Chapter Five : Connected Spaces

Definition : Disconnected & Connected Spaces

The space (X, τ) is **disconnected** iff there exist two open disjoint nonempty sets A and B such that $A \bigcup B = X$. i.e.,

$$X \text{ is disconnected } \Leftrightarrow X = A \bigcup B \quad ; \quad A, B \in \tau \; , \; A \bigcap B = \phi \; , A \neq \phi \neq B.$$

The sets A and B form a **separation** of X.

The space (X, τ) is **connected** iff it is not disconnected.

$$X \text{ is connected } \Leftrightarrow X \neq A \bigcup B \quad ; \quad A, B \in \tau \; , \; A \bigcap B = \phi \; , A \neq \phi \neq B.$$

Remark : The connected spaces is the spaces required in topology, but the definition dependent on disconnected spaces for simply.

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}\}$. Is X is connected or disconnected space ??.

Solution : X is connected, since the only one case to represented on X as a union of nonempty open sets is $X = X \bigcup \{1\}$, but this sets is joint.

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$. Is X is connected or disconnected space ??.

Solution : X is disconnected, since :

$X = \{1\} \bigcup \{2, 3\}$ and $\{1\}, \{2, 3\} \in \tau$ and $\{1\} \bigcap \{2, 3\} = \phi$ and $\{1\} \neq \phi, \{2, 3\} \neq \phi$.

Remark : There are 29 difference topology on set contains 3 elements, we can testes this topologies connected or disconnected.

Remarks :

[1] (X, D) is disconnected if X contains more than one element, since :
$\exists A \; ; \; \phi \neq A \subsetneq X \;\; \Rightarrow \;\; X = A \bigcup A^c \, , \; A, A^c \in D, \; A \bigcap A^c = \phi \, , A \neq \phi$ and $A^c \neq \phi$ since $A \neq X$.

[2] (X, I) is connected spaces always since the only open sets are X and ϕ and this sets not make the space is disconnected.

[3] If $\tau = \mathscr{F}$ and $\tau \neq I$, then (X, τ) is disconnected since :

$$\exists A \in \tau \; ; \; \phi \neq A \subsetneq X \;\; \Rightarrow \;\; X = A \bigcup A^c.$$

Example : (X, τ_{cof}) is connected space, if X is infinite set since there are not exist nonempty disjoint open sets.

Example : $(\mathbb{R}, \tau_u)$ is connected space, since $\mathbb{R}$ not equal union of two nonempty disjoint open sets,
but $\mathbb{R} \setminus \{x_0\}$; $x_0 \in \mathbb{R}$ is separated since $\mathbb{R} \setminus \{x_0\} = (-\infty, x_0) \bigcup (x_0, \infty)$.

Example : The metric space is either connected or is disconnected, for example :
$(\mathbb{R}, \tau_u)$ generated from metric space $(\mathbb{R}, | |)$ is connected space.
But, (X, D) ; X contains more than one element is disconnected space which is generated from metric space.

Example : Let X = {a, b, c, d}. Define a topology τ on X and another topology τ' on X such that (X, τ) is connected and (X, τ') is disconnected.
Solution : Let $\tau = \{X, \phi, \{a, b\}\}$ and $\tau' = \{X, \phi, \{a, b\}, \{c, d\}\}$, then (X, τ) is connected and (X, τ') is disconnected.

We introduce some theorems to equivalent properties for a space being connected :

Theorem : (X, τ) is connected space iff X cannot be written as a union of two nonempty disjoint closed sets.
Proof : ($\Rightarrow$) Suppose that X is connected
To prove $X \neq A \bigcup B$; $A, B \in \mathcal{F}$, $A \bigcap B = \phi$, $A \neq \phi \neq B$
Suppose that $X = A \bigcup B$; $A, B \in \mathcal{F}$, $A \bigcap B = \phi$, $A \neq \phi \neq B$

$\Rightarrow A = B^c \wedge B = A^c$
$\Rightarrow A \in \tau \wedge B \in \tau$ (since $A = B^c \wedge B \in \mathcal{F}$ and $B = A^c \wedge A \in \mathcal{F}$)
$\Rightarrow X = A \bigcup B$; $A, B \in \tau$, $A \bigcap B = \phi$, $A \neq \phi \neq B$
$\Rightarrow$ X disconnected C!! Contradiction !! since X is connected
$\Rightarrow X \neq A \bigcup B$; $A, B \in \mathcal{F}$, $A \bigcap B = \phi$, $A \neq \phi \neq B$.

