Chapter Three : Compact Spaces

<u>Definition :</u> Cover & **Finite Cover & Open (resp., Closed) Cover**

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of subsets of the space (X, τ). We called the family $\{A_\alpha\}_{\alpha\in\Lambda}$ **<u>cover</u>** of X iff X equal the union of elements of the family $\{A_\alpha\}_{\alpha\in\Lambda}$.

$$\text{(i.e., } X = \bigcup_{\alpha\in\Lambda} A_\alpha)$$

If $\{A_\alpha\}_{\alpha\in\Lambda}$ is finite and cover X, then $\{A_\alpha\}_{\alpha\in\Lambda}$ is called a **<u>finite cover</u>** of X.

If each A_α, $\alpha\in\Lambda$, is open (resp., closed) in X and $\{A_\alpha\}_{\alpha\in\Lambda}$ cover X, then $\{A_\alpha\}_{\alpha\in\Lambda}$ is called an **<u>open (resp., closed) cover</u>** of X.

<u>Definition :</u> Subcover

Let $C = \{A_\alpha\}_{\alpha\in\Lambda}$ be a cover of X and $\{B_i\}_{i\in\Lambda}$ be a sub family of C and cover X, then $\{B_i\}_{i\in\Lambda}$ is called **<u>subcover</u>** from C.

<u>Definition :</u> Compact Space

A space X is called **<u>compact</u>** iff each open cover of X has a finite subcover for X.
i.e.,

$$X \text{ is compact } \Leftrightarrow \forall\ C = \{U_\alpha\}_{\alpha\in\Lambda}\ ;\ U_\alpha \in \tau\ \forall\alpha \ \wedge\ X = \bigcup_{\alpha\in\Lambda} U_\alpha$$
$$\Rightarrow \exists\ \alpha_1, \alpha_2, \ldots, \alpha_n\ ;\ X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$
$$X \text{ is not compact } \Leftrightarrow \exists\ C = \{U_\alpha\}_{\alpha\in\Lambda}\ ;\ U_\alpha \in \tau\ \forall\alpha \ \wedge\ X = \bigcup_{\alpha\in\Lambda} U_\alpha$$
$$\Rightarrow \nexists\ \alpha_1, \alpha_2, \ldots, \alpha_n\ ;\ X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

<u>Example :</u> Take $X = \mathbb{R}$ and $\tau = \{\mathbb{R}, \phi, \mathbb{Q}, Irr\}$

The open set in τ are $\mathbb{R}, \phi, \mathbb{Q}, Irr$.

Take, $C_1 = \{\mathbb{Q}, Irr\}$ is open cover for $\mathbb{R}$ (i.e., $\mathbb{R} = \mathbb{Q} \bigcup Irr$) and it's a finite subcover of $\mathbb{R}$, so this cover satisfy the definition of compact space.

Now, we introduce all open cover for $\mathbb{R}$ as follow :

$$C_2 = \{\mathbb{R}, \phi, \mathbb{Q}\},\ C_3 = \{\mathbb{R}, \phi, Irr\},\ C_4 = \{\mathbb{R}, \mathbb{Q}, Irr\},\ C_5 = \{\phi, \mathbb{Q}, Irr\},$$
$$C_6 = \{\mathbb{R}, \phi\},\ C_7 = \{\mathbb{R}, \mathbb{Q}\},\ C_8 = \{\mathbb{R}, Irr\},\ C_9 = \{\mathbb{R}, \phi, \mathbb{Q}, Irr\} = \tau$$

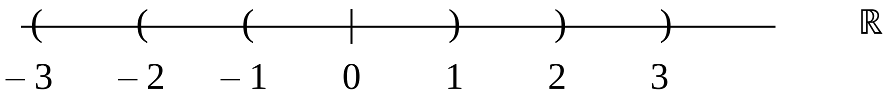
and every cover of them has a finite subcover, hence $(\mathbb{R}, \tau)$ is compact.

<u>Remark :</u> If we want to show that the space is not compact it's enough give one open cover, but has not finite subcover. The following example show this :

Example : Is $(\mathbb{R}, \tau_u)$ compact space ???

Answer : **No**

Take the open cover $C = \{(-n, n)\ ;\ n \in \mathbb{N}\}$ for $\mathbb{R}$ (i.e., $\mathbb{R} = \bigcup_{i=1}^{n}(-n, n)$).



