Chapter Four : Separation Axioms

Definition : **T_0 – Space**

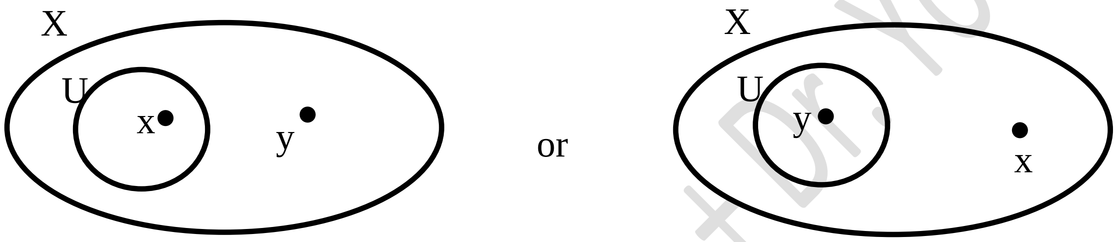
Let (X, τ) be a topological space. Then the space (X, τ)is called **T_0 – space** iff for each pair of distinct points x, y ∈ X, there is either an open set containing x but not y or an open set containing y but not x. i.e.,

X is T_0 – space $\Leftrightarrow \forall\, x, y \in X\ ;\ x \neq y\ \exists\, U \in \tau\ ;\ (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$.

If (X, τ) is not T_0 – space, we define,

X is not T_0 – space $\Leftrightarrow \exists\, x, y \in X\ ;\ x \neq y\ \forall\, U \in \tau\ ;\ (x, y \in U) \vee (x, y \notin U)$.

The following figure show the definition of T_0 – Space :



Example : Let X = {1, 2, 3} and τ = {X, ϕ, {1}, {1, 2}}. Is (X, τ), T_0 – space.

Solution : Must test every deference elements in X, satisfy the definition or not as follows :

$1 \neq 2 \Rightarrow \exists$ open set $\{1\} \in \tau\ ;\ 1 \in \{1\}\ \wedge\ 2 \notin \{1\}$

$1 \neq 3 \Rightarrow \exists$ open set $\{1\} \in \tau\ ;\ 1 \in \{1\}\ \wedge\ 3 \notin \{1\}$

$2 \neq 3 \Rightarrow \exists$ open set $\{1, 2\} \in \tau\ ;\ 2 \in \{1, 2\}\ \wedge\ 3 \notin \{1, 2\}$

∴ The definition is satisfy ⇒ (X, τ) is T_0 – space.

Example : Let X = {a, b, c} and τ = {X, ϕ, {a}}. Is (X, τ), T_0 – space.

Solution : No, since b ≠ c ⇒ ∄ open set containing b but not c or an open set containing c but not b.

Clear that X containing b, c and ϕ open set not containing b, c and {a} open set not containing b, c. Therefore, (X, τ) is not T_0 – space.

Example : Is $(\mathbb{N}, \tau_{cof})$ T_0 – space.

Solution : Yes, we prove that in general since ℕ containing infinite numbers :

Let $n, m \in \mathbb{N}\ ;\ n \neq m$, to prove $\exists$ open set $U \in \tau_{cof}\ ;\ m \in U\ \wedge\ n \notin U$ or vice versa.

Take $U = \mathbb{N} \setminus \{n\} \Rightarrow m \in U \wedge n \notin U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

By similar way we can take : $U = \mathbb{N} \setminus \{m\} \Rightarrow m \notin U \wedge n \in U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

The two cases similar and satisfy the definition $\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_0 – space.

Example : In the space (X, I) if X is any set containing more than one element, then (X, I) is not T_0 – space, since X contains more than one element we take $x, y \in X$; $x \neq y$ and $\nexists$ open set containing x but not y or an open set containing y but not x and ϕ open set not containing x, y.

Example : The space (X, D) is T_0 – space.

Solution : Let $x, y \in X$; $x \neq y$, then $\{x\} \in D$ i.e., $\{x\}$ open set (by definition D since $D = IP(X)$), hence $x \in \{x\}$ and $y \notin \{x\}$. We can take $\{y\}$ replace of $\{x\}$ and $y \in \{y\}$ and $x \notin \{y\}$. Therefore, (X, D) is T_0 – space.

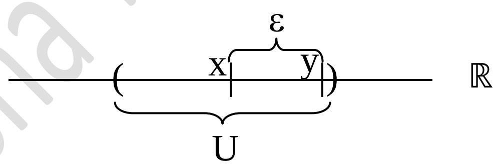
Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_0 – space.

Solution : Let $x, y \in \mathbb{R}$; $x \neq y$

Take $U = (x - \varepsilon , x + \varepsilon)$; $\varepsilon = | x - y |$

$\Rightarrow U \in \tau_u \wedge x \in U \wedge y \notin U$

$\therefore (\mathbb{R}, \tau_u)$ is T_0 – space.



Theorem : Every metric space is T_0 – space.

Proof : Let (X, d) be a metric space and $x, y \in X$; $x \neq y$

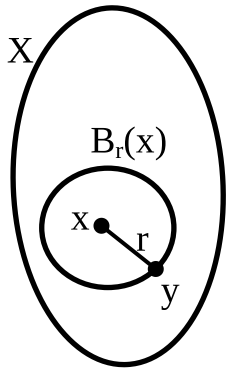
Let $r = d(x, y)$; $d(x, y)$ is the distinct between x and y

Take $U = B_r(x)$; $U = B_r(x)$ is open ball with center x and radius r

$\therefore U \in \tau_d$; $x \in U \wedge y \notin U$;

τ_d is a topology on X induced by d (see page 33)

$\therefore$ (X, d) is T_0 – space.



Now, we introduce theorem gave an equivalent modules for definition T_0 – space.

Theorem : (X, τ) is T_0 – space iff $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall x, y \in X$; $x \neq y$.

i.e., (X, τ) is T_0 – space iff the closure of singleton sets is deference if the elements are deference.

Proof : ($\Rightarrow$) Suppose that X is T_0 – space, to prove $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall x, y \in X$; $x \neq y$

$\because$ X is T_0 – space and $x \neq y \Rightarrow \exists U \in \tau$; $(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

Suppose that $(x \in U \wedge y \notin U) \Rightarrow (x \in U \wedge y \in X - U)$

$X - U$ closed set since U open set $\Rightarrow \{y\} \subseteq X - U$

$\Rightarrow \overline{\{y\}} \subseteq \overline{X - U} = X - U$

(since $X - U$ closed and $\overline{X - U} = X - U$)

$\Rightarrow \overline{\{y\}} \subseteq X - U \wedge x \in U$

$\Rightarrow \{x\} \not\subseteq X - U$

$\Rightarrow \overline{\{x\}} \not\subseteq X - U$

$\therefore \overline{\{x\}} \neq \overline{\{y\}}$

By similar way if we take $(x \notin U \wedge y \in U)$

$(\Leftarrow)$ suppose that $\overline{\{x\}} \neq \overline{\{y\}} \quad \forall\ x \neq y \in X$, to prove X is T_0 – space

Suppose that X is not T_0 – space $\Rightarrow (\exists\ x, y \in X ; \forall\ U \in \tau ; x \in U \Rightarrow y \in U)$ (def of T_0 – space) (i.e., every open set containing x its containing y)

Let $z \in X$; $z \in \overline{\{x\}}$ ---------------(*)

$\Rightarrow \forall\ U \in \tau$; $z \in U \wedge U \cap \{x\} \neq \phi$

(since by true : $z \in \overline{A} \Leftrightarrow \forall\ U \in \tau$; $z \in U \wedge U \cap A \neq \phi$)

But, $U \cap \{x\} \neq \phi \Rightarrow x \in U$ (since the only element in $\{x\}$ is x)

$\therefore$ every set contains z must contains x. So, we have the following two statements : every open set contains z must contains x and every open set contains x must contains y.

$\therefore$ every open set contains z must contains y.

$\Rightarrow \forall\ U \in \tau$; $z \in U \wedge U \cap \{y\} \neq \phi$

$\Rightarrow z \in \overline{\{y\}}$ ---------------(*)

$\Rightarrow \forall\ z \in \overline{\{x\}} \Rightarrow z \in \overline{\{y\}} \Rightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$

By similar way we prove $\overline{\{y\}} \subseteq \overline{\{x\}}$

$\therefore \overline{\{x\}} = \overline{\{y\}}$ C!! contribution (since $\overline{\{x\}} \neq \overline{\{y\}}$)

$\therefore$ X is T_0 – space.

<u>Theorem :</u> The property of being a T_0 – space is a hereditary property.

<u>Proof :</u>

Let (X, τ) T_0 – space and (W, τ_W) subspace of X, to prove (W, τ_W) is T_0 – space

Let $x, y \in W$; $x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_0 – space $\Rightarrow \exists\ U \in \tau$; $(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

$\Rightarrow U \cap W \in \tau_W$ (by def. of τ_W)

; $(x \in U \cap W \wedge y \notin U \cap W) \vee (x \notin U \cap W \wedge y \in U \cap W)$

$\therefore (W, \tau_W)$ is T_0 – space.

Theorem : The property of being a T_0 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and suppose that X is T_0 – space, to prove Y is T_0 – space

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow f^{-1}(y_1), f^{-1}(y_2) \in X$

$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi, f^{-1}(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists! x_1 \in X ; f^{-1}(y_1) = x_1$ and $\exists! x_2 \in X ; f^{-1}(y_2) = x_2$

and $x_1 \neq x_2$ and $x_1, x_2 \in X$

$\because$ X is T_0 – space $\Rightarrow \exists U \in \tau ; (x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because f^{-1}$ is cont. or f open $\Rightarrow f(U) \in \tau'$; $(f(x_1) \in f(U) \wedge f(x_2) \notin f(U))$

$\vee (f(x_1) \notin f(U) \wedge f(x_2) \in f(U))$

$\therefore$ Y is T_0 – space

By similar we prove, if Y is T_0 – space, then X is T_0 – space.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_0 – space iff each X and Y are T_0 – space.

Proof :

($\Rightarrow$) Suppose that $X \times Y$ is a T_0 – space, to prove that X and Y are T_0 – space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow (x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because X \times Y$ is a T_0 – space $\Rightarrow \exists$ a basic open set $U \times V \in \tau_{X \times Y}$;

$((x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V) \vee ((x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V)$

$\Rightarrow \exists U \in \tau ; (x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U) \Rightarrow$ X is a T_0 – space

and $\exists V \in \tau' ; (y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V) \Rightarrow$ Y is a T_0 – space.