($\Leftarrow$) Suppose that $X \neq A \bigcup B$; $A, B \in \mathcal{F}$, $A \bigcap B = \phi$, $A \neq \phi \neq B$
To prove X is connected
Suppose that X is disconnected

$\Rightarrow X = U \bigcup V$; $U, V \in \tau$, $U \bigcap V = \phi$, $U \neq \phi \neq V$
$\Rightarrow U = V^c \wedge V = U^c$
$\Rightarrow U, V \in \mathcal{F}$ C!! Contradiction !!

since the complement of every one of them is open set and this contradiction with hypotheses

$\therefore$ X connected space.

Theorem : (X, τ) is connected space iff the only subsets of the space X which are open and closed are X and ϕ.

Proof : ($\Rightarrow$) Suppose that X is connected

To prove, if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$.

Suppose that $A, A^c \in \tau$ and $A \neq X$ and $A \neq \phi$

$$\Rightarrow X = A \bigcup A^c \wedge A, A^c \in \tau \wedge A \bigcap A^c = \phi \text{ , } A \neq \phi \neq A^c \text{ (since } A \neq X)$$

$$\Rightarrow X \text{ disconnected} \quad \text{C!! Contradiction !!}$$

$\therefore$ if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$.

($\Leftarrow$) Suppose that, if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$, i.e., $\tau \bigcap \mathcal{F} = \{X, \phi\}$.

To prove X is connected

Suppose that X is disconnected

$$\Rightarrow X = U \bigcup V \text{ ; } U, V \in \tau \text{ , } U \bigcap V = \phi \wedge U \neq \phi \neq V$$

$$\Rightarrow U = V^c \wedge V = U^c \Rightarrow U, V \in \mathcal{F}$$

$$\Rightarrow U, V \in \tau \bigcap \mathcal{F} \quad \text{C!! Contradiction !!}$$

Since U and V are open and closed and not equal X and ϕ.

$\therefore$ X connected space.

Theorem : (X, τ) is connected space iff the only subsets of the space X which have empty boundary sets are X and ϕ. i.e.,

$$[X \text{ is connected} \Leftrightarrow (\phi \neq A \subsetneq X \Rightarrow A^b \neq \phi)]$$

on the other hand :

$$[X \text{ is connected} \Leftrightarrow (A^b = \phi \Rightarrow A = \phi \vee A = X)]$$

Proof : ($\Rightarrow$) Suppose that X is connected

To prove, if $A^b = \phi \Rightarrow A = \phi \vee A = X$.

Suppose that $A \subseteq X$ and $A^b = \phi \wedge A \neq \phi \wedge A \neq X$

$$\Rightarrow \phi \subseteq A \Rightarrow A^b \subseteq A \Rightarrow A \in \mathcal{F} \quad \text{(by theorem : } A \in \mathcal{F} \Leftrightarrow A^b \subseteq A)$$

On the other hand :

$$\phi \bigcap A = \phi \Rightarrow A^b \bigcap A = \phi \Rightarrow A \in \tau \quad \text{(by theorem : } A \in \tau \Leftrightarrow A^b \bigcap A = \phi)$$

$$\Rightarrow A, A^c \in \tau \text{ and } A \neq \phi \wedge A \neq X$$

$$\Rightarrow X \text{ disconnected}$$

(by theorem : (X, τ) is connected space iff the only subsets of the space X which are open and closed are X or ϕ)

(this contradiction with hypotheses : X is connected $\Rightarrow A^b \neq \phi$ if $\phi \neq A \neq X$)

$\therefore$ if $A^b = \phi \Rightarrow A = \phi \vee A = X$.

($\Leftarrow$) Suppose that, if $A^b = \phi \Rightarrow A = \phi \vee A = X$.

To prove X is connected

Suppose that X is disconnected

$$\Rightarrow X = U \bigcup V \ ; U, V \in \tau \ , U \bigcap V = \phi \ \wedge \ U \neq \phi \neq V$$

$$\Rightarrow U = V^c \ \wedge \ V = U^c \quad \Rightarrow U, V \in \mathscr{F}$$

$$\Rightarrow U^b = \phi \ \wedge \ V^b = \phi \qquad \text{(by theorem : } A, A^c \in \tau \Leftrightarrow A^b = \phi)$$

(this contradiction since $U^b = \phi$, but $U \neq X$ and $U \neq \phi$)

$\therefore$ X connected space.

<u>Theorem :</u> (X, τ) is connected space iff every continuous function from domain X to codomain $(\{1, 2\}, D)$ is constant function.

<u>Proof :</u> Let $f : (X, \tau) \rightarrow (\{1, 2\}, D)$ be continuous function.

($\Rightarrow$) Suppose that X connected space

To prove, f is constant function

Suppose that f not constant

$$\Rightarrow \exists A \subseteq X \ ; f(a) = 1 \quad \forall a \in A$$

$$\text{and } \exists B \subseteq X \ ; f(b) = 2 \quad \forall b \in B$$

Notes that,

(1) $X = A \bigcup B$, since if $X \neq A \bigcup B \Rightarrow \exists x \in X$; x has no image $\Rightarrow$ f not funct.

