times Y}$ ,

**Proof.** To prove any topology is discrete topology it's enough to prove every singleton set is open i.e., $(\forall \{(x, y)\}$ singleton set $\Rightarrow \{(x, y)\} \in \tau_{X\times Y})$

$\{(x, y)\} = \{x\} \times \{y\}$ (def. Cartesian product of X by Y)

$\{x\} \in \tau$ (since $\tau = D_X$)

$\{y\} \in \tau'$ (since $\tau' = D_Y$)

$\{(x, y)\} \in \beta_{X\times Y}$

$\Rightarrow \{(x, y)\} \in \tau_{X\times Y}$ (def. product spaces)

**[4]** If $\tau \neq D_X$ or $\tau' \neq D_Y$, then $\tau_{X\times Y} \neq D_{X\times Y}$, and the following example show that :
If $X = \{1, 2\}$, $\tau = \{X, \phi, \{1\}, \{2\}\} = D_X$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi\} \neq D_Y$, then

$\beta = \{X\times Y, \phi, \{1\}\times Y, \{2\}\times Y\}$

$= \{X\times Y, \phi, \{(1, a), (1, b)\}, \{(2, a), (2, b)\}\} = \beta_{X\times Y} = \tau_{X\times Y} \neq D_{X\times Y}$

**[5]** If $A \subseteq X$ and $B \subseteq Y$, then $A\times B \subseteq X\times Y$ and we can compute the closure of $A\times B$ in $X\times Y$ (i.e., $\overline{A \times B}$), on the other hand there are $\overline{A}$ in X and $\overline{B}$ in Y, also we can compute $\overline{A} \times \overline{B}$ and the question what relation between $\overline{A \times B}$ and $\overline{A} \times \overline{B}$ and the answer $\overline{A \times B} = \overline{A} \times \overline{B}$.
Also, by similar way we can conclusion $(A\times B)^o = A^o\times B^o$.

**[6]** There are two natural projection functions from product space $X\times Y$ to codomain X and others to codomainY and denoted by $P_X$ and $P_Y$ and called the first project $X\times Y$ on X and called the second project $X\times Y$ on Y. We will show that the two functions are surjective , continuous and open as follows :

$P_X : X\times Y \to X$ ; $P_X((x, y)) = x$ and

$P_Y : X\times Y \to Y$ ; $P_Y((x, y)) = y$

; the first projection map the order pair (x, y) to first coordinate while the second projection map the order pair (x, y) to the second coordinate.

**<u>To prove, $P_X$ is continuous function</u>**

We must prove, if $U \in \tau \Rightarrow P_X^{-1}(U) \in \tau_{X\times Y}$

Let $U \in \tau \Rightarrow P_X^{-1}(U) = U\times Y$ (By def. of $P_X$)

$\because U \in \tau \wedge Y \in \tau' \Rightarrow U\times Y \in \beta_{X\times Y}$

$\Rightarrow U\times Y \in \tau_{X\times Y}$ (since $\beta_{X\times Y} \subset \tau_{X\times Y}$)

$\Rightarrow P_X^{-1}(U) \in \tau_{X\times Y}$

$\therefore P_X$ is continuous functions

By similar way we prove $P_Y$ is continuous functions

Let $V \in \tau' \Rightarrow P_Y^{-1}(V) = X\times V$ (By def. of $P_Y$)

$\because X \in \tau \wedge V \in \tau' \Rightarrow X\times V \in \beta_{X\times Y}$

$\Rightarrow X\times V \in \tau_{X\times Y}$ (since $\beta_{X\times Y} \subset \tau_{X\times Y}$)

$\Rightarrow P_Y^{-1}(V) \in \tau_{X\times Y}$

$\therefore P_Y$ is continuous functions

**<u>To prove, $P_X$ is open function</u>**

Let $U\times V \in \beta_{X\times Y} \Rightarrow U\times V$ open set in $X\times Y$ ; $U \in \tau \wedge V \in \tau'$

$\Rightarrow P_X(U\times V) = U$

$\because U \in \tau \Rightarrow P_X(U\times V) \in \tau$

$\therefore P_X$ is open functions

By similar way we prove $P_Y$ is open functions

Let $U\times V \in \beta_{X\times Y} \Rightarrow U\times V$ open set in $X\times Y$ ; $U \in \tau \wedge V \in \tau'$

$\Rightarrow P_Y(U\times V) = V$

$\because V \in \tau' \Rightarrow P_Y(U\times V) \in \tau$

$\therefore P_Y$ is open functions.