($\Leftarrow$) Suppose that X and Y are T_0 – space, to prove $X \times Y$ is a T_0 – space

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow (x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_0 – space $\Rightarrow \exists U \in \tau ; (x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ Y is a T_0 – space $\Rightarrow \exists V \in \tau' ; (y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V)$

$\Rightarrow \exists U \times V$ is a basic open set ; $((x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V)$

$\vee ((x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V)$

$\therefore X \times Y$ is a T_0 – space.

<u>Definition :</u> T_1 – Space

Let (X, τ) be a topological space. Then the space $(X, \tau)$is called **<u>T_1 – space</u>** iff for each pair of distinct points $x, y \in X$, there exists an open set in X containing x but not y, and an open set in X containing y but not x. i.e.,

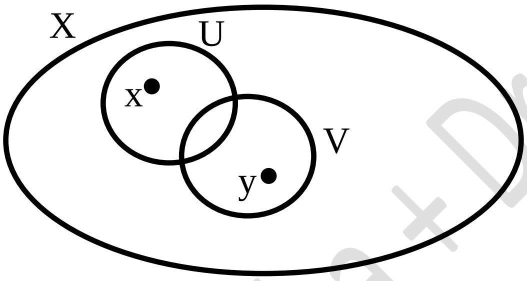
X is T_1 – space $\Leftrightarrow \forall x, y \in X ; x \neq y \ \exists U, V \in \tau ; (x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

If (X, τ) is not T_1 – space, we define,

X is not T_1 – space $\Leftrightarrow \exists x, y \in X ; x \neq y \ \forall U, V \in \tau$

$$; (x \in U \wedge y \in U) \vee (x \notin U \wedge y \notin U)$$
$$(x \in V \wedge y \in V) \vee (x \notin V \wedge y \in V).$$

The following figure show the definition of T_1 – space :



<u>Remark :</u> Every T_1 – space is T_0 – space (i.e., $T_1 \Rightarrow T_0$). But the reverse implications does not hold (i.e., $T_0 \not\Rightarrow T_1$) and the following example show that :

<u>Example :</u> Let (X, τ) be a topological space such that $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{1, 2\}\}$.

<u>Solution :</u> Clear (X, τ) is T_0 – space (see page 76).

But, (X, τ) is not T_1 – space, since $2 \neq 3$ and $\exists$ open set $\{1, 2\}$ in X containing 2 but not 3, but $\nexists$ open set in X containing 3 but not 2, since the only open set containing 3 is X and X containing 2 too.

<u>Remark :</u> If (X, τ) is T_1 – space, then its not necessary to test that the space is T_0 – space, since every T_1 – space is a T_0 – space.

<u>Example :</u> The space (X, D) is T_1 – space.

<u>Solution :</u>

Let $x, y \in X ; x \neq y \Rightarrow \exists \{x\}, \{y\} \in D ; (x \in \{x\} \wedge y \notin \{x\}) \wedge (x \notin \{y\} \wedge y \in \{y\})$

$\Rightarrow$ (X, D) is T_1 – space.

Example : Is $(\mathbb{N}, \tau_{cof})$ T_1 – space ??

Solution : Yes,

Let n, m $\in \mathbb{N}$; $n \neq m$, take $U = \mathbb{N} \setminus \{m\}$, $V = \mathbb{N} \setminus \{n\}$

$\Rightarrow U, V \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

(since $V^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

$\Rightarrow (n \in U = \mathbb{N} \setminus \{m\} \wedge m \notin U) \wedge (n \notin V = \mathbb{N} \setminus \{n\} \wedge m \in V)$

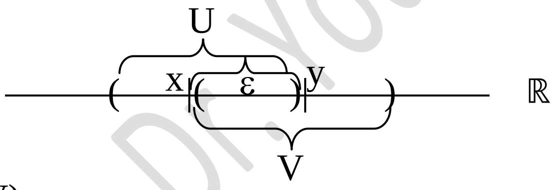
$\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_1 – space.

Example : The usual topological space $(\mathbb{R}, \tau_u)$ is a T_1 – space.

Solution :

Let $x, y \in \mathbb{R}$; $x \neq y$, $\varepsilon = | x - y |$

Take $U = (x - \varepsilon , x + \varepsilon)$, $V = (y - \varepsilon, y + \varepsilon)$



$\therefore$ $U, V \in \tau_u$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

$\therefore$ $(\mathbb{R}, \tau_u)$ is T_1 – space.

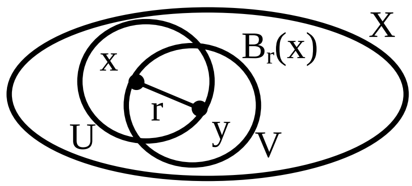
Theorem : Every metric space is a T_1 – space.

Proof : Let (X, d) be a metric space and $x, y \in X$; $x \neq y$

Take $U = B_r(x)$, $V = B_r(y)$; $r = d(x, y)$

$\therefore$ $U, V \in \tau_d$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$

$\therefore$ (X, d) is T_1 – space.



Theorem : (X, τ) is T_1 – space iff $\{x\}$ is closed $\forall$ $x \in X$.

i.e., (X, τ) is T_1 – space iff every singleton set in X is closed.

Proof : ($\Rightarrow$) Suppose that X is T_1 – space, to prove $\{x\}$ closed $\forall$ $x \in X$

i.e., $X - \{x\}$ open set, we must prove $X - \{x\}$ contains a nbhd $\forall$ $y \in X - \{x\}$

Let $y \in X - \{x\}$ $\Rightarrow$ $x \neq y$

$\because$ X is T_1 – space $\Rightarrow$ $\exists$ $U, V_y \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V_y \wedge y \in V_y)$

$\Rightarrow$ $y \in V_y$ $\wedge$ $x \notin V_y$

$\Rightarrow$ $\{x\} \bigcap V_y = \phi$

$\Rightarrow$ $V_y \subseteq X - \{x\}$ $\wedge$ $y \in V_y$

$\Rightarrow$ $V_y \subseteq X - \{x\}$ $\forall$ $y \in X - \{x\}$

$\therefore$ $X - \{x\}$ contains a nbhd $\forall$ $y \in X - \{x\}$.

$\therefore$ $X - \{x\}$ open set $\Rightarrow$ $\{x\}$ closed $\forall$ $x \in X$.

($\Leftarrow$) suppose that $\{x\}$ closed $\forall\ x \in X$, to prove X is T_1 – space

Let $x, y \in X$; $x \neq y \Rightarrow \{x\}, \{y\}$ are closed sets

$\Rightarrow X - \{x\}, X - \{y\}$ are open sets

Say $U = X - \{y\}, V = X - \{x\} \Rightarrow (x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$

$\therefore$ (X, τ) is T_1 – space.

Corollary : If X is a T_1 – space, then every finite set is closed.

Proof : Let A be a finite set in X

$\Rightarrow A = \{x_1, \ldots . x_n\} = \bigcup_{i=1}^{n}\{x_i\}$

$\because$ (X, τ) is T_1 – space $\Rightarrow \{x_i\} \in \mathcal{F}\ \forall\ i$

$\Rightarrow \bigcup_{i=1}^{n}\{x_i\}$ closed

$\Rightarrow$ A is a closed

Corollary : If X is finite set and (X, τ) is a T_1 – space, then $\tau = D$.

Proof : To prove $\tau = D$ must we prove $(\forall\ x \in X \Rightarrow \{x\} \in \tau)$, i.e., every singleton set $\{x\}$ is open.

Let $x \in X$

$\because$ X finite set $\Rightarrow X - \{x\}$ finite set

$\because$ X is T_1 – space $\Rightarrow X - \{x\}$ closed set

(by previous corollary : If X is T_1 – space, then every finite set is closed)

$\Rightarrow \{x\}$ open

$\therefore$ $\tau = D$.

Remark : From the previous corollary the only topology that make the space (X, τ) is T_1 – space when X is finite set is D. For example if $X = \{1, 2, 3\}$ and we know there is 29 deference topology on X (see page 2) so that there is 28 topology on X is not T_1 – space except one topology is D. Therefore, we not try to give an example for space is T_1 – space on finite set and the topology not D.

Now, we introduce some corollaries on the theorem in page 81 and your proves is directed from theorem.

Corollary (1) : (X, τ) is T_1 – space iff $\overline{\{x\}} = \{x\}\ \forall\ x \in X$.

Corollary (2) : (X, τ) is T_1 – space iff $\{x\} = \bigcap\{F ; F \in \mathcal{F} \wedge x \in F\}\ \forall\ x \in X$.

Corollary (3) : (X, τ) is T_1 – space iff $\{x\}^b = \phi \ \ \forall x \in X$.

Corollary (4) : (X, τ) is T_1 – space iff $\{x\}^b \subseteq \{x\} \ \ \forall x \in X$.

Corollary (5) : (X, τ) is T_1 – space iff $\{x\}^/ \subseteq \{x\} \ \ \forall x \in X$.

Corollary (6) : (X, τ) is T_1 – space iff $\{x\}^/ = \phi \ \ \forall x \in X$.

Theorem : The property of being a T_1 – space is a hereditary property.

Proof :

Let (X, τ) T_1 – space and (W, τ_W) subspace of X, to prove (W, τ_W) T_1 – space

Let $x, y \in W$; $x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_1 – space $\Rightarrow \exists U, V \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

$\Rightarrow U \cap W \wedge V \cap W \in \tau_W$ (by def. τ_W)

$\Rightarrow (x \in U \cap W \wedge y \notin U \cap W) \wedge (x \notin V \cap W \wedge y \in V \cap W)$.

$\therefore$ (W, τ_W) is a T_1 – space.