(2) $A \neq \phi$, since if $A = \phi \Rightarrow$ f constant (the prove end)

and, $B \neq \phi$, since if $B = \phi \Rightarrow$ f constant (the prove end)

(3) $A \bigcap B = \phi$, since if $A \bigcap B \neq \phi \Rightarrow \exists x \in X$; x has two image $\Rightarrow$ f not funct.

Now,

$\{1\}, \{2\} \in D$ (by def. of D) $\Rightarrow \{1\}, \{2\}$ open set in $(\{1, 2\}, D)$

$\because$ f continuous $\Rightarrow A = f^{-1}(\{1\}) \in \tau \ \wedge \ B = f^{-1}(\{2\}) \in \tau$

$$\Rightarrow X = A \bigcup B \ \wedge \ A, B \in \tau \ \wedge \ A \bigcap B = \phi \ \wedge A \neq \phi \neq B$$

$$\Rightarrow X \text{ disconnected} \qquad \text{C!! Contradiction !!}$$

$\therefore$ f is constant function

($\Leftarrow$) Suppose that f is constant function

To prove, X connected space

Suppose that X disconnected

$\Rightarrow X = A \bigcup B$; $A, B \in \tau$, $A \bigcap B = \phi$, $A \neq \phi \neq B$

Define $f : (X, \tau) \to (\{1, 2\}, D)$; $f(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 2 & \text{if} \quad x \in B \end{cases}$

Clear that f is continuous, since $D = \{\{1, 2\}, \phi, \{1\}, \{2\}\}$ and

$f^{-1}(\{1, 2\}) = \{1, 2\} \in \tau$, $f^{-1}(\phi) = \phi \in \tau$, $f^{-1}(\{1\}) = A \in \tau$, $f^{-1}(\{2\}) = B \in \tau$

$\Rightarrow$ f not constant C!! Contradiction !!

$\therefore$ X connected space.

Remark : If (X, τ) is topological space and (W, τ_W) is a subspace of X, then the space W being disconnected or connected not directly relation by X and the open sets in X, but dependent on the open sets in W. i.e., its dependent on τ_W so that : W is connected space iff there exist two open disjoint nonempty sets A and B in W such that $A \bigcup B = W$.

i.e., W is disconnected $\Leftrightarrow$ $W = A \bigcup B$; $A, B \in \tau_W$, $A \bigcap B = \phi$, $A \neq \phi \neq B$.

The space (W, τ_W) is **connected** iff it is not disconnected.

W is connected $\Leftrightarrow$ $W \neq A \bigcup B$; $A, B \in \tau_W$, $A \bigcap B = \phi$, $A \neq \phi \neq B$.

Remark : The property of being a connected space is not a hereditary property and the following example show that :

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1, 2\}, \{1, 3\}, \{1\}\}$. Let $W \subseteq X$; $W = \{2, 3\}$. Is W is connected space ??

Solution : Compute τ_W :

$$\tau_W = \{ W \bigcap U ; U \in \tau\} = \{W, \phi, \{2\}, \{3\},\}$$

Notes that $\tau_W = D$, then W is disconnected space but not connected since :

$W = \{2\} \bigcup \{3\}$ and $\{2\}, \{3\} \in \tau_W$ and $\{2\} \bigcap \{3\} = \phi$ and $\{2\} \neq \phi$, $\{3\} \neq \phi$.

Notes that X is connected space but not disconnected, while it's have disconnected subspace.

Theorem : continuous image of connected space is connected. i.e.,

If $f : (X, \tau) \to (Y, \tau')$ is continuous and onto function and X is connected, then Y is connected.

Proof : Suppose that Y is disconnected

$\Rightarrow \exists A, B \in \tau'$; $A \bigcap B = \phi \wedge A \neq \phi \neq B \wedge Y = A \bigcup B$

$\Rightarrow f^{-1}(Y) = f^{-1}(A \bigcup B)$

$\Rightarrow X = f^{-1}(A) \bigcup f^{-1}(B)$ (since f onto $\Rightarrow f^{-1}(Y) = X \wedge f^{-1}(A \bigcup B) = f^{-1}(A) \bigcup f^{-1}(B)$)

$\Rightarrow f^{-1}(A), f^{-1}(B) \in \tau$ (since f continuous)

$\wedge f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$

$\wedge A \neq \phi \wedge B \neq \phi \Rightarrow f^{-1}(A) \neq \phi \neq f^{-1}(B)$

$\therefore$ X is disconnected C!! Contradiction !!

$\therefore$ Y is connected.

Remark : If $f : (X, \tau) \to (Y, \tau')$ is continuous and onto function and Y is connected space, then X not necessary connected space and the following example show that :

Example : Let $f : (\mathbb{R}, D) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall \; x \in \mathbb{R}$.