Notes that the open cover C has no finite subcover, because if we assumption there exists a finite open interval cover $\mathbb{R}$, then their union is the large interval, for example $(-m, m)$; $m \in \mathbb{N}$, this means $\mathbb{R} = (-m, m)$, $m \neq \infty$ and this contradiction!!!

Example : Show that $(\mathbb{N}, \tau_{cof})$ is compact space.

Solution : Let $C = \{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover for $\mathbb{N}$, then

$$\mathbb{N} = \bigcup_{\alpha\in\Lambda} U_\alpha \ ;\ U_\alpha \in \tau_{cof} \ \ \forall\ \alpha \in \Lambda$$

since $U_\alpha \in \tau_{cof}$, then $\mathbb{N} - U_\alpha$ if finite , for all $U_\alpha \in \tau_{cof}$

take arbitrary set say U_{α_n}, then $\mathbb{N} - U_{\alpha_n}$ if finite,

let $\mathbb{N} - U_{\alpha_n} = \{x_1, \ldots, x_{n-1}\}$; $x_1, \ldots, x_{n-1} \in \mathbb{N}$

this means that U_{α_n} contains all natural numbers excepts $x_1, \ldots, x_{n-1}$

take another set contains x_1 say U_{α_1} and set contains x_2 say U_{α_2} …. etc set contains x_{n-1} say $U_{\alpha_{n-1}}$. So we have n set which are U_{α_n} ,......, U_{α_1} such that $\mathbb{N} = \bigcup_{i=1}^{n} U_{\alpha_i}$. therefore, the open cover $C = \{U_\alpha\}_{\alpha\in\Lambda}$ has a finite subcover $\{U_{\alpha_i}\}_{i=1}^{n}$, hence $(\mathbb{N}, \tau_{cof})$ is compact.

Definition : Compact Subspace

Let (X, τ) be a topological space and W be a subspace of X. We called a space W is **compact space** iff every open cover from X cover W has a finite subcover. i.e.,

$$W \text{ is compact} \Leftrightarrow \forall\ \{U_\alpha\}_{\alpha\in\Lambda}\ ;\ U_\alpha \in \tau\ \ \forall\ \alpha\ \wedge\ W \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$$
$$\Rightarrow\ \exists\ \alpha_1, \alpha_2, \ldots, \alpha_n\ ;\ W \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Theorem : (Heine-Borel Theorem)

The subset $A \subseteq X$ is compact iff A is closed and bounded.

Remarks : The previous theorem is one of theorems which study in mathematical analysis which is specific to subsets of Euclidean space $\mathbb{R}$ with usual topology and special case in $(\mathbb{R}, \tau_u)$. So, every subset of $\mathbb{R}$ is compact iff its closed and bounded.

Example : In $(\mathbb{R}, \tau_u)$ show that any set from the following sets is compact by using Heine-Borel Theorem.

$A = (2, 3)$, , $B = [5, 7]$, $C = [-2, 1)$, $D = \mathbb{N}$, , $E = \{2, 3, 4\}$, $F = \mathbb{Q}$

Solution :

A not closed and bounded $\Rightarrow$ not compact.

B closed and bounded $\Rightarrow$ compact.

C not closed and bounded $\Rightarrow$ not compact.

D closed and not bounded $\Rightarrow$ not compact.

E closed and bounded $\Rightarrow$ compact.

F not closed and not bounded $\Rightarrow$ not compact.

In general in usual topology $(\mathbb{R}, \tau_u)$,

every closed intervals is compact sets.

every half closed (open) interval and open intervals is not compact sets.

every finite sets of points is compact sets.

$\mathbb{Q}$ and Irr not compact.

Definition : Hereditary Property

We call a property "P" of a space (X, τ) **hereditary property** iff every subspace of a space X with the property must have the property.

Notes that if there exists at least one subspace not satisfy this property, then this property not hereditary property.

Remark : Compactness is not hereditary property. For example :

Example : Take $X = [0,1]$ with induce topology from $(\mathbb{R}, \tau_u)$ and take $W = (0, 1)$. Clear that $W \subseteq X$ and X is compact space, but W is not compact.

Theorem : If A and B are compact sets in a space (X, τ), then $A \bigcup B$ is compact set.