Theorem : The property of being a T_1 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and X is a T_1 – space, to prove Y is a T_1 – space

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow f^{-1}(y_1), f^{-1}(y_2) \in X$

$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi, \ f^{-1}(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\, x_1 \in X$; $f^{-1}(y_1) = x_1$ and $\exists!\, x_2 \in X$; $f^{-1}(y_2) = x_2$

and $x_1 \neq x_2$ and $x_1, x_2 \in X$

$\because$ X is T_1 – space $\Rightarrow \exists U, V \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \wedge (x_1 \notin V \wedge x_2 \in V)$

$\because f^{-1}$ is cont. or f open $\Rightarrow f(U), f(V) \in \tau'$; $(f(x_1) \in f(U) \wedge f(x_2) \notin f(U))$

$\wedge \ (f(x_1) \notin f(V) \wedge f(x_2) \in f(V) \ )$

$\therefore$ Y is T_1 – space

By similar way we prove, if Y is T_1 – space, then X is T_1 – space.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X\times Y$ is a T_1 – space iff each X and Y are T_1 – space.

Proof :

$(\Leftarrow)$ Suppose that X and Y are T_1 – space, to prove $X\times Y$ is a T_1 – space

Let $(x_1, y_1), (x_2, y_2) \in X\times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow (x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_1 – space $\Rightarrow \exists\, U_1, U_2 \in \tau$; $(x_1 \in U_1 \wedge x_2 \notin U_1) \wedge (x_1 \notin U_2 \wedge x_2 \in U_2)$

$\because$ Y is a T_1 – space $\Rightarrow \exists\, V_1, V_2 \in \tau'$; $(y_1 \in V_1 \wedge y_2 \notin V_1) \wedge (y_1 \notin V_2 \wedge y_2 \in V_2)$

$\Rightarrow \exists$ basic open sets $U_1\times V_1, U_2\times V_2$;

$((x_1, y_1) \in U_1\times V_1 \wedge (x_2, y_2) \notin U_1\times V_1) \wedge ((x_1, y_1) \notin U_2\times V_2 \wedge (x_2, y_2) \in U_2\times V_2)$

$\therefore$ $X\times Y$ is a T_1 – space.

$(\Rightarrow)$ Suppose that $X\times Y$ is a T_1 – space, to prove X and Y are T_1 – space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow (x_1, y_1), (x_2, y_2) \in X\times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because X\times Y$ is a T_1–space $\Rightarrow \exists\, U, V \in \tau_{X\times Y}$; $(x_1, y_1) \in U \wedge (x_2, y_2) \notin U \wedge (x_2, y_2) \in V \wedge (x_1, y_1) \notin V$ that is mean $\exists$ basic open sets $U_1\times V_1, U_2\times V_2 \in \tau_{X\times Y}$;

$((x_1, y_1) \in U_1\times V_1 \wedge (x_2, y_2) \notin U_1\times V_1) \wedge ((x_1, y_1) \notin U_2\times V_2 \wedge (x_2, y_2) \in U_2\times V_2)$

$\Rightarrow \exists\, U_1, U_2 \in \tau$; $(x_1 \in U_1 \wedge x_2 \notin U_1) \wedge (x_1 \notin U_2 \wedge x_2 \in U_2) \Rightarrow$ X is a T_1 – space

and $\exists\, V_1, V_2 \in \tau'$; $(y_1 \in V_1 \wedge y_2 \notin V_1) \wedge (y_1 \notin V_2 \wedge y_2 \in V_2) \Rightarrow$ Y is a T_1 – space.

Definition : **T_2 – Space or Hausdorff Space**

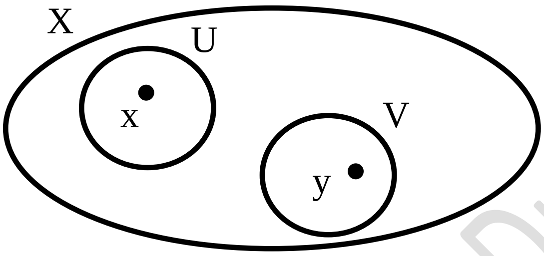
Let (X, τ) be a topological space. Then the space (X, τ) is called a **T_2 – space or Hausdorff space** iff for each pair of distinct points x, y ∈ X, there exist open sets U and V such that x ∈ U, y ∈ V, and $U \cap V = \phi$. i.e.,

$$X \text{ is } T_2 - \text{space} \Leftrightarrow \forall\, x, y \in X \,;\, x \neq y\ \exists\, U, V \in \tau \,;\, (x \in U \wedge y \in V),\ U \cap V = \phi$$

If (X, τ) is not T_2 – space, we define,

$$X \text{ is not } T_2\text{–space} \Leftrightarrow \exists\, x, y \in X \,;\, x \neq y\ \forall\, U, V \in \tau;\ U \cap V = \phi,\ (x, y \in U \vee x, y \in V)$$

The following figure show the definition of T_2 – space :



Remark : Every T_2 – space is T_1 – space (i.e., $T_2 \Rightarrow T_1$). But the reverse implications do not hold (i.e., $T_1 \not\Rightarrow T_2$) and the following example show that :

Example : Take cofinite topology $(\mathbb{N}, \tau_{cof})$.

Solution : Clear $(\mathbb{N}, \tau_{cof})$ is T_1 – space (see page 81).

But, $(\mathbb{N}, \tau_{cof})$ is not T_2 – space, since if n ≠ m , take $U = \mathbb{N} \setminus \{m\}$, $V = \mathbb{N} \setminus \{n\}$, but $U \cap V \neq \phi$. Therefore, $T_1 \not\Rightarrow T_2$.

Remark : If (X, τ) is T_2 – space, then not necessary test that the space is T_1 – space and T_0 – space, since every T_2 – space is T_1 – space and every T_1 – space is T_0 – space i.e., ($T_2 \Rightarrow T_1 \Rightarrow T_0$).

Example : In the space (X, I) if X is any set containing more than one element, then (X, I) is not T_0 – space (see page 77), so that it's not T_1 – space and not T_2 – space.

Example : The space (X, D) is T_2 – space.

Solution :

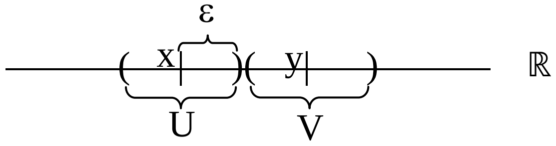
Let $x, y \in X \,;\, x \neq y \Rightarrow \{x\}, \{y\} \in D \,;\, \{x\} \cap \{y\} = \phi$, $(x \in \{x\} \wedge y \in \{y\})$

$\Rightarrow$ (X, D) is T_2 – space.

Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_2 – space.

Solution : Let $x, y \in \mathbb{R}$; $x \neq y$, $\varepsilon = \frac{1}{2} | x - y |$

Take $U = (x - \varepsilon, x + \varepsilon)$, $V = (y - \varepsilon, y + \varepsilon)$



$\therefore$ $U, V \in \tau_u$; $U \cap V = \phi$, $(x \in U \wedge y \in V)$

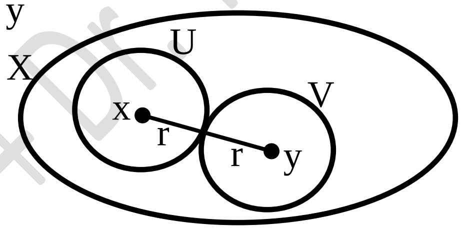
$\Rightarrow$ $(\mathbb{R}, \tau_u)$ is T_2 – space.

Remark : In the previous remark (p. 82) we show that if X is finite set and $\tau \neq D$, then (X, τ) is not T_1 – space and we say her is not T_2 – Space. i.e., the only topology make (X, τ) is T_2 – space if X is finite set is D.

Theorem : Every metric space is T_2 – space.

Proof : Let (X, d) be a metric space and $x, y \in X$; $x \neq y$

Take $U = B_r(x)$, $V = B_r(y)$; $r = \frac{1}{2}\ d(x, y)$



$\therefore$ $U, V \in \tau_d$; $U \cap V = \phi$, $(x \in U \wedge y \in V)$

$\therefore$ (X, d) is T_2 – space.

Theorem : (X, τ) is a T_2 – space iff the diagonal $\Delta = \{(x, x) \in X \times X ; x \in X\}$ is a closed subset of the product $X \times X$.

Proof : $(\Rightarrow)$ Suppose that X is T_2 – space, to prove Δ closed in $X \times X$

i.e., $X \times X - \Delta$ open set, we must prove $X \times X - \Delta$ contains a nbhd $\forall$ $(x, y) \in X \times X - \Delta$

Let $(x, y) \in X \times X - \Delta \Rightarrow (x, y) \notin \Delta$ (def. of deference)

$\Rightarrow x \neq y$ (since Δ has equal coordinate)

$\because$ X is a T_2 – space $\Rightarrow \exists\, U, V \in \tau$; $U \cap V = \phi$, $(x \in U \wedge y \in V)$

$\Rightarrow U \times V \in \beta_{X \times X} \subseteq \tau_{X \times X}$ (by def. product space)

$\Rightarrow U \times V$ open set in $X \times X$ and

$U \times V \subseteq X \times X - \Delta \ \wedge\ (x, y) \in U \times V$ (since $U \cap V = \phi$)

Since, if $U \times V \not\subseteq X \times X - \Delta \Rightarrow \exists\, (x, x) \in \Delta \Rightarrow x \in U \wedge x \in V$ C!! (contridition)

$\therefore$ $X \times X - \Delta$ contains a nbhd $\forall$ $x \in X \times X - \Delta$

$\Rightarrow X \times X - \Delta \in \tau_{X \times X}$

$\Rightarrow \Delta$ closed in $X \times X$

$(\Leftarrow)$ Suppose that Δ closed in $X \times X$, to prove X is T_2 – space

Let $x, y \in X$; $x \neq y \Rightarrow (x, y) \notin \Delta$ (by def. of Δ)

$\Rightarrow (x, y) \in \Delta^c = X \times X - \Delta$

$\because$ Δ closed set $\Rightarrow X \times X - \Delta$ open set

$\Rightarrow \exists\ U\times V\ ;\ U, V \in \tau \wedge (x, y) \in U\times V\ ,\ U\times V \subseteq X\times X - \Delta,\ x \in U,\ y \in V$

$\Rightarrow U\times V \bigcap \Delta = \phi$ (i.e., $\nexists$ element in $U\times V$ has equal coordinate)

$\Rightarrow U \bigcap V = \phi$

$\therefore$ (X, τ) is T_2 – space.

<u>Theorem :</u> The property of being a T_2 – space is a hereditary property.