Clear that f is continuous and onto function and $(\mathbb{R}, I)$ is connected, but $(\mathbb{R}, D)$ is not connected (disconnected).

Corollary (1) : The property of being a connected space is a topological property.

Proof :

Let (X, τ) and (Y, τ') be topological space ; $X \cong Y$

$\because X \cong Y \Rightarrow \exists f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Suppose that X is connected, to prove Y is connected

$\because$ f continuous, onto and X connected $\Rightarrow f(X) = Y$ connected

(by theorem : continuous image of connected space is connected)

Now, suppose that Y is connected, to prove X is connected

$\because f^{-1}$ continuous, onto and Y connected $\Rightarrow f^{-1}(Y) = X$ connected

(by same theorem : continuous image of connected space is connected)

Corollary (2) : Let (X, τ) and (Y, τ') be two topological spaces. If the product space $X \times Y$ is a connected space then each X and Y are connected spaces.

Proof :

The projection function $P_X : X \times Y \to X$ is continuous and onto

$\because X \times Y$ connected and P_X continuous $\Rightarrow P_X(X \times Y)$ connected

(by theorem : continuous image of connected space is connected)

$\because P_X$ onto $\Rightarrow P_X(X \times Y) = X$

$\Rightarrow$ X connected

By the similar way we prove Y connected.

The projection function $P_Y : X \times Y \to Y$ is continuous and onto

$\because X \times Y$ connected and P_Y continuous $\Rightarrow P_Y(X \times Y)$ connected

(by theorem : continuous image of connected space is connected)

$\because$ P_X onto $\Rightarrow$ $P_Y(X\times Y) = Y$

$\Rightarrow$ Y connected

We can use previous theorem and their corollaries to known some spaces either connected or disconnected as the following examples :

Example : If we know the function $f : (\mathbb{R}, \tau_u) \rightarrow (Y, \tau)$ is continuous onto. Is Y connected space ??

Solution : Yes, since $(\mathbb{R}, \tau_u)$ is connected space and f is continuous onto function, then $f(\mathbb{R}) = Y$ is connected space.

Example : If we know that $(\mathbb{R}, D) \cong (Y, \tau)$. Is Y connected space ??

Solution : No, since the equivalent topological spaces either connected spaces or not connected spaces, because the property of being a connected space is a topological property and since $(\mathbb{R}, D)$ is disconnected space, then $f(\mathbb{R}) = Y$ is disconnected space.

Example : If we know the product space $X\times Y$ for spaces X and Y is indiscreet topology i.e., $\tau_{X\times Y} = I$. What we say about X and Y ??

Solution : Yes, since $(X\times Y, I)$ is connected space, then X and Y is connected space.

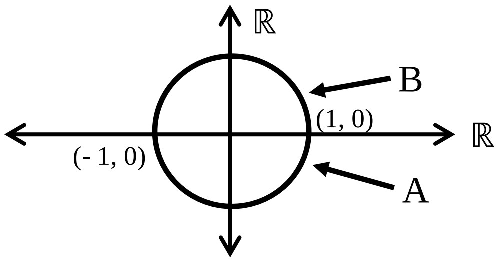
Remark : If A and B are connected subsets of a space (X, τ), then $A \cap B$ is not necessary connected and the following example show that :

Example : Let $X = \mathbb{R}^2$ and its product space $(\mathbb{R}, \tau_u) \times (\mathbb{R}, \tau_u)$ such that the open neighhoods in this space is a disc her center is a point (for example $\{(x, y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ is an open neighhood for the point (0, 0) and geometrical it's a disc with center (0, 0) and radius 1.

Let A and B define as follows :

$A = \{(x, y) \in \mathbb{R}^2 \ ; \ x^2+y^2 = 1 \wedge y \leq 0\}$

$B = \{(x, y) \in \mathbb{R}^2 \ ; \ x^2+y^2 = 1 \wedge y \geq 0\}$



such that the geometric representation of A as lower half for circle circumference with radius 1 and center (0, 0) as the above figure.

and the geometric representation of B as upper half for circle circumference with radius 1 and center (0, 0) as the above figure.

$A \bigcap B = \{(1, 0), (-1, 0)\}$

Notes that every one of A and B are connected, but $A \bigcap B$ not connected set.

Remark : If A and B are connected subsets of a space (X, τ), then $A \bigcup B$ is not necessary connected and the following example show that :

Example : Let $(X, \tau) = (\mathbb{R}, \tau_u)$ and $A = (1, 2)$ and $B = (3, 4)$

Clear that A and B are connected sets since we cannot represented as a union of nonempty disjoint open intervals subsets of A or B, but $A \bigcup B$ is not connected (disconnected) since

$A \bigcup B = (1, 2) \bigcup ((3, 4)$ and clear that every one of (1, 2) and (3, 4) nonempty disjoint open sets in $A \bigcup B$.