**[7]** Notes that $X\times Y \neq Y\times X$ since $(x, y) \neq (y, x)$ in general, but $X\times Y \cong Y\times X$ (i.e., $X\times Y$, $Y\times X$ are Homeomorphic), to prove this :

Define $f : X\times Y \to Y\times X$ ; $f((x, y)) = (y, x)$

f is 1-1 function since,

Let $f((x_1, y_1)) = f((x_2, y_2)) \Rightarrow (y_1, x_1) = (y_2, x_2)$

$\Rightarrow x_1 = x_2 \wedge y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$.

f is onto function since,

$\forall (y, x) \in Y\times X \ \exists (x, y) \in X\times Y ; f((x, y)) = (y, x)$.

f is continuous function since,

Let $\beta$ be a base of $X\times Y$ and $\beta'$ be a base of $Y\times X$

Let $V\times U \in \beta' \Rightarrow V \in \tau' \wedge U \in \tau$

$\Rightarrow V\times U \in \tau_{Y\times X}$

$\Rightarrow f^{-1}(V\times U) = U\times V$ open set in $X\times Y$

f is open function since, the image of every open set in domain is open set in codomain ;

Let $U\times V \in \beta \Rightarrow U\times V$ open set in $X\times Y$ ; $U \in \tau \wedge V \in \tau'$

$\Rightarrow f(U\times V) = V\times U \in \tau_{Y\times X}$

$\therefore$ f is homeomorphism function.

**[8]** If $y_0 \in Y$, then the product space $X\times\{y_0\}$ topological equivalent the space X. i.e., $X\times\{y_0\} \cong X$ ; $X\times\{y_0\} = \{(x, y_0) : x \in X\}$

To prove this :

Define $f : X \to X\times\{y_0\}$ ; $f(x) = (x, y_0) \quad \forall x \in X$

f is 1-1 function since,

Let $f(x_1) = f(x_2) \Rightarrow (x_1, y_0) = (x_2, y_0)$

$\Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X$

f is onto function since,

$\forall (x, y_0) \in X\times\{y_0\} \ \exists x \in X ; f(x) = (x, y_0)$

f is continuous function, since the sets in the base of the space $X\times\{y_0\}$ is $U\times\{y_0\}$ ; $U \in \tau$ or $\phi$, then

$f^{-1}(U\times\{y_0\}) = U \in \tau$ and $f^{-1}(\phi) = \phi \in \tau$

f is open function, since if U is open in domain X, then $f(U) = U\times\{y_0\}$ and $U\times\{y_0\}$ is open in codomain $X\times\{y_0\}$.

$\therefore$ f is homeomorphism function.

**[9]** If $x_0 \in X$, then the product space $\{x_0\}\times Y$ topological equivalent the space Y. i.e., $\{x_0\}\times Y \cong Y$ ; $\{x_0\}\times Y = \{(x_0, y) : y \in Y\}$

To prove this : (By a similar way of [8])

Define $f : Y \to \{x_0\}\times Y$ ; $f(y) = (x_0, y)$ $\forall\ y \in Y$

f is 1-1 function since,

Let $f(y_1) = f(y_2) \Rightarrow (x_0, y_1) = (x_0, y_2)$

$\Rightarrow y_1 = y_2$ $\forall\ y_1, y_2 \in Y$

f is onto function since,

$\forall\ (x_0, y) \in \{x_0\}\times Y\ \exists\ y \in Y ; f(y) = (x_0, y)$

f is continuous function, since the sets in the base of the space $\{x_0\}\times Y$ is $\{x_0\}\times V$ ; $V \in \tau'$ or $\phi$, then

$f^{-1}(\{x_0\}\times V) = V \in \tau'$ and $f^{-1}(\phi) = \phi \in \tau'$

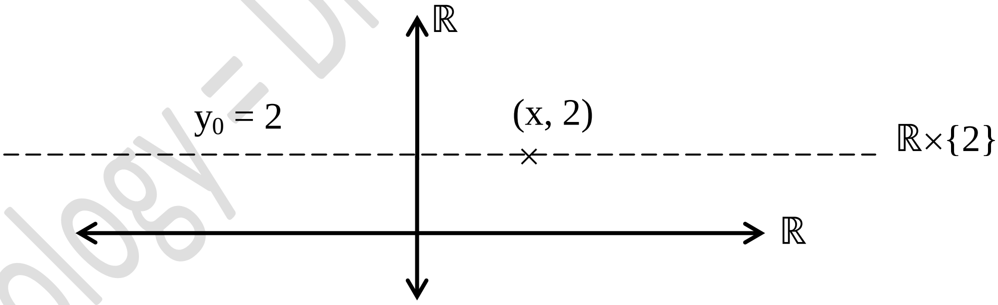
f is open function, since if V is open in domain Y, then $f(V) = \{x_0\}\times V$ and $\{x_0\}\times V$ is open in codomain $\{x_0\}\times V$.

$\therefore$ f is homeomorphism function.