<u>Proof :</u>

Let (X, τ) T_2 – space and (W, τ_W) subspace of X, to prove (W, τ_W) T_2 – space

Let $x, y \in W\ ;\ x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_2 – space $\Rightarrow \exists\ U, V \in \tau\ ;\ U \bigcap V = \phi\ ,\ (x \in U \wedge y \in V)$.

$\Rightarrow U \bigcap W \wedge V \bigcap W \in \tau_W$, (by def. of τ_W)

$(U \bigcap W) \bigcap (U \bigcap W) = (U \bigcap V) \bigcap W = \phi \bigcap W = \phi$

and $(x \in U \bigcap W \wedge y \in V \bigcap W)$.

$\therefore$ (W, τ_W) is T_2 – space.

<u>Theorem :</u> The property of being a T_2 – space is a topological property.

<u>Proof :</u>

Let $(X, \tau) \cong (Y, \tau')$ and suppose that Y is T_2 – space, to prove X is T_2 – space

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists\ f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $x_1, x_2 \in X\ ;\ x_1 \neq x_2 \Rightarrow f(x_1), f(x_2) \in Y$

$\because$ f onto function $\Rightarrow f(x_1) \neq \phi,\ f(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\ y_1 \in Y\ ;\ f(x_1) = y_1$ and $\exists!\ y_2 \in Y\ ;\ f(x_2) = y_2$

and $y_1 \neq y_2$, $y_1, y_2 \in Y$

$\because$ Y is T_2 – space $\Rightarrow \exists\ V_1, V_2 \in \tau'\ ;\ V_1 \bigcap V_2 = \phi\ ,\ (y_1 \in V_1 \wedge y_2 \in V_2)$

$\because$ f is continuous $\Rightarrow f^{-1}(V_1) = U_1,\ f^{-1}(V_2) = U_2 \in \tau$;

$U_1 \bigcap U_2 = f^{-1}(V_1) \bigcap f^{-1}(V_2) = f^{-1}(V_1 \bigcap V_2) = f^{-1}(\phi) = \phi$,

$(x_1 \in U_1 \wedge x_2 \in U_2)$

$\therefore$ X is T_2 – space

By similar way we prove, if X is T_2 – space, then Y is T_2 – space.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X\times Y$ is a T_2 – space iff each X and Y are T_2 – space.

Proof :

($\Leftarrow$) Suppose that X and Y are T_2 – space, to prove $X\times Y$ is T_2 – space

Let $(x_1, y_1), (x_2, y_2) \in X\times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow (x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_2 – space $\Rightarrow \exists\, U_1, U_2 \in \tau$; $U_1 \cap U_2 = \phi$, $(x_1 \in U_1 \wedge x_2 \in U_2)$

$\because$ Y is a T_2 – space $\Rightarrow \exists\, V_1, V_2 \in \tau'$; $V_1 \cap V_2 = \phi$, $(y_1 \in V_1 \wedge y_2 \in V_2)$

$\Rightarrow \exists$ basic open sets $U_1\times V_1, U_2\times V_2$;

$(U_1\times V_1) \cap (U_2\times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \phi \times \phi = \phi$,

$((x_1, y_1) \in U_1\times V_1 \wedge (x_2, y_2) \in U_2\times V_2)$

$\therefore$ $X\times Y$ is a T_2 – space.

($\Rightarrow$) Suppose that $X\times Y$ is a T_2 – space, to prove X and Y are T_2 – space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow (x_1, y_1), (x_2, y_2) \in X\times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because X\times Y$ is T_2 – space $\Rightarrow \exists\, U, V \in \tau_{X\times Y}$; $(x_1, y_1) \in U \wedge (x_2, y_2) \in V \wedge U \cap V = \phi$,

that is mean $\exists$ basic open sets $U_1\times V_1, U_2\times V_2 \in \tau_{X\times Y}$;$U_1\times V_1, U_2\times V_2 \in \tau_{X\times Y}$;

$(U_1\times V_1) \cap (U_2\times V_2) = \phi$, $((x_1, y_1) \in U_1\times V_1 \wedge (x_2, y_2) \in U_2\times V_2)$

$\Rightarrow \exists\, U_1, U_2 \in \tau$; $U_1 \cap U_2 = \phi$, $(x_1 \in U_1 \wedge x_2 \in U_2) \Rightarrow$ X is a T_2 – space

and $\exists\, V_1, V_2 \in \tau'$; $V_1 \cap V_2 = \phi$, $(y_1 \in V_1 \wedge y_2 \in V_2) \Rightarrow$ Y is a T_2 – space.

Remark : We study in mathematical analysis real sequences and their convergence and we know if the real sequences is convergence, then has a **unique limit** and this is especial case since their domain is real numbers with usual topology ($\mathbb{R}$, τ_u). But, if we go to the topological spaces in general and define the convergence sequences in topological space we found the sequences is convergence but their **limit not unique**. In mathematical analysis (3^{th} stage) we use the usual topology only and this T_2 – space and we will introduce illustrate to this problem with definitions and theorems.

Definition : Let (X, d) be a metric space. We called the function $S : \mathbb{N} \to X$ is a sequences in X and denoted to image of element n in $\mathbb{N}$ which is S(n) by S_n , so that the sequence is $S_1, S_2, \ldots, S_n, \ldots$; $n \in \mathbb{N}$ or $(S_n)_{n\in\mathbb{N}}$.
We called the sequence is convergence to element x_0 in X (denoted by $S_n \longrightarrow x_0$) if the following condition satisfy :

$$\forall\ \varepsilon > 0 \quad \exists\ k \in \mathbb{N} \text{ s.t } n \geq k\ ;\ S_n \in N_\varepsilon(x_0)$$

such that $N_\varepsilon(x_0)$ is open ball in X with center x_0 and radius ε.
this means that the sequences (S_n) convergence to limit x_0 in X if every open ball with center x_0 contains all elements of sequences except finite numbers of elements.

Notes that the definition especial of metric space, so we now generalization this definition to topological space such that we replaces open ball by open set since the open ball no exist in topological space because there is not distant.

Definition : Let $(S_n)_{n\in\mathbb{N}}$ be a sequences in topological space (X, τ) (i.e., $S : \mathbb{N} \longrightarrow X$).
We called the sequence $(S_n)_{n\in\mathbb{N}}$ is convergence to x_0 in X (denoted by $S_n \longrightarrow x_0$) if the following condition satisfy :

$$S_n \longrightarrow x_0 \Leftrightarrow \forall\ U \in \tau\ ;\ x_0 \in U \quad \exists\ k \in \mathbb{N}\ ;\ S_n \in U \quad \forall\ n \geq k$$

i.e., every open nbhd of x_0 contains all elements of sequences except finite numbers.

Remark : If (X, τ) is not T_2 – space, then the convergence sequences may be has more than one limit point.

Example : Let X = {1, 2, 3} and τ = I = {X, ϕ} such that (X, τ) is topological space. Let $(S_n)_{n\in\mathbb{N}}$ be a sequences in X such that $S_n = 1$ for all n

Solution : Clear $S_n \longrightarrow 1$, $S_n \longrightarrow 2$, and $S_n \longrightarrow 3$??
To clear that apply the definition ; $S_n \longrightarrow 1$, since the open nbhds of 1 is X only because it's the unique open set contains 1 and $S_n \longrightarrow 1$ since X contains all elements so its contain the sequence. Therefore the definition satisfy.

By similar way $S_n \longrightarrow 2$ and $S_n \longrightarrow 3$ since X the unique open set that contains 2 and also contains 3 and X contains the sequences too.

Question : Give an example to convergence sequence in topological space has five deference limit points.

Answer : Let X = {1, 2, 3, 4, 5, 6} and $\tau = \{X, \phi, \{6\}\}$ such that (X, τ) is topological space.

Define $(S_n)_{n\in\mathbb{N}}$ as follows $S_n = 3\ \forall\ n \in \mathbb{N}$, so that

$$S_n \longrightarrow 1,\ S_n \longrightarrow 2,\ S_n \longrightarrow 3,\ S_n \longrightarrow 4,\ S_n \longrightarrow 5, \text{ but } S_n \not\longrightarrow 6$$

Since {6} is open nbhd for 6 , but $S_n \notin \{6\}\ \forall\ n \in \mathbb{N}$, because $S_n = 3$ and $3 \notin \{6\}$.

On the other hand, $S_n \longrightarrow 1, 2, 3, 4, 5$ since X the unique open set that contains 1, 2, 3, 4, 5 and X contains 3 i.e., $S_n \in X\ \forall\ n \in \mathbb{N}$.

We can change the previous question to make give an example to convergence sequence has ten or seven or any known number deference limit point. By taken X has 11 element (if require 10 deference limit point) and define topology on X contains X, ϕ, and singleton set contain one of the elements of X and define constant sequence such that the constant number is one number of elements of X not in singleton set. Therefore, we have the require.

Question : Give an example to convergence sequence in topological space has infinite number of deference limit point.

Answer : Let $X = \mathbb{R}$ (or X = any infinite set) and take $\tau = I = \{X, \phi\}$ and take any sequence in $\mathbb{R}$ for example :

$$S_n = \begin{cases} \sqrt{2} & \text{if} \quad n \in E \\ 0 & \text{if} \quad n \in O \end{cases}$$

Notes that the sequence $(S_n)_{n\in\mathbb{N}}$ is not convergence in $(\mathbb{R}, \tau_u)$.

But in $(\mathbb{R}, I)$ is convergence and has infinite numbers of limit points, since every real number is limit point because $\mathbb{R}$ the unique open set and $\mathbb{R}$ contains $\sqrt{2}$ and 0 and $\mathbb{R}$ contains all elements of the sequences (i.e., every open nbhd for all element in $\mathbb{R}$ contains all elements of the sequences) and this means in the definition of convergence in metric space that for all real number is limit point of the sequence $(S_n)_{n\in\mathbb{N}}$.

Remark : If (X, τ) is T_2 – space, then every convergence sequence in X has unique limit point and this illustrate that consider the limit point if exist, then its unique in mathematical analysis (3^{th} stage), because we study the metric space only and special case ($\mathbb{R}$, | |) and we prove that every metric space is T_2 – space (see page 86), so that every convergence sequence in metric space has unique limit point and we will introduce the theorem show this :

Theorem : If (X, τ) is T_2 – space, then every convergence sequence in X has unique limit point.