Notes that in general in a space $(\mathbb{R}, \tau_u)$: If A is subset of $\mathbb{R}$, then A is connected iff A is interval i.e.,

$$A \text{ is connected } \Leftrightarrow A = (a, b) \text{ or } A = [a, b] \text{ or } A = (a, b] \text{ or } A = [a, b)$$

Theorem : Let (X, τ) be a topological space. If W is a connected subsets of X and $X = A \bigcup B$ such that $A, B \in \tau$ and $A \bigcap B = \phi$ and $A \neq \phi \neq B$, then $W \subseteq A$ or $W \subseteq B$.

Proof :

Suppose that $W \not\subseteq A$ and $W \not\subseteq B$

$\Rightarrow W \bigcap A \neq \phi$ and $W \bigcap B \neq \phi$

$\because A, B \in \tau \Rightarrow W \bigcap A, W \bigcap B \in \tau_W$ (by def. of subspace topology)

Notes that,

$W \bigcap A \neq \phi$ (since, if $W \bigcap A = \phi \Rightarrow W \subseteq B$)

$W \bigcap B \neq \phi$ (since, if $W \bigcap B = \phi \Rightarrow W \subseteq A$)

Also,

$(W \bigcap A) \bigcap (W \bigcap B) = W \bigcap (A \bigcap B) = W \bigcap \phi = \phi$

and $W = (W \bigcap A) \bigcup (W \bigcap B)$

$\Rightarrow$ W is disconnected C!! Contradiction !!

$\therefore W \subseteq A \vee W \subseteq B$.

Remark : Notes that $A \bigcup B$ may be not connected in spite of A connected and B connected, but if we add a condition $A \bigcap B \neq \phi$, then $A \bigcup B$ is connected set and this show in the next theorem :

Theorem : If A and B are connected subsets of a space (X, τ) and $A \bigcap B \neq \phi$, then $A \bigcup B$ is connected.

Proof :

Let (X, τ) topological space and $A, B \subseteq X$; A, B connected and $A \bigcap B \neq \phi$

To prove $A \bigcup B$ connected ??

Suppose that $A \bigcup B$ disconnected

$\Rightarrow A \bigcup B = U \bigcup V$; $U, V \in \tau_{A\bigcup B}$, $U \bigcap V = \phi$, $U \neq \phi \neq V$

$\Rightarrow A \subseteq A \bigcup B \Rightarrow A \subseteq U \bigcup V$ and A connected

$\Rightarrow A \subseteq U \vee A \subseteq V$ (by previous theorem)

By similar way $\Rightarrow B \subseteq A \bigcup B \Rightarrow B \subseteq U \bigcup V$ and B connected

$\Rightarrow B \subseteq U \vee B \subseteq V$ (by previous theorem)

Now,

$\Rightarrow$ either $A \subseteq U \wedge B \subseteq U \Rightarrow A \bigcup B \subseteq U \Rightarrow V = \phi$ C!!

or $A \subseteq V \wedge B \subseteq V \Rightarrow A \bigcup B \subseteq V \Rightarrow U = \phi$ C!!

or $A \subseteq U \wedge B \subseteq V \Rightarrow A \bigcap B \subseteq U \bigcap V = \phi \Rightarrow A \bigcap B = \phi$ C!!

or $A \subseteq V \wedge B \subseteq U \Rightarrow A \bigcap B \subseteq U \bigcap V = \phi \Rightarrow A \bigcap B = \phi$ C!!

$\therefore A \bigcup B$ connected.

Remark : We can generalization the above theorem to family of connected sets as follows :

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of connected subsets of a space (X, τ) and $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is connected set.

We prove previously, if the product space $X\times Y$ is connected space, then each X and Y is connected space and we question the converse is true (i.e., Let (X, τ) and (Y, τ') be a topological spaces. If each X and Y is connected space, then the product space $X\times Y$ is connected space) and the answer is true and we postpone the prove for this problem until the availability of basic prove from previous theorem as follows :

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. If each X and Y are connected space, then the product space $X\times Y$ is a connected space.