Proof : Let $C = \{U_\alpha\}_{\alpha\in\Lambda}$; $U_\alpha \in \tau \;\; \forall \alpha$ open cover of $A \bigcup B$.

To prove C has a finite subcover

$\because \; A \bigcup B \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow A \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \wedge B \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

(since $A \subseteq A \bigcup B$, $B \subseteq A \bigcup B$)

$\therefore$ C cover of A and B, but A and B are compact

$\Rightarrow \exists \, \alpha_1, \dots, \alpha_n$; $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ and $\exists \, \alpha_1, \dots, \alpha_m$; $B \subseteq \bigcup_{j=1}^{m} U_{\alpha_j}$

$\Rightarrow A \bigcup B \subseteq \bigcup_{k=1}^{n+m} U_{\alpha_k}$

$\therefore$ C has finite subcovet for $A \bigcup B \Rightarrow A \bigcup B$ compact set.

Remark : If A and B are compact sets in a space (X, τ), then $A \bigcap B$ is not necessary compact set.

Example : Let $X = \mathbb{N} \bigcup \{o, -1\}$, $\tau = \{U \subseteq X : -1, 0 \in U \wedge U^c \text{ finite}\} \bigcup P(\mathbb{N})$
$A = \{0\} \bigcup \mathbb{N}$, $B = \{-1\} \bigcup \mathbb{N}$
Notice that A and B are compact subsets of X, but $A \bigcap B$ is not compact, since there exists $\mathbb{G} = \{\{n\} : n \in \mathbb{N}\}$ is an open cover of $A \bigcap B$, but it has no finite subcover.

Remarks :

[1] If τ is finite set, then (X, τ) is compact space, since every open cover heir being finite, so every open cover has a finite subcover.
Special case : (X, I) is compact space for any X (finite or infinite), for example $(\mathbb{R}, I)$ and $(\mathbb{N}, I)$ are compact spaces … etc.
Another special case : if X is finite, then τ is finite set and $\tau \subseteq IP(X)$, therefore (X, τ) is compact space.

[2] If X is infinite set, then (X, D) is not compact space, since the open cover $C = \{\{x\} ; x \in X\}$ has no finite subcover. If X is finite, then (X, D) is compact space (by Remark [1]).

Theorem : The continuous image of compact space is compact. i.e.,
If $f : (X, \tau) \to (Y, \tau')$ is continuous function and X is compact space, then f(X) is compact.

Proof : Let $f : (X, \tau) \to (Y, \tau')$ be continuous and X compact space.
To prove, f(X) compact in Y
Let $C = \{V_\alpha\}_{\alpha\in\Lambda}$ open cover for f(X)

$\Rightarrow f(X) \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$; $V_\alpha \in \tau'$ $\forall \alpha \in \Lambda$

$\Rightarrow f^{-1}(f(X)) \subseteq f^{-1}(\bigcup_{\alpha\in\Lambda} V_\alpha)$

$\Rightarrow X \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$ (since $f^{-1}(f(X)) = X$ and $f^{-1}(\bigcup_{\alpha\in\Lambda} A_\alpha) = \bigcup_{\alpha\in\Lambda} f^{-1}(A_\alpha)$)

Since f is continuous $\Rightarrow f^{-1}(V_\alpha) \in \tau$ $\forall \alpha \in \Lambda$

$\Rightarrow \{f^{-1}(V_\alpha)\}_{\alpha\in\Lambda}$ is open cover for X

$\because$ X is compact $\Rightarrow \exists\, \alpha_1, \dots, \alpha_n$; $X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$

$\Rightarrow f(X) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}))$

$\Rightarrow \quad = \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i}))$ (since $f(A \bigcup B) = f(A) \bigcup f(B)$)

$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$ (since $f(f^{-1}(A)) \subseteq A$)

$\therefore$ f(X) compact set.

<u>Corollary :</u> If the product space X×Y is compact, then X and Y are compact spaces.

<u>Proof :</u> The projection function $P_X : X\times Y \to X$ is continuous and onto

$\because$ X×Y compact $\Rightarrow$ $P_X(X\times Y)$ compact (by previous theorem)

$\because$ P_X onto $\Rightarrow$ $P_X(X\times Y) = X$

$\Rightarrow$ X compact

By the similar way we prove Y compact.