Proof : Let $(S_n)_{n\in\mathbb{N}}$ be convergence sequences in X

Suppose that $S_n \longrightarrow x$ and $S_n \longrightarrow y$; $x \neq y \in X$

$\because$ X is T_2 – space $\Rightarrow \exists\, U, V \in \tau$; $U \cap V = \phi$, $(x \in U \wedge y \in V)$.

$\because$ $S_n \longrightarrow x$ and $x \in U \in \tau \Rightarrow \exists\, k_1 \in \mathbb{N}$; $S_n \in U \;\; \forall\, n \geq k_1$

$\because$ $S_n \longrightarrow y$ and $y \in V \in \tau \Rightarrow \exists\, k_2 \in \mathbb{N}$; $S_n \in V \;\; \forall\, n \geq k_2$

$\because$ elements of sequence is infinite (since domain = $\mathbb{N}$), then there are elements common between U and V (i.e., $U \cap V \neq \phi$). C!! contradiction

$\therefore$ every convergence sequence in X has unique limit point.

We will use idea of convergence sequence in topological space and idea axiom of T_2 – space with important continuous concept in topology :

Theorem : If $f : (X, \tau) \rightarrow (Y, \tau')$ is continuous function and $(S_n)_{n\in\mathbb{N}}$ is convergence sequence in X such that $S_n \longrightarrow x$, then $f(S_n) \longrightarrow f(x)$.

i.e., The continuous function maps convergence sequence in domain to convergence sequence in codomain and their limit point is image of limit point in domain.

Proof : To prove $f(S_n) \longrightarrow f(x)$ in Y

Let V be an open nbhd of f(x) in Y, i.e., $f(x) \in V \in \tau'$

$\because$ f continuous $\Rightarrow f^{-1}(V) \in \tau$, i.e., $f^{-1}(V)$ open set in X

$\because$ $f(x) \in V \Rightarrow x \in f^{-1}(V)$

$\Rightarrow f^{-1}(V)$ be an open nbhd of x in X

$\because$ $S_n \longrightarrow x$ and $x \in f^{-1}(V) \in \tau \Rightarrow \exists\, k \in \mathbb{N}$; $S_n \in f^{-1}(V) \;\; \forall\, n \geq k$

Take image for S_n and for $f^{-1}(V)$, we have

$\Rightarrow \exists\, k \in \mathbb{N}$; $f(S_n) \in V \;\; \forall\, n \geq k$

i.e., V contains all elements of sequence except k of these element.

$\therefore$ $f(S_n) \longrightarrow f(x)$.

Now, we prove one of the important theorem which connected between the concept axiom T_2 – space and concept compactness.

Theorem : Every compact set in T_2 – space is closed.

Proof : Let (X, τ) be T_2 – space and $A \subseteq X$; A compact in X

To prove A is closed in X, i.e., X – A open set in X

we must prove X – A contains an open nbhd $\forall\ x \in X - A$

(i.e., $\forall\ x \in X - A\ \exists\ U \in \tau$; $x \in U \subseteq X - A$)

Let $x \in X - A \Rightarrow x \notin A \Rightarrow x \neq a\ \forall\ a \in A$

$\because$ X is T_2 – space $\Rightarrow \exists\ U_a, V_a \in \tau$; $U_a \bigcap V_a = \phi$, $(x \in U_a \wedge a \in Va)\ \forall\ a \in A$.

We have two family of open sets are $\{U_a\}_{a\in A}$ (every element in this family contains x) and $\{V_a\}_{a\in A}$ (every element in this family contains one of the elements A) and every U_a corresponding V_a such that $U_a \bigcap V_a = \phi$.

$\Rightarrow \{V_a\}_{a\in A}$ open cover of A, i.e., $A \subseteq \bigcup_{a\in A} V_a$

$\because$ A is compact set $\Rightarrow \exists\ a_1, \dots, a_n$; $A \subseteq \bigcup_{i=1}^{n} V_{a_i}$

Therefore, there is a finite family $\{U_a\}_{a\in A}$ corresponding the finite family $\{V_{a_i}\}_{i=1}^{n}$ which is $\{U_{a_i}\}_{i=1}^{n}$.

$\because$ every U_a contain x, then $x \in \bigcap_{i=1}^{n} U_{a_i}$

$\because$ every U_{a_i} is open set, then $\bigcap_{i=1}^{n} U_{a_i}$ is open set contain x

(second condition of def of top.)

Say, $U = \bigcap_{i=1}^{n} U_{a_i} \Rightarrow x \in U \in \tau$

On the other hand $\bigcup_{i=1}^{n} V_{a_i}$ is open set (third condition of def. of top.)

Say, $V = \bigcup_{i=1}^{n} V_{a_i} \Rightarrow A \subseteq V \in \tau$

Notes that,

$U \bigcap V = \phi$ (since $U_a \bigcap V_a = \phi$)

$\Rightarrow A \bigcap U = \phi$ (since $A \subseteq V$ and $U \bigcap V = \phi$)

$\Rightarrow U \subseteq X - A$

$\Rightarrow x \in U \subseteq X - A \wedge U \in \tau$

$\Rightarrow X - A$ open set in X $\forall\ x \in X - A$

$\Rightarrow$ A closed set in X.

Finally, we introduce one of important theorem which connected the concepts continuous, T_2 – space, and compactness with homomorphism concept.

Theorem : If $f : (X, \tau) \to (Y, \tau')$ is continuous bijective function and X is compact space and Y is T_2 – space, the f is homomorphism.

Proof : It is enough to prove f is closed function

Let F be a closed set in X

$\because$ X is compact and F closed in X $\Rightarrow$ F compact in X

(by theorem : A closed subset of compact space is compact)

$\because$ f is continuous and F is compact in X $\Rightarrow$ f(F) is compact in Y

(by theorem : A continuous image of compact set is compact set)

$\because$ Y is T_2 – space and f(F) compact in Y $\Rightarrow$ f(F) is closed in Y

(by previous theorem : Every compact set in T_2 – space is closed)

$\therefore$ f is closed function $\Rightarrow$ f is Homeomorphism.

In chapter three (compact space) we say the intersection of two compact sets not necessary compact set (see page 68) and intersection closed set and compact sets not necessary compact set, but this statements are satisfy if we add the condition that T_2 – space and the following theorems show this.

Theorem : If A is closed set and B is compact set in T_2 – space (X, τ), then $A \cap B$ is compact.

Proof :

$\because$ X is T_2 – space and B compact in X $\Rightarrow$ B is closed in X

(by theorem : Every compact set in T_2 – space is closed)

$\because$ A and B closed sets $\Rightarrow$ $A \cap B$ closed set i.e., $A \cap B \in \mathcal{F}$

(second condition of def of top.)

$\because$ $A \cap B \subseteq B$ i.e., $A \cap B$ subspace of B and

$A \cap B$ closed in B and B compact $\Rightarrow$ $A \cap B$ compact

(by theorem : A closed subset of compact space is compact)

Corollary : If A and B are compact sets in T_2 – space (X, τ), then $A \cap B$ is compact.

Proof :

$\because$ X is T_2 – space and A compact in X $\Rightarrow$ A is closed in X

(by theorem : Every compact set in T_2 – space is closed)

$\because$ X is T_2 – space and A closed and B compact in X $\Rightarrow$ $A \cap B$ compact

(by previous theorem)

Remark : If X is T_2 – space and compact, then $X \supseteq A$ closed $\Leftrightarrow$ A compact.

<u>Definition :</u> Regular Space

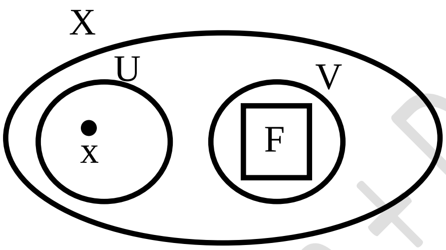
Let (X, τ) be a topological space. Then the space (X, τ) is called a **<u>Regular Space</u>** iff for each closed set $F \subset X$ and each point $x \notin F$, there exist open sets U and V such that $x \in U$, $F \subset V$, and $U \cap V = \phi$ (denoted by R – space). i.e.,

$$X \text{ is R – space} \Leftrightarrow \forall\, x \in X\ \forall\, F \in \mathscr{F};\ x \notin F\ \exists\, U, V \in \tau\ ;\ U \cap V = \phi\ ,\ (x \in U \wedge F \subset V).$$

If (X, τ) is not R – space, we define,

$$X \text{ is not R – space} \Leftrightarrow \exists\, x \in X\ \exists\, F \in \mathscr{F}\ ;\ x \notin F \wedge \forall\, U, V \in \tau\ ;\ U \cap V = \phi\ ,$$
$$(x \in U \wedge F \subseteq U) \vee (x \in V \wedge F \subseteq V) \vee$$
$$(x \notin U \wedge F \not\subseteq V) \vee (x \notin V \wedge F \not\subseteq V)\ .$$

The following figure show the definition of R – space :



<u>Example :</u> Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$. Is (X, τ) R – space.

<u>Solution :</u> First, find the family of closed sets :

$$\mathscr{F} = \{X, \phi, \{2, 3\}, \{1, 3\}, \{3\}\}$$

Second, take every closed set and every element not belong to it as follow :

Take X , but every element belong to X

Take ϕ and $1, 2, 3 \notin \phi \Rightarrow \exists\ U = X$ and $V = \phi$ such that $1, 2, 3 \in X = U$ and $\phi \subseteq \phi = V$ and $X \cap \phi = \phi$, so the definition satisfy.

Take $F = \{2, 3\}$ and $1 \notin F \Rightarrow$ the only open set that contains F is X, but $X \cap U \neq \phi$, so the definition not satisfy. $\therefore$ (X, τ) is not R – space.

<u>Example :</u> Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$. Is (X, τ) R – space.

<u>Solution :</u> First, find the family of closed sets :

$$\mathscr{F} = \{X, \phi, \{2, 3\}, \{1\}\} = \tau$$

and X, ϕ in previous example satisfy the definition (in general X, ϕ satisfy the definition in every example).

Take $F = \{1\}$ closed set ; $2, 3 \notin F \Rightarrow \exists\ U = \{2, 3\}$ and $V = \{1\} = F$; $U, V \in \tau$

$\Rightarrow\ U \cap V = \phi\ ;\ 2, 3 \in U\ \wedge\ F \subseteq V$ (the definition satisfy)

Take $F = \{2, 3\}$ closed set ; $1 \notin F \Rightarrow \exists\ U = \{1\}$ and $V = \{2, 3\} = F$; $U, V \in \tau$

$\Rightarrow\ U \cap V = \phi\ ;\ 1 \in U\ \wedge\ F \subseteq V$ (the definition satisfy)

$\therefore$ (X, τ) is R – space.