Proof :

Let (X, τ) and (Y, τ') connected spaces, to prove $(X\times Y, \tau_{X\times Y})$ connected

$\because X \cong X\times\{y\}$; $y \in Y$ and X connected $\Rightarrow X\times\{y\}$ connected

(connected is topological property)

$\because Y \cong \{x\}\times Y$; $x \in X$ and Y connected $\Rightarrow \{x\}\times Y$ connected

(connected is topological property)

Fixed $y_0 \in Y \Rightarrow X\times\{y_0\}$ connected (this is true $\forall\, y \in Y$)

Clear, $X\times\{y_0\} \bigcap \{x\}\times Y \neq \phi$ (since $(x, y_0) \in X\times\{y_0\} \bigcap \{x\}\times Y$)

$\Rightarrow X\times\{y_0\} \bigcup \{x\}\times Y$ connected $\forall\, x \in X$

(by previous theorem: If A and B are connected and $A\bigcap B\neq\phi$, then $A\bigcup B$ is connected)

The family $\{X\times\{y_0\} \bigcup \{x\}\times Y\}_{x\in X}$ of connected sets in $X\times Y$

Clear that $X\times Y = \bigcup_{x\in X} (X\times\{y_0\} \bigcup \{x\}\times Y)$ and $\bigcap_{x\in X} (X\times\{y_0\} \bigcap \{x\}\times Y) \neq \phi$

$\therefore$ $X\times Y$ connected

(by previous theorem since it's a union of family of intersection connected sets)

Remark : Take the following example to show the Cartesian product which are $X\times\{y\}$ and $\{x\}\times Y$ and the union and intersection in the previous theorem :

Example : Let X = {1, 2, 3} and Y = {a, b, c}, then

$X\times Y$ = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)} and,

$X\times\{a\}$ = {(1, a), (2, a), (3, a)}

$X\times\{b\}$ = {(1, b), (2, b), (3, b)}

$X\times\{c\}$ = {(1, c), (2, c), (3, c)} also,

$\{1\}\times Y$ = {(1, a), (1, b), (1, c)}

$\{2\}\times Y$ = {(2, a), (2, b), (2, c)}

$\{3\}\times Y$ = {(3, a), (3, b), (3, c)}

Clear that the intersection any two sets her is nonempty and

$X\times Y = \bigcup_{x\in X} (X\times\{a\} \bigcup \{x\}\times Y)$.

Definition : Component of x

Let (X, τ) be a topology space and $x \in X$. We say the set which is a union of every connected sets that contains x is a **component of x** and denoted by C(x). i.e.,

$$C(x) = \bigcup\{A \subseteq X : x \in A \ \wedge\ A \text{ is connected}\}$$

This means that C(x) is the large connected set contains x.

Example : Let X = {1, 2, 3} and $\tau = \{X, \phi, \{3\}, \{1, 2\}\}$. Compute the component for every element in the space (X, τ).

Solution :

C(1) = {1, 2}, since {1, 2} is a large connected set contains 1.

Notes that {1, 3} contains 1 too, but it's not connected since the induce topology on {1, 3} is $\tau_{\{1, 3\}}$ = {{1, 3}, ϕ, {3}, {1}}, then {1, 3} = {1} ⋃ {3} this means it's not connected.
On the other hand {1} is connected set (since every singleton set in any topology is connected set) and its contains 1 but not largest set.
By similar way : C(2) = {1, 2} while C(3) = {3} since any set contains 3 except {3} is not connected.

Example : Let X = {a, b, c} and τ = {X, ϕ, {a}}. Compute the component for every element in the space (X, τ).
Solution :
C(a) = C(b) = C(c) = X, since X is connected space and its large connected set contains any element in this space.

Remark : (X, τ) is connected space iff C(x) = X for all x ∈ X.
If X is connected, then X is large connected set for every element in X and C(x) = X.
On the other hand, if C(x) = X for every element x ∈ X, then X is connected set since C(x) is connected set (by definition).

Remarks :
[1] The space (ℝ, τ_u) is connected space, then C(x) = ℝ for all x ∈ ℝ.
[2] In the space (X, D), if X contains more than one element, then C(x) = {x} for all x ∈ X, since the only unique sets her is singleton sets.
[3] The space (X, I) is connected space, then C(x) = X for all x ∈ X.
[4] The space (X, τ_{cof}); X infinite set is connected space, then C(x)= X for all x ∈ X.

Theorem : The component C(x) for every element x is closed set.
Proof :
$C(x) \subseteq \overline{C(x)}$ (since $A \subseteq \overline{A}$)
∵ C(x) is connected set ⇒ $\overline{C(x)}$ is connected set
(by theorem : If $A \subseteq B \subseteq \overline{A}$ and A connected, then B connected and so $\overline{A}$ connected)
∵ C(x) the large connected set that contains x and $\overline{C(x)}$ connected ; $C(x) \subseteq \overline{C(x)}$,
So $C(x) = \overline{C(x)}$ ⇒ C(x) is closed set (by theorem : A closed ⇔ $A = \overline{A}$)

Remark : C(x) ≠ ϕ for all x ∈ X, since x ∈ C(x).

Theorem : If C(x) ⋂ C(y) ≠ ϕ, then C(x) = C(y).