The projection function $P_Y : X\times Y \to Y$ is continuous and onto

$\because$ X×Y compact $\Rightarrow$ $P_Y(X\times Y)$ compact (by previous theorem)

$\because$ P_Y onto $\Rightarrow$ $P_Y(X\times Y) = Y$

$\Rightarrow$ Y compact

<u>Remark :</u> If X and Y are compact spaces, , then X×Y is compact spaces (i.e., the converse of the above theorem is true in general), and it's theorem one of the important theorem in topology called **<u>Tichonov theorem</u>** (without prove) and we will introduce some examples to user of this theorem.

$(\mathbb{R}, \tau_u)\times(\mathbb{N}, \tau_{cof})$ is not compact space, since $(\mathbb{R}, \tau_u)$ not compact.

$(\mathbb{N}, \tau_{cof})\times(\mathbb{N}, \tau_{cof})$ is compact space, since $(\mathbb{N}, \tau_{cof})$ compact.

$(\mathbb{N}, \tau_{cof})\times(\mathbb{R}, I)$ is compact space, since $(\mathbb{N}, \tau_{cof})$ compact and $(\mathbb{R}, I)$ compact.

$(\mathbb{R}, D)\times(\mathbb{R}, I)$ is not compact space, since $(\mathbb{R}, D)$ not compact.

$(\{1, 2, 3\}, \tau)\times(X, I)$ is compact space for any τ and for any X, since {1, 2, 3} is finite set and (X, I) compact for any X.

<u>Definition :</u> Topological Property

A property "P" of a topological space (X, τ) is called a **<u>topological property</u>** iff every topological space (Y, τ') homeomorphic to (X, τ) also has the same property. i.e.,
if $(X, \tau) \cong (Y, \tau')$ and (X, τ) has a property "P", then (Y, τ') has the same property and vise versa.

<u>Theorem :</u> Compactness is a topological property.

Proof : Let (X, τ) and (Y, τ') be topological space ; $X \cong Y$

Suppose that X is compact, To prove Y is compact

$\because$ $X \cong Y \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

$\because$ f continuous, onto and X compact $\Rightarrow$ $f(X) = Y$ compact

(by theorem : The continuous image of compact space is compact)

Now, suppose that Y is compact, To prove X is compact

$\because$ f^{-1} continuous, onto and Y compact $\Rightarrow$ $f^{-1}(Y) = X$ compact

(by same theorem : The continuous image of compact space is compact)

Remark : By using compactness as a topological property, we can decided the known space is equivalent another known space or not. Also, we can decided the space unknown (compact or not compact) if its equivalent another known space compact or not compact. For example :

$(\mathbb{R}, I) \ncong (\mathbb{R}, \tau_u)$ since $(\mathbb{R}, I)$ is compact, but $(\mathbb{R}, \tau_u)$ not compact. Also,

If $(\mathbb{N}, \tau_{cof}) \cong (Y, \tau)$ and since $(\mathbb{N}, \tau_{cof})$ is compact, then (Y, τ) is compact (since compactness is topological property).

Remark : Compactness is not hereditary property (remark p.67), but if we add a condition for the subset from compact space become a compact set and the following theorem show that :

Theorem : A **closed** subset of a compact space is compact.

Proof :

Let (X, τ) compact space and F closed set in X

To prove, F compact set

Let $C = \{U_\alpha\}_{\alpha\in\Lambda}$ open cover of F

$\Rightarrow F \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \; ; U_\alpha \in \tau \;\; \forall \alpha \in \Lambda$

$\because \; X = F \bigcup F^c \Rightarrow X = \bigcup_{\alpha\in\Lambda} U_\alpha \bigcup F^c$ (since $F \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$)

$\because \; U_\alpha \in \tau \;\; \forall \alpha \in \Lambda \;\wedge\; F^c \in \tau$ (since F closed set)

$\Rightarrow \{U_\alpha\}_{\alpha\in\Lambda} \bigcup \{F^c\}$ open cover of X

$\because$ X compact $\Rightarrow \exists\, \alpha_1, \dots, \alpha_n \; ; X = (\bigcup_{i=1}^{n} U_{\alpha_i}) \bigcup F^c$

But, $F \subseteq X \Rightarrow F \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i}) \bigcup F^c$

Since $F \bigcap F^c = \phi \Rightarrow F \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

$\therefore$ F compact set.