Remark : In previous example notes that X is not T_0 – space, not T_1 – space, and not T_2 –space, so the R –space is not necessarily T_0 –space or T_1 –space or T_2 – space. i.e.,

$$(R \nRightarrow T_0 \ \wedge \ R \nRightarrow T_1 \ \wedge \ R \nRightarrow T_2)$$

Remark : If $\tau = \mathcal{F}$ in any topological space (X, τ), then its R – space, since :

If F closed set in X and $x \notin F \Rightarrow \exists\ U = F$ and $V = F^c$; $F \subseteq F = U$, $x \in V = F^c$, $F \cap F^c = \phi$. So the definition of R – space satisfy.

From this remark we have (X, I) and (X, D) are R – space.

Example : Is cofinite topology $(\mathbb{N}, \tau_{cof})$ R – space.

Solution : No. Since $\nexists$ two nonempty disjoint open sets satisfy the definition satisfy of R – space, for example :

If $X = \mathbb{N}$ and $F = \{1, 2, 3\}$ and $x = 4$, then $x \notin F$ and if we assume there exists U, V $\in \tau_{cof}$ and $U \cap V = \phi$, then

$(U \cap V)^c = \phi^c \ \Rightarrow \ U^c \cup V^c = X$

finite finite finite C!! contradiction

Theorem : The space (X, τ) is regular (R – space) iff for each $x \in X$ and each open set W containing x, there exists an open set U such that $x \in U \subseteq \overline{U} \subseteq W$.

Proof : $(\Rightarrow)$ Suppose that X is R – space.

Let $x \in X$, $W \in \tau$; $x \in W \Rightarrow x \notin X - W$ and $X - W \in \mathcal{F}$

$\because$ X is R – space $\Rightarrow \exists\ U, V \in \tau$; $U \cap V = \phi$, $(x \in U \wedge X - W \subseteq V)$

$\because\ U \cap V = \phi \quad \Rightarrow\ U \subseteq X - V$

We have, $U \subseteq X - V$ and $X - V \subseteq W$

$\Rightarrow\ \overline{U} \subseteq \overline{X - V}$ (since $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$)

$\Rightarrow\ \overline{U} \subseteq X - V$ (since $X - V$ closed $\Rightarrow X - V = \overline{X - V}$)

$\Rightarrow\ \overline{U} \subseteq X - V \ \wedge \ X - V \subseteq W$

$\Rightarrow\ \overline{U} \subseteq W$

$\Rightarrow x \in U \subseteq \overline{U} \subseteq W$ (since $A \subseteq \overline{A}$)

$(\Leftarrow)$ Suppose the condition of theorem satisfy, to prove X is R – space

Let $x \in X$ and F closed set in X ; $x \notin F$

$\Rightarrow x \in X - F \in \tau$ (since F closed)

$\Rightarrow \exists\ U \in \tau$; $x \in U \subseteq \overline{U} \subseteq X - F$ (by hypothesis)

$\Rightarrow\ \overline{U} \subseteq X - F$

$\Rightarrow\ F \subseteq X - \overline{U}$ (since $A \subseteq B \Leftrightarrow B^c \subseteq A^c$)

But, $X - \overline{U}$ open since $\overline{U}$ closed, say $X - \overline{U} = V$

$\Rightarrow x \in U \wedge F \subseteq V \wedge U \bigcap V = \phi$, (since $U \subseteq \overline{U}$ and $\overline{U} \bigcap X - \overline{U} = \phi \Rightarrow U \bigcap V = \phi$)

$\therefore$ X is R – space.

Theorem : Every metric space is R – space.

Proof : Let (X, d) be a metric space $\Rightarrow$ (X, τ_d) the topology derivative from this metric. We will prove this theorem by using previous theorem :

Let $x \in X$ and W open set ; $x \in W$

To prove $\exists$ U open set ; $x \in U \subseteq \overline{U} \subseteq W$ or $x \in U \subseteq Cl(U) \subseteq W$

$\because$ $x \in W \Rightarrow \exists$ open ball of x contains in W (since $W \in \tau_d$)

$\Rightarrow \exists\, p \in \mathbb{R}^+$; $N(x, p) \subseteq W$

(since the set is open $\Leftrightarrow$ contains open nbhd for every element)

Take $q \in \mathbb{R}^+$; $0 < q < p$ $\Rightarrow N(x, q) \subseteq N(x, p)$

(since the first half ball similar than second half ball and the center is unique)

$\Rightarrow N(x, q) \subseteq Cl(N(x, q)) \subseteq N(x, p)$ (since $q < p$)

since $N(x, q) = \{y \in X ; d(x, y) < q\}$ and $Cl(N(x, q)) = \{y \in X ; d(x, y) \leq q\}$

$\Rightarrow N(x, q) \subseteq Cl(N(x, q)) \subseteq W$ (since $N(x, p) \subseteq W$ and $Cl(N(x, q) \subseteq N(x, p))$

$\because$ every open ball in metric space is open set $\Rightarrow$ N(x, q) open set, say $U = N(x, q)$

$\Rightarrow x \in U \subseteq Cl(U) \subseteq W$

$\therefore$ (X, d) is R – space.

Remark : There is a method to prove previous theorem by using definition of R – space, but this prove is shorter.

Remark : Since $(\mathbb{R}, \tau_u)$ is metric space, then $(\mathbb{R}, \tau_u)$ is R – space.

Theorem : The property of being a R – space is a hereditary property.

Proof : Let (X, τ) R – space and (W, τ_W) subspace of X, to prove (W, τ_W) R – space

Let $x \in W$ and E closed set in W ; $x \notin E$

$\Rightarrow x \in X$ (since $W \subseteq X$) $\wedge\ \exists\, F \in \mathcal{F}$; $E = F \bigcap W$ (i.e., F closed in X)

$\because$ X is R – space $\Rightarrow \exists\, U, V \in \tau$; $U \bigcap V = \phi$, $(x \in U \wedge F \subseteq V)$

$\Rightarrow$ $U \bigcap W \wedge V \bigcap W \in \tau_W$, (by def of τ_W)

$(U \bigcap W) \bigcap (V \bigcap W) = (U \bigcap V) \bigcap W = \phi \bigcap W = \phi$

and $(x \in U \bigcap W$ (since $x \in U \wedge x \in W$)

$\wedge$ $E \subseteq V \bigcap W$). (since $E = F \bigcap W \Rightarrow E \subseteq F \wedge E \subseteq W$

$\Rightarrow$ $E \subseteq V \wedge E \subseteq W$

$\Rightarrow$ $E \subseteq V \bigcap W$

$\therefore$ (W, τ_W) is R – space.

Theorem : The property of being a R – space is a topological property.

Proof : Let $(X, \tau) \cong (Y, \tau')$ and suppose that X is R – space, to prove Y is R – space

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f open

Let $y \in Y$ and $F \in \mathcal{F}'$ i.e., F closed in Y ; $y \notin F$

$\because$ f onto function $\Rightarrow \exists x \in X ; f(x) = y$

$\because$ f continuous $\Rightarrow f^{-1}(F) \in \mathcal{F}$ i.e., $f^{-1}(F)$ closed in X ; $x \notin f^{-1}(F)$ (since $f(x) = y \notin F$)

$\because$ X is R – space $\Rightarrow \exists U, V \in \tau ; U \cap V = \phi , (x \in U \wedge f^{-1}(F) \subseteq V)$

$\because$ f open $\Rightarrow f(U), f(V) \in \tau'$

$\because$ f is 1-1 $\wedge$ onto $\Rightarrow f(x) \in f(U) \wedge f(f^{-1}(F) \subseteq f(V)$

$\Rightarrow y \in f(U) \wedge F \subseteq f(V)$ (since $y = f(x) \wedge f(f^{-1}(F)) = F$)

$\because U \cap V = \phi \Rightarrow f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$

$\therefore$ X is R – space.

By similar way we prove, if Y is R – space, then X is R – space.

Remark : The continuous image of R – space is not necessarily R – space. i.e., if $f : (X, \tau) \to (Y, \tau')$ is continuous onto function and X is R – space, then Y not necessarily R – space and the following example show this :

Example : Let $f : (\mathbb{R}, D) \to (\mathbb{R}, \tau_{cof})$; $f(x) = x \quad \forall x \in \mathbb{R}$.

f is continuous function since the domain $(\mathbb{R}, D)$ is discrete topology (see page 37) and clear f is onto and in general (X, D) is R – space (i.e., $(\mathbb{R}, D)$ is R – space), but in general (X, τ_{cof}) is not R – space (see page 95) (i.e., $(\mathbb{R}, \tau_{cof})$ is not R – space).

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a R – space iff each X and Y are R – space.

Proof : ($\Leftarrow$) Suppose that X and Y are R – space, to prove $X \times Y$ is R – space

Let $(x, y) \in X \times Y$ and A closed set in $X \times Y$; $(x, y) \notin A$

$\Rightarrow \exists$ F closed set in X and F' closed set in Y ; $F \times F' \subseteq A$ and $(x, y) \notin F \times F'$

$\Rightarrow x \notin F \in \mathcal{F} \wedge y \notin F' \in \mathcal{F}'$

$\because$ X is a R – space $\Rightarrow \exists U, V \in \tau ; U \cap V = \phi , (x \in U \wedge F \subseteq V)$

$\because$ Y is a R – space $\Rightarrow \exists U', V' \in \tau' ; U' \cap V' = \phi , (y \in U' \wedge F' \subseteq V')$

$\Rightarrow$

$U \times Y, V \times Y \in \tau_{X \times Y}$; $(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \phi \times Y = \phi$,

$((x, y) \in U \times Y \wedge F \times F' \subseteq V \times Y)$

Or

$X\times U', X\times V' \in \tau_{X\times Y}$; $(X\times U') \bigcap (X\times V') = X \times (U' \bigcap V') = X \times \phi = \phi$,

$$((x, y) \in X\times U' \wedge F\times F' \subseteq X\times V')$$

In both cases we have $X\times Y$ is R – space.