Proof :

$\because$ C(x) $\bigcap$ C(y) $\neq \phi$ and C(x), C(y) the connected sets $\Rightarrow$ C(x) $\bigcup$ C(y) is connected set (by theorem : Union of connected sets is connected if their intersection nonempty)

We get : C(x) $\bigcup$ C(y) connected set ; C(x) $\subseteq$ C(x) $\bigcup$ C(y) and C(y) $\subseteq$ C(x) $\bigcup$ C(y)

Since C(x) and C(y) are the largest connected sets ; x $\in$ C(x) and y $\in$ C(y)

$\Rightarrow$ C(x) $\bigcup$ C(y) = C(x) = C(y).

Remark : Family of components elements in the space (X, τ) being a partition for X.

(1) C(x) $\neq \phi$ $\forall$ x $\in$ X.

(2) if C(x) $\neq$ C(y), then C(x) $\bigcap$ C(y) = ϕ (by previous theorem)

(3) X = $\bigcup_{x\in X}$ C(x) i.e., $\bigcup_{x\in X}$ C(x) $\subseteq$ X and X $\subseteq$ $\bigcup_{x\in X}$ C(x)

since C(x) $\subseteq$ X $\forall$ x $\Rightarrow$ $\bigcup_{x\in X}$ C(x) $\subseteq$ X

since $\forall$ x $\in$ X $\Rightarrow$ x $\in$ C(x)

$\Rightarrow$ x $\in$ $\bigcup_{x\in X}$ C(x)

So, X $\subseteq$ $\bigcup_{x\in X}$ C(x)

Example : Let X = {1, 2, 3} and τ = {X, ϕ, {1}, {2, 3}}.

Her C(1) = {1} and C(2) = C(3) = {2, 3}

Notes that C(1) $\neq \phi$, C(2) $\neq \phi$ and C(3) $\neq \phi$

Also, C(1) $\neq$ C(2) $\Rightarrow$ C(1) $\bigcap$ C(2) = ϕ

and C(1) $\neq$ C(3) $\Rightarrow$ C(1) $\bigcap$ C(3) = ϕ

on the other hand, the union of components equal X i.e., X = C(1) $\bigcup$ C(2) $\bigcup$ C(3) and this clear since C(1) = {1} and C(2) = C(3) = {2, 3}, then X = {1} $\bigcup$ {2, 3}.

Definition : Locally Connected

The space (X, τ) is **locally connected at a point** x $\in$ X iff there exists an open connected of a point x. If (X, τ) is locally connected at each point x $\in$ X, then X is called a **locally connected space.** i.e.,

X is locally connected $\Leftrightarrow$ $\forall$ x $\in$ X $\exists$ U $\in \tau$; x $\in$ U and U is connected.

Remark : There is no relation between the concepts connected and locally connected i.e., connected and locally connected are independent concepts and we show that by the following example :

Example : (A space that is locally connected but not connected)

Let X = (– 3, 0) $\bigcup$ (3, 8)

X is a subspace of ($\mathbb{R}$, τ_u) : Clear that X is not connected since it's a union of nonempty disjoint open intervals. On the other hand X is locally connected space since every element in X either in (– 3, 0) and its connected intervals and it's a connected open neighhood for every element in (– 3, 0) and by similar way if the element contain in (3, 8).

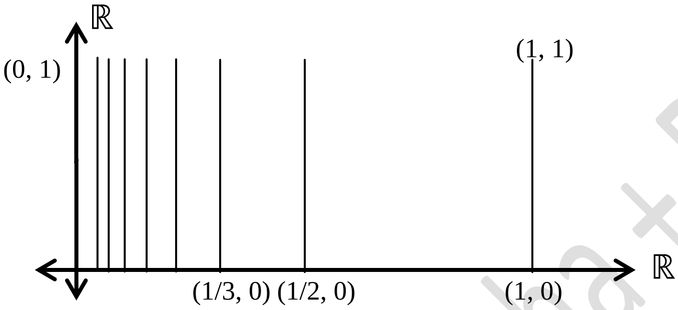
i.e., Locally Connected $\not\Rightarrow$ Connected.

<u>Example :</u> Comb Space (A space that is connected but not locally connected)

$X = A \bigcup B$; $A = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant y \leqslant 1 \ , x = 0 \ \text{ or } \ x = \frac{1}{n} \ , n \in \mathbb{N}\}$

$B = \{(x, 0) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 1\}$

The following figure show the comb space :

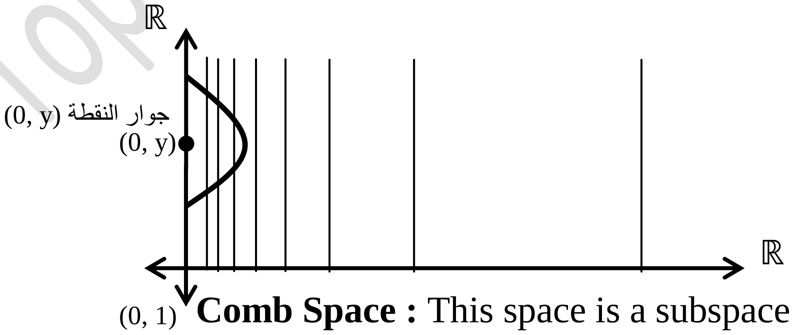


Comb Space : This space is a subspace of ($\mathbb{R}^2$, τ_u)

Notes that this space consist of infinite sets of line segment length every one 1 and when converge to y-axis , then this line segment is adherent to each other i.e., the distant between them is reduced when they converge from Y.