Notes that the condition being F closed is very important and the theorem is not true if the condition deleted.

Definition : Finite Intersection Property

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of sets. We call this family satisfy the **finite intersection property** and denoted by (f.i.p) if the intersection of any finite number of elements of this family is nonempty. i.e.,

$$\{A_\alpha\}_{\alpha\in\Lambda} \text{ has f.i.p.} \Leftrightarrow \bigcap_{i=1}^{n} A_{\alpha_i} \neq \phi \;\; \forall\, n$$

Example : Let $A_n = (-\frac{1}{n}, \frac{1}{n})$; $n \in \mathbb{N}$

Notes that, the intersection of any finite numbers of elements of this family is nonempty, so its satisfy (f.i.p).

Remark : If the family $\{A_\alpha\}_{\alpha\in\Lambda}$ satisfy (f.i.p), then it is not necessarily that the intersection every elements of the family is nonempty. i.e., not necessarily $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$.

Example : Let $A_n = \{n, n+1, \ldots\}$; $n \in \mathbb{N}$

$$A_1 = \mathbb{N},\ A_2 = \mathbb{N} \setminus \{1\},\ A_3 = \mathbb{N} \setminus \{1, 2\}, \ldots$$

Notes that, intersection every finite numbers of the family is nonempty while $\bigcap_{n=1}^{\infty} A_n = \phi$ and this family satisfy (f.i.p). i.e., in general

$$\bigcap_{i=1}^{n} A_{\alpha_i} \neq \phi\ \forall\ n \not\Rightarrow \bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$$

Theorem : A space (X, τ) is compact iff every family of closed subsets of X satisfy (f.i.p) being intersection nonempty.

Proof : ($\Rightarrow$) Suppose that X is compact space and $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of closed sets satisfy (f.i.p.), to prove $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

Suppose $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$

$\Rightarrow (\bigcap_{\alpha\in\Lambda} F_\alpha)^c = \phi^c$

$\Rightarrow \bigcup_{\alpha\in\Lambda} F_\alpha^c = X$

Since F_α is closed $\forall\ \alpha \Rightarrow F_\alpha^c \in \tau\ \forall\ \alpha$

$\Rightarrow \{F_\alpha^c\}_{\alpha\in\Lambda}$ is open cover for X

Since X is compact $\Rightarrow \exists\ \alpha_1, \ldots, \alpha_n$; $X = \bigcup_{i=1}^{n} F_{\alpha i}^c$

$\Rightarrow X^c = (\bigcup_{i=1}^{n} F_{\alpha i}^c)^c$

$\Rightarrow \phi = \bigcap_{i=1}^{n} F_{\alpha i}$ C!!

Since this family satisfy (f.i.p), then $\bigcap_{i=1}^{n} F_{\alpha i} \neq \phi$.

So, $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

($\Leftarrow$) Suppose $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of closed sets satisfy (f.i.p.) and $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$. ((for any family of closed sets satisfy (f.i.p.))), to prove X is compact.

Suppose that X is not compact $\Rightarrow \exists$ open cover for X has not finite subcover

i.e., $X = \bigcup_{\alpha\in\Lambda} U_\alpha \wedge X \neq \bigcup_{i=1}^{n} U_{\alpha i}\ \forall\ n$

$\Rightarrow X^c \neq (\bigcup_{i=1}^{n} U_{\alpha i})^c \Rightarrow \phi \neq \bigcap_{i=1}^{n} U_{\alpha i}^c$

$U_{\alpha i}^c \in \mathcal{F}$ since $U_{\alpha i} \in \tau$

we have, the family of closed sets $\{U_{\alpha}^{c}\}_{\alpha\in\Lambda}$ satisfy (f.i.p), but intersection this family is empty since

$X = \bigcup_{\alpha\in\Lambda} U_{\alpha} \Rightarrow X^{c} = (\bigcup_{\alpha\in\Lambda} U_{\alpha})^{c}$

$\Rightarrow \phi = \bigcap_{\alpha\in\Lambda} U_{\alpha}^{c}$ C!! (with hypothesis)

$\therefore$ X compact space.