(⇒) Suppose that $X\times Y$ is R – space, to prove X and Y are R – space

Let $x \in X$ and $F \in \mathscr{F}$; $x \notin F$ and $y \in Y$ and $F' \in \mathscr{F}'$; $y \notin F'$

$$\Rightarrow (x, y) \in X\times Y \text{ and } F\times F' \in \mathscr{F}_{X\times Y} ; (x, y) \notin F\times F'$$

$\because$ $X\times Y$ is a R – space $\Rightarrow \exists\ U\times V, U'\times V' \in \tau_{X\times Y}$; $(U\times V) \bigcap (U'\times V') = \phi$,

$$((x, y) \in U\times V \wedge F\times F' \subseteq \mathscr{F}\times\mathscr{F}')$$

$\Rightarrow \exists\ U, U' \in \tau$; $U \bigcap U' = \phi$, $(x \in U \wedge F \subseteq U')$ $\Rightarrow$ X is R – space

and $\exists\ V, V' \in \tau'$; $V \bigcap V' = \phi$, $(y \in V \wedge F' \subseteq V')$ $\Rightarrow$ Y is R – space.

Remark : Notes that : $(T_0 \not\Rightarrow R \ \wedge\ T_1 \not\Rightarrow R \ \wedge\ T_2 \not\Rightarrow R)$

Example : Let X = {1, 2, 3} and τ = {X, ϕ, {1}, {2}, {1, 2}} ; (X, τ) topological space.

Solution : Clear (X, τ) is not R – space (see page 94)

On the other hand, (X, τ) is T_0 – space (Check that !!)

So, we have T_0 – space, but not R – space.

Example : Take cofinite topology ($\mathbb{N}$, τ_{cof}).

Solution : Clear ($\mathbb{N}$, τ_{cof}) not R – space (see page 95).

On the other hand, ($\mathbb{N}$, τ_{cof}) is T_1 – space (see page 81).

So, we have T_1 – space, but not R – space.

Definition : T_3 – Space

Let (X, τ) be a topological space. Then the space (X, τ) is called a **T_3 – Space** iff its regular and T_1 – space. i.e.,

$$T_3 \text{ – space} = T_1 \text{ – space} + R \text{ – space}$$

Example : The space (X, D) is T_3 – space, since its T_1 – space and R – space.

Example : The space (X, I) ; X contains more than one element is not T_3 – space, since its not T_1 – space and R – space.

Example : In the cofinite topology (X, τ_{cof}), if X is infinite set, then its not T_3 – space, since its T_1 – space and not R – space.

Example : Let X = {1, 2, 3} and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$. The space (X, τ) is not T_3 – space, since its not T_1 – space and R – space.

Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_3 – space, since its T_1 – space and R – space.

Theorem : Every metric space is T_3 – space.
Proof : Since every metric space is T_1 – space and R – space.

Theorem : The property of being a T_3 – space is a hereditary property.
Proof : Since the property T_1 – space and R – space are a hereditary property.
Then T_3 – space is a hereditary property.

Theorem : The property of being a T_3 – space is a topological property.
Proof : Since the property T_1 – space and R – space are a topological property.
Then T_3 – space is a topological property.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_3 – space iff each X and Y are T_3 – space.
Proof : We prove in previous theorems : That the product space $X \times Y$ is T_1 – space and R –space iff each X and Y is T_1 –space (see page 84) and R –space (see page 97). Hence, we have the product space $X \times Y$ is a T_3 – space iff each X and Y are T_3 – space.

Remark : The continuous image of T_3 – space is not necessarily T_3 – space. i.e., if $f : (X, \tau) \to (Y, \tau')$ is continuous onto function and X is T_3 – space, then Y not necessarily T_3 – space and the following example show this :

Example : Let $f : (\mathbb{R}, D) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall\, x \in \mathbb{R}$.
f is continuous function since the domain $(\mathbb{R}, D)$ is discrete topology (see page 37) and clear f is onto and in general (X, D) is T_3 – space (i.e., $(\mathbb{R}, D)$ is T_3 – space), but (X, I) is not T_3 – space (see page 98).

Theorem : If (X, τ) is a T_3 – space, then X is a T_2 – space.

Proof : Suppose that X is a T_3 – space (i.e., ,T_1 – space and R – space), to prove X is T_2 – space.

Let $x, y \in X$; $x \neq y$

$\because$ X is T_1 – space $\Rightarrow \{x\}, \{y\} \in \mathscr{F}$ (by theorem, X is T_1-space $\Leftrightarrow \{x\}$ closed $\forall\, x \in X$)

$\Rightarrow x \notin \{y\}$ (since $x \neq y$)

$\because$ X is R – space $\Rightarrow \exists\, U, V \in \tau$; $U \bigcap V = \phi$, $(x \in U \wedge \{y\} \subseteq V)$

$\Rightarrow x \in U \wedge y \in V$

$\therefore$ (X, τ) is a T_2 – space.

In previous theorem we take $x \notin \{y\}$ and by the similar way we can take $y \notin \{x\}$ and we have the same result.

Remark : From above theorem we have :

$$T_3\text{ – space} \Rightarrow T_2\text{ – space} \Rightarrow T_1\text{ – space} \Rightarrow T_0\text{ – space}$$
$$\not\Leftarrow \qquad \not\Leftarrow \qquad \not\Leftarrow$$

Remark : There is another method to express on the above theorem as follows :

If (X, τ) is R – space and every singleton set in X is closed, then X is T_2 – space.

Example : Let $X = \mathbb{N}$ and $\tau = \{U \subseteq X ; 1 \in U\} \bigcup \{\phi\}$. Is $(\mathbb{N}, \tau)$ T_3 – space ??

Solution : We test $(\mathbb{N}, \tau)$ is T_1 – space ?? and R – space ??

Let $x, y \in \mathbb{N}$; $x \neq y$, to find open set containing x but not y, and open set containing y but not x.

Suppose $x = 1$, then for any $y \in \mathbb{N}$ such that $x \neq y$ there is no open set contains y but not x (since definition τ is every open set must contains 1 if it's not empty set), so X not T_1 – space. Furthermore, X is not R – space.

Definition : Normal Space

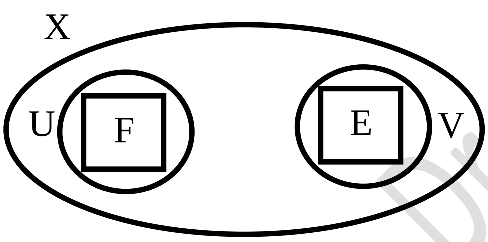
Let (X, τ) be a topological space. Then the space (X, τ) is called a **Normal Space** (denoted by N – space) iff for each pair of closed disjoint subsets F and E of X, there exist open sets U and V such that $F \subseteq U$, $E \subseteq V$, and $U \cap V = \phi$. i.e.,

$$X \text{ is N – Space} \Leftrightarrow \forall\, F, E \in \mathscr{F} ;\ F \cap E = \phi\ \ \exists\, U, V \in \tau ;\ U \cap V = \phi ,\ (F \subseteq U \wedge E \subseteq V)$$

If (X, τ) is not N – space, we define,

$$X \text{ is not N – Space} \Leftrightarrow \exists\, F, E \in \mathscr{F} ;\ F \cap E = \phi\ \ \forall\, U, V \in \tau ;\ U \cap V = \phi ,$$
$$(F \not\subseteq U \wedge E \not\subseteq V) \vee (F \not\subseteq V \wedge E \not\subseteq U) .$$

The following figure show the definition of N – space :



Example : Let X = {1, 2, 3} and τ = {X, ϕ, {1}, {2}, {1, 2}}. Is (X, τ) N – space.

Solution : First, find the family of closed sets :

$$\mathscr{F} = \{X, \phi, \{2, 3\}, \{1, 3\}, \{3\}\}$$

Second, take every two closed sets there intersection is empty :

Notes that any two closed sets their intersection is nonempty, since all closed sets contains 3 except ϕ. Therefore, the definition of N – space is satisfy.

We can prove this as follows :

$$[\forall\, F, E \in \mathscr{F} ;\ F \cap E = \phi] \Rightarrow [\ \exists\, U, V \in \tau ;\ U \cap V = \phi ,\ (F \subseteq U \wedge E \subseteq V)]$$

False statement either false statement or true statement

In two cases we have $(F \Rightarrow F = T)$ and $(F \Rightarrow T = T)$, therefore the definition of N – space is satisfy.

Remark : In the previous example notes that (X, τ) is not R – space and its N – space so that :

$$(\text{N – space} \not\Rightarrow \text{R – space})$$

Also, in this example (X, τ) is not T_1 – space and not T_2 – space so that :

$$(\text{N – space} \not\Rightarrow T_1\text{ – space}) \wedge (\text{N – space} \not\Rightarrow T_2\text{ – space})$$

Furthermore,

$$(\text{R – space} \not\Rightarrow \text{N – space}) \wedge (T_1\text{ – space} \not\Rightarrow \text{N – space}) \wedge (T_2\text{ – space} \not\Rightarrow \text{N – space})$$

Remark : $(T_0\text{ – space} \not\Rightarrow \text{N – space}) \wedge (\text{N – space} \not\Rightarrow T_0\text{ – space})$

Example : The space ($\mathbb{N}$, τ_{cof}) is T_0 – space and not N – space, since there is two nonempty disjoint closed sets, but there is no two nonempty disjoint open sets.
Notes that too ($\mathbb{N}$, τ_{cof}) is T_1 – space and not N – space.

Example : The space ($\mathbb{R}$, I) is not T_0 – space, since $\mathbb{R}$ is the only open set contains elements and its contains all elements. But ($\mathbb{R}$, I) is N – space since the closed sets are $F = \mathbb{R}$ and $E = \phi$ only, and $\mathbb{R} \bigcap \phi = \phi$ and the open sets are $\mathbb{R}$ and ϕ and $\mathbb{R} \subseteq \mathbb{R}$ and $\phi \subseteq \phi$.

Example : The space (X, D) is N – space, since every sets her is open and closed then: If $F, E \in \mathcal{F}$; $F \bigcap E = \phi$, then $F, E \in \tau$; ($F \subseteq F \wedge E \subseteq E$).