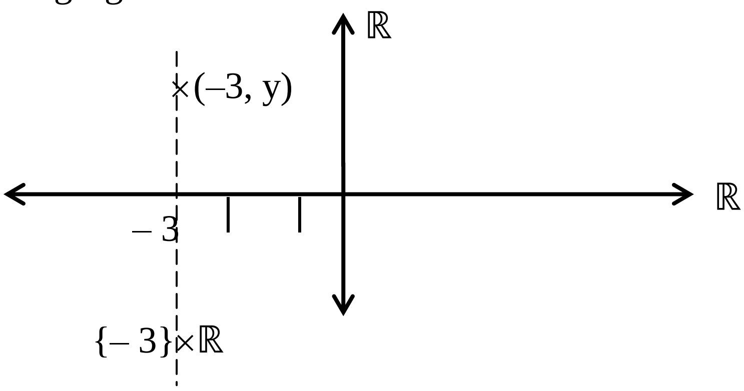
Notes that $X\times\{y_0\}$ is a sub space of the space $X\times Y$ and represented horizontal section in the space $X\times Y$ at the point $y_0$ . Also, $\{x_0\}\times Y$ is a sub space of the space $X\times Y$ and represented vertical section in the space $X\times Y$ at the point $x_0$.

**For example**, take $X = Y = \mathbb{R}$ and $\tau = \tau' = \tau_u$ , then the product space $X\times Y$ is the known plane $\mathbb{R}^2$.

Let $y_0 = 2$ , then $X\times\{y_0\} = \mathbb{R}\times\{2\}$ is subspace from $\mathbb{R}^2$ and represented as horizontal line segment and the following figure show this :




Let $x_0 = -3$ , then $\{x_0\}\times Y = \{-3\}\times\mathbb{R}$ is subspace from $\mathbb{R}^2$ and represented as vertical line segment and the following figure show this :



**Definition : Quotient Space**

Let $(X, \tau_X)$ be a topological space and Y be any set. Let $f : X \to Y$ be a surjective function, then the set

$$\tau_f = \{G \subseteq Y \, ; \, f^{-1}(G) \in \tau_X\}$$

Is a topology on Y this topology called **quotient topology** on Y generated by f and $(X, \tau_X)$.

**Question :** The topology $\tau_f = \{G \subseteq Y \, ; \, f^{-1}(G) \in \tau_X\}$ is the largest topology on Y make the function f continuous.

**Answer :** Let $\tau$ be another topology on Y making f continuous.

$\Rightarrow$ $f^{-1}(G)$ is open in X $\forall$ $G \in \tau$.

$\Rightarrow$ $G \in \tau_f$ (def. of $\tau_f$ )

$\Rightarrow$ $\tau \subseteq \tau_f$

$\Rightarrow$ $\tau_f$ is the largest topology on Y making f is continuous.

**Theorem :** Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous surjective function, if f either open or closed, then $\tau_f = \tau_Y$.

**Proof :**

Clearly, $\tau_Y \subseteq \tau_f$ (by previous question)

Now, to show that $\tau_f \subseteq \tau_Y$

Let $G \in \tau_f \Rightarrow f^{-1}(G) \in \tau_X$

$\Rightarrow$ $f(f^{-1}(G)) = G$ is open in Y (since f is open)

$\Rightarrow$ $G \in \tau_Y$

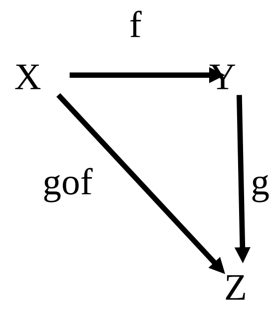
$\Rightarrow$ $\tau_f \subseteq \tau_Y$

So, $\tau_f = \tau_Y$.

By similar way if f is closed.

**Theorem :** Let Y has the quotient space generated by the surjective function $f : X \to Y$, then $g : Y \to Z$ is continuous function if and only if gof is continuous function.

**Proof :**




($\Rightarrow$) The composition of continuous functions is continuous.

($\Leftarrow$) Let G be open set in Z

Since gof is cont. $\Rightarrow$ $(gof)^{-1}(G) = f^{-1}(g^{-1}(G))$ is open in X

But $g^{-1}(G) \subseteq Y \wedge f^{-1}(g^{-1}(G))$ is open in X

$\Rightarrow$ $g^{-1}(G)$ is open in Y (by definition of $\tau_f$ , $g^{-1}(G) \in \tau_f$ )

$\Rightarrow$ g is continuous.

**Remarks :**

**[1]** Let X be a nonempty set. The **partition or decomposition** on X with the relation R is the family of disjoint nonempty subsets of X and their union equal X. The elements of this partition called **equivalence classes** and denoted by [x].

**[2]** The set of equivalence classes for X is called **quotient set** for X with the relation R and denoted by $X|R = \{[x] : x \in X\}$.

**[3]** The mapping $p : X \to X/R$ ; $p(x) = [x]$ is called **quotient mapping**.

**Definition : Quotient Space**

Let $(X, \tau)$ be a topological space and R be equivalence relation on X. Let $p : X \to X/R$ ; $p(x) = [x]$ be surjective quotient mapping from X to X|R, then the quotient topology on X|R is the largest topology make the function f continuous and the space $(X|R, \tau_{X|R})$ is called **quotient space**.