<u>Definition :</u> Lindelöf Space

A space (X, τ) is called **<u>Lindelöf space</u>** iff each open cover of X has a countable subcover for X. i.e.,

$$X \text{ is Lindelöf} \Leftrightarrow \forall\ C = \{U_{\alpha}\}_{\alpha\in\Lambda}\ ;\ U_{\alpha} \in \tau\ \forall\ \alpha\ \wedge\ X = \bigcup_{\alpha\in\Lambda} U_{\alpha}$$

$$\Rightarrow \exists\ \alpha_1, \alpha_2, \ldots..\ ;\ X = \bigcup_{i=1}^{\infty} U_{\alpha_i}.$$

<u>Question : Prove or disprove :</u>

(1) Every compact space is Lindelöf space.

(2) Every Lindelöf space is compact space.

(3) Every finite space is Lindelöf space.

(4) Every countable space is Lindelöf space.

<u>Solution :</u>

(1) **<u>Yes, prove,</u>** i.e., Compact $\Rightarrow$ Lindelöf.

Let X be a compact space $\Rightarrow$ every open cover of X has a finite subcover

$\because$ every finite set is countable set

$\Rightarrow$ every open cover of X has a countable subcover

$\Rightarrow$ X is Lindelöf space.

(2) **<u>No, disprove,</u>** i.e., Lindelöf $\not\Rightarrow$ Compact. For example :

$(\mathbb{R}, \tau_u)$ is not compact space (see page 66), but its Lindelöf space :

Since every open cover of $\mathbb{R}$ contains of open intervals (by definition of τ_u) and every open interval contains at least one rational number (since $\mathbb{Q}$ is dense set in $\mathbb{R}$), so we can use this rational numbers to numerical the open intervals, so this cover became countable (since the rational numbers is countable).

$\because$ every set is subset of itself, so the countable open cover we search it is itself,

$\therefore (\mathbb{R}, \tau_u)$ is Lindelöf.

(3) **<u>Yes, prove,</u>**

Since every finite space is compact space and every compact space is Lindelöf.

(4) **<u>Yes, prove,</u>**

Since every open cover is countable, so it's the subcover required.

<u>Example :</u> $(\mathbb{N}, \tau)$ and $(\mathbb{Q}, \tau)$ is Lindelöf for any topology τ.

Notes that, the Lindelöf space not nessarily countable (example : $(\mathbb{R}, \tau_u)$ is Lindelöf, but not countable)

Example : (X, I) is Lindelöf, since its compact space and the only open sets are X, ϕ. For examples of this space $(\mathbb{R}, I)$, $(\mathbb{C}, I)$ and $(\mathbb{Q}, I)$ are Lindelöf space (we can replace X by any set).

Example : (X, D) is Lindelöf if X is countable and not Lindelöf if X is uncountable. For examples of this space $(\mathbb{N}, D)$ and $(\mathbb{Q}, D)$ are Lindelöf space, but $(\mathbb{R}, D)$ and $(\mathbb{C}, D)$ are not Lindelöf space.

Example : (X, τ_{cof}) is Lindelöf if X any infinite set since its compact space. For example of this space $(\mathbb{N}, \tau_{cof})$, $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{C}, \tau_{cof})$ … etc.

Example : (X, τ_{cof}) is Lindelöf if X any uncountable set since its compact space.

Remark : Lindelöfness is not hereditary property, but if we add a condition for the subset from Lindelöf space become a Lindelöf set and the following theorem show that :

Theorem : A **closed** subset of a Lindelöf space is Lindelöf.
Proof : Similarly of prove Compactness (see page 70).

Theorem : The continuous image of Lindelöf space is Lindelöf. i.e.,
If $f : (X, \tau) \to (Y, \tau')$ is continuous function and X is Lindelöf space, then f(X) is Lindelöf.
Proof : Similarly of prove Compactness (see page 68).

Theorem : Lindelöfness is topological property.
Proof : Let (X, τ) and (Y, τ') be topological spaces ; $X \cong Y$
Suppose that X is Lindelöf, to prove Y is Lindelöf
$\because$ $X \cong Y \Rightarrow \exists f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous
$\because$ f continuous, onto and X Lindelöf $\Rightarrow$ f(X) = Y Lindelöf
(by theorem : The continuous image of Lindelöf space is Lindelöf)
Now, suppose that Y is Lindelöf, To prove X is Lindelöf
$\because$ f^{-1} continuous, onto and Y Lindelöf $\Rightarrow$ $f^{-1}(Y) = X$ Lindelöf
(by same theorem : The continuous image of Lindelöf space is Lindelöf)

Examples of this theorems :

Example (1) : Let f : ($\mathbb{R}$, τ_u) → (X, τ) be continuous onto function. What about the space (X, τ) ??