Notes that the shape this space similarity the comb such that the line segment on x-axis represented the base of the comb while the vertical line represented teeth comb.

We know this space is **<u>connected space</u>** since its consist of one piece, but it's **<u>not locally connected</u>** and in true its not locally connected only on every point on y-axis since every neighhood for any point (0, y) on y-axis consist of sets of disjoint line segment to each other, so that the open neighhood is not connected i.e., we cannot find connected open neighhood for any point from type (0, y) ; $y \neq 0$ and $0 < y \leqslant 1$ and this show by the following figure which is intersection open ball in $\mathbb{R}^2$ with the space X.



Comb Space : This space is a subspace of ($\mathbb{R}^2$, τ_u)

Connected $\not\Rightarrow$ Locally Connected

Remark : The property of being a locally connected space is not a hereditary property and the following example show that :

Example : In the above example, $(\mathbb{R}^2, \tau_u)$ is locally connected space while the **comb space W** is a sub space of $(\mathbb{R}^2, \tau_u)$ which is not locally connected space.

Example : the space (X, D) is locally connected space since every element x in X, then {x} is a connected open neighhood.

Remark : continuous image of locally connected space is not necessary locally connected and the following example show that :

Example : Take comb space W in previous example and take difference topologies one of them discrete topology D and the other is the induce topology τ_W from $(\mathbb{R}^2, \tau_u)$ and define the function f as follows :

$$f : (W, D) \to (W, \tau_w) \ ; \ f(x) = x \ \forall \ x \in W$$

clear that f is continuous since its domain D and its onto since its identity function. Also, we know (W, D) is locally connected space while (W, τ_W) is not locally connected space (we clear that in illustrate previous).

Example : the space (X, I) is locally connected space since the only unique for every element x in X is itself X and X her is a connected set.

Example : $(\mathbb{R}^2, \tau_u)$ is locally connected space since there is always open internals corresponding every real number contains it and since every open interval is connected and it's aa open neighthood for a point, then $(\mathbb{R}^2, \tau_u)$ is locally connected space.

Example : (X, τ_{cof}) is locally connected space since every open set in this space is connected set, therefore there is an open neighthood for every element contains it.

Example : Let X = {1, 2, 3} and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$ such that τ is a topology on X. Is (X, τ) connected space ?? locally connected space ??

Solution :

(X, τ) is not connected (disconnected) space since $X = \{1\} \bigcup \{2, 3\}$ such that {1}, {2, 3} are nonempty disjoint open sets.

(X, τ) is locally connected space since every element have connected open neighthood ; $1 \in \{1\} \in \tau$ and $\{1\}$ is connected set, also $2, 3 \in \{2, 3\} \in \tau$ and $\{2, 3\}$ is connected set.

Theorem : If f $f : (X, \tau) \to (Y, \tau')$ is onto, continuous and open function and X is locally connected, then Y is locally connected.

Proof :

Let $y \in Y \Rightarrow \exists\, x \in X$; $f(x) = y$ (since f is onto)

$\because$ X is locally connected $\Rightarrow \exists$ connected open nbd for x

i.e., $\exists\, U \in \tau$; $x \in U \wedge U$ is connected

$\because$ f is continuous $\Rightarrow f(U)$ is connected

(by theorem : continuous image of connected space is connected)

$\because$ f is open $\Rightarrow f(U)$ is open i.e., $f(U) \in \tau'$ also $y \in f(U)$

We get, f(U) is connected open nbd for y $\Rightarrow$ Y is locally connected.

Corollary : The property of being a locally connected space is a topological property.

Proof :

Let (X, τ) and (Y, τ') be topological spaces ; $X \cong Y$

$\because X \cong Y \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Suppose that X is locally connected, to prove Y is locally connected

$\because$ f onto, continuous, open and X locally connected $\Rightarrow f(X) = Y$ locally connected

(by previous theorem)

Now, suppose that Y is locally connected, to prove X is locally connected

$\because f^{-1}$ onto, continuous, open and Y locally connected $\Rightarrow f^{-1}(Y) = X$ locally connected

(by previous theorem)

Remark : Let (X, τ) and (Y, τ') be a topological spaces. If the product space $X \times Y$ is locally connected space then each X and Y is locally connected space.