Solution : Since ($\mathbb{R}$, τ_u) is Lindelöf space, then (X, τ) is Lindelöf (by theorem : The continuous image of Lindelöf space is Lindelöf)

Example (2) : Let f : ($\mathbb{R}$, D) → (X, τ) be continuous onto function. What about the space (X, τ) ??

Solution : Since ($\mathbb{R}$, D) is not Lindelöf space, then we cannot decided the space (X, τ) is not Lindelöf because theorem tell us : The continuous onto image of Lindelöf space is Lindelöf, but the domain is not Lindelöf, for example :

f : ($\mathbb{R}$, D) → ($\mathbb{R}$, I) such that f(x) = x

f is continuous onto and domain ($\mathbb{R}$, D) is not Lindelöf, but codomain (X, I) is Lindelöf.

f : ($\mathbb{R}$, D) → ($\mathbb{R}$, D) such that f(x) = x

f is continuous onto and domain ($\mathbb{R}$, D) is not Lindelöf and codomain (X, D) is not Lindelöf.

Remark : Let f : (X, τ) → (Y, τ') be continuous onto function and Y is Lindelöf, but not necessarily X is Lindelöf, for example :

Example : Let f : ($\mathbb{R}$, D) → ($\mathbb{R}$, τ_u) ; f(x) = x, be continuous onto function and ($\mathbb{R}$, τ_u) is Lindelöf, but ($\mathbb{R}$, D) X is not Lindelöf

Corollary : If the product space X×Y is Lindelöf, then X and Y are Lindelöf spaces.

Proof : The projection function P_X : X×Y → X is continuous and onto

∵ X×Y Lindelöf ⇒ P_X(X×Y) Lindelöf

(by theorem : The continuous image of Lindelöf space is Lindelöf)

∵ P_X onto ⇒ P_X(X×Y) = X

⇒ X Lindelöf

By the similar way we prove Y Lindelöf.

The projection function P_Y : X×Y → Y is continuous and onto

∵ X×Y Lindelöf ⇒ P_Y(X×Y) Lindelöf

(by same theorem : The continuous image of Lindelöf space is Lindelöf)

$\because$ P_Y onto $\Rightarrow$ $P_Y(X\times Y) = Y$

$\Rightarrow Y$ Lindelöf

Remark : If X and Y are Lindelöf spaces, then X×Y is Lindelöf space (i.e., the converse of the above corollary is true in general), i.e.,

X and Y are Lindelöf $\Leftrightarrow$ X×Y is Lindelöf

and we will introduce some user of this theorem.

$(\mathbb{R}, \tau_u)\times(\mathbb{R}, D)$ is not Lindelöf space, since $(\mathbb{R}, D)$ not Lindelöf.

$(\mathbb{N}, D)\times(\mathbb{N}, \tau_{cof})$ is Lindelöf space, since $(\mathbb{N}, D)$ Lindelöf and $(\mathbb{N}, \tau_{cof})$ Lindelöf.

Example : If $(\mathbb{R}, I)\times(\mathbb{R}, \tau)$ is Lindelöf space, what about the space $(\mathbb{R}, \tau)$??

Solution : The space $(\mathbb{R}, \tau)$ must be Lindelöf.

Corollary : Every quotient space from a Lindelöf space is Lindelöf.

Proof : let (X, τ) be a Lindelöf space and ~ be equivalent relation on X, then the quotient space is $(X/\sim, \tau_p)$ such that $p : X \to X/\sim$; $p(x) = [x]$.

clear that p is continuous onto (see quotient space)

since X is Lindelöf $\Rightarrow$ X/~ is Lindelöf

(by theorem : The continuous image of Lindelöf space is Lindelöf)

Examples :

Every quotient space from (X, I) is Lindelöf space.

Every quotient space from $(\mathbb{R}, \tau_u)$ is Lindelöf space.

Every quotient space from (X, τ_{cof}) is Lindelöf space.

Every quotient space from $(\mathbb{R}, D)$ is not necessarily Lindelöf space.