Remark : If (X, τ) is topological space and $\tau = \mathcal{F}$ (i.e., every closed set is open) or $U \in \tau \iff U \in \mathcal{F}$, then (X, τ) is N – space.
The spaces (X, I) and (X, D) is special case from this spaces.
We can use this remark to have infinite numbers of N – space simply by take any set X and make topology τ on X as follows : $\tau = \{X, \phi, A, A^c\}$; $A \subseteq X$.

Example : Let X = {1, 2, 3} and $\tau = \{X, \phi, \{1\}\}$. Show that (X, τ) is N – space.
Solution : First, find the family of closed sets :

$$\mathcal{F} = \{X, \phi, \{2, 3\}\}$$

Second, take every two closed sets there intersection is empty as follows :
Take, $\phi, X \in \mathcal{F}$; $\phi \bigcap X = \phi \Rightarrow \exists\, U = \phi \wedge V = X \in \tau$; $U \bigcap V = \phi$, ($\phi \subseteq U \wedge X \subseteq V$).
Take, $\phi, \{2, 3\} \in \mathcal{F}$; $\phi \bigcap \{2, 3\} = \phi \Rightarrow \exists\, U = \phi \wedge V = X \in \tau$; $U \bigcap V = \phi$, ($\phi \subseteq U \wedge \{2, 3\} \subseteq V$).
$\therefore$ (X, τ) is N – space.
Notes that this space not T_0 – space, not T_1 – space, not T_2 – space, not R – space, and not T_3 – space.

Remark : The continuous image of N – space is not necessarily N – space. i.e., if f : (X, τ) $\to$ (Y, τ') is continuous onto function and X is N – space, then Y not necessarily N – space and the following example show this :

Example : Let f : (ℕ, D) → (ℕ, τ_{cof}) ; f(x) = x ∀ x ∈ ℕ.
f is continuous function since the domain (ℕ, D) is discrete topology (see page 37) and clear f is onto and (ℕ, D) is N – space, but (ℕ, τ_{cof}) is not N – space.

Remark : The property of being a N – space is not a hereditary property and the following example show that :

Example : Let X = {1, 2, 3, 4, 5} and τ = {X, ϕ, {1, 4, 5}, {1, 3, 5}, {1, 5}, {1, 3, 4, 5}}. Clear that (X, τ) is N – space, since : the family of closed sets :

$$\mathcal{F} = \{X, \phi, \{2, 3\}, \{2, 4\}, \{2, 3, 4\}, \{2\}\}$$

Notes that any two closed sets except ϕ contains 2, so there is no nonempty disjoint closed sets and the sets are disjoint are ϕ and any other closed. Also, there is disjoint open sets to solve this case which are ϕ and X.
Now, take the subspace W ⊆ X ; W = {3, 4, 5} such that :

$$\tau_W = \{ W \cap U ; U \in \tau\} = \{W, \phi, \{4, 5\}, \{5\}, \{3, 5\}\}$$

$$\text{and, } \mathcal{F}_W = \{W, \phi, \{3\}, \{4\}, \{3, 4\}\}$$

notes that : {3} and {4} closed sets in W and {3} ⋂ {4} = ϕ. But, there is no disjoint open sets in τ_W such that one of them contains {3} and the other contain {4}, so that W is not N – space, while X is N – space.

Theorem : The property of being a N – space is a topological property.
Proof : Let (X, τ) ≅ (Y, τ') and suppose that X is N – space, to prove Y is N – space
∵ (X, τ) ≅ (Y, τ') ⇒ ∃ f : (X, τ) → (Y, τ') ; f 1-1, f onto, f continuous, f open
Let F, E ∈ $\mathcal{F}$ ' ; F ⋂ E = ϕ
∵ f continuous ⇒ $f^{-1}(F), f^{-1}(E) \in \mathcal{F}$ and $f^{-1}(F) \cap f^{-1}(E) = f^{-1}(F \cap E) = f^{-1}(\phi) = \phi$
(by theorem : the function f is continuous ⇔ the inverse image of every closed set in codomaun is closed in domain)
∵ X is N – space ⇒ ∃ U, V ∈ τ ; U ⋂ V = ϕ , ($f^{-1}(F) \subseteq U \wedge f^{-1}(E) \subseteq V$)
∵ f open ⇒ f(U), f(V) ∈ τ'
∵ f is onto ⇒ $f(f^{-1}(F)) \subseteq f(U) \wedge f(f^{-1}(E)) \subseteq f(V)$
⇒ F ⊆ f(U) ∧ E ⊆ f(V) (since $f(f^{-1}(F)) = F \wedge f(f^{-1}(E)) = E$)
∵ U ⋂ V = ϕ ⇒ f(U) ⋂ f(V) = f(U ⋂ V) = f(ϕ) = ϕ
∴ Y is N – space.
By similar way we prove, if Y is N – space, then X is N – space.

Theorem : The space (X, τ) is normal (N – space) iff for each closed subset $F \subseteq X$ and open set W containing F (i.e., $F \subseteq W$), there exists an open set U such that $F \subseteq U \subseteq \overline{U} \subseteq W$.

Proof : $(\Rightarrow)$ Suppose that X is N – space and $F \subseteq X$; $F \in \mathcal{F}$.

Let $W \in \tau$; $F \subseteq W \Rightarrow F \cap X - W = \phi \wedge X - W \in \mathcal{F}$ (since $W \in \tau$)

$\because$ X is N – space $\Rightarrow \exists\, U, V \in \tau$; $U \cap V = \phi$, $(\underline{\underline{F \subseteq U}} \wedge X - W \subseteq V)$

$\Rightarrow \underline{\underline{X - V \subseteq W}}$ (since $A \subseteq B \Rightarrow B^c \subseteq A^c$)

$\because U \cap V = \phi \Rightarrow U \subseteq X - V$

$\Rightarrow \overline{U} \subseteq \overline{X - V}$ (since $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$)

$\Rightarrow \overline{U} \subseteq X - V$ (since $X - V$ closed $\Rightarrow X - V = \overline{X - V}$)

$\Rightarrow \underline{\underline{U \subseteq \overline{U} \subseteq X - V}}$ (since $A \subseteq \overline{A}$)

$\Rightarrow F \subseteq U \wedge U \subseteq \overline{U} \subseteq X - V \wedge X - V \subseteq W$

$\Rightarrow F \subseteq U \subseteq \overline{U} \subseteq W$

$(\Leftarrow)$ Suppose the condition of theorem satisfy, to prove X is N – space

Let $F, E \in \mathcal{F}$; $F \cap E = \phi \Rightarrow F \subseteq X - E \in \tau$ (since $E \in \mathcal{F}$).

$\Rightarrow \exists\, U \in \tau$; $F \subseteq U \subseteq \overline{U} \subseteq X - E$ (by hypothesis)

$\Rightarrow \overline{U} \subseteq X - E$

$\Rightarrow E \subseteq X - \overline{U}$ (since $A \subseteq B \Leftrightarrow B^c \subseteq A^c$)

But, $X - \overline{U}$ open since $\overline{U}$ closed, say $X - \overline{U} = V$

$\Rightarrow E \subseteq V = X - \overline{U} \wedge F \subseteq U$

$\wedge\, U \cap V = \phi$, (since $U \subseteq \overline{U}$ and $\overline{U} \cap X - \overline{U} = \phi \Rightarrow U \cap V = \phi$)

$\therefore$ X is N – space.

Remark : Let (X, τ) and (Y, τ') be two topological spaces. If the product space $X \times Y$ is a N – space, then each X and Y are N – space.

But, the conversely is not true in general i.e.,. If each X and Y are N – space, then not necessary that the product space $X \times Y$ is a N – space.

Remark : Every metric space is N – space. Therefore, since $(\mathbb{R}, \tau_u)$ is metric space, then its N – space.

Theorem : A **closed** subspace of N – space is N – space.

Proof :

Let (X, τ) N – space and (W, τ_W) closed subspace of X, to prove (W, τ_W) N – space

Let F_W, E_W are closed sets in W ; $F_W \cap E_W = \phi$

$\because$ F_W, E_W are closed sets in X and $F_W = F_W \cap W \;\wedge\; E_W = E_W \cap W$,
then $F_W \cap E_W = \phi$ in X
$\because$ X is N – space $\Rightarrow \exists\, U, V \in \tau \,;\, U \cap V = \phi$, $(F \subseteq U \wedge E \subseteq V)$

$$\Rightarrow U \cap W \;\wedge\; V \cap W \in \tau_W , \qquad \text{(by def. of } \tau_W)$$

$$(U \cap W) \cap (V \cap W) = (U \cap V) \cap W = \phi \cap W = \phi ,$$

since $F_W = F \cap W \Rightarrow F_W \subseteq F \;\wedge\; F_W \subseteq W \;\Rightarrow\; F_W \subseteq U \wedge F_W \subseteq W \Rightarrow F_W \subseteq U \cap W$
since $E_W = E \cap W \Rightarrow E_W \subseteq E \;\wedge\; E_W \subseteq W \;\Rightarrow\; E_W \subseteq V \wedge E_W \subseteq W \Rightarrow E_W \subseteq V \cap W$
$\therefore$ (W, τ_W) is N – space.

Definition : T_4 – Space
Let (X, τ) be a topological space. Then the space (X, τ) is called a **T_4 – Space** iff its normal and T_1 – space. i.e.,

$$T_4 \text{ – space} = T_1 \text{ – space} + N \text{ – space}$$

Example : Let X = {1, 2, 3} and τ = {X, ϕ, {1}, {2, 3}}. Then the space (X, τ) is not T_4 – space, since its N – space but not T_1 – space.

Remark : If X is finite space, then (X, τ) is T_4 – space iff τ = D, (because if X is finite space, then its T_1 – space iff τ = D and if τ = D, then X is N – space).

Example : The space (X, D) is T_4 – space, since its T_1 – space and N – space.

Example : The space (X, I) ; X contains more than one element is not T_4 – space, since its not T_1 – space.

Example : The space $(\mathbb{N}, \tau_{cof})$ is not T_4 – space, since its T_1 – space but not N – space.

Remark : The property of being a T_4 – space is not a hereditary property, since the normality is not a hereditary property.

Theorem : The property of being a T_4 – space is a topological property.
Proof : Since the property T_1 – space and N – space are a topological property. Then T_4 – space is a topological property.

Theorem : A **closed** subspace of T_4 – space is T_4 – space.

Proof : Let (X, τ) T_4 – space and W closed set in X, to prove W is T_4 – space

$\because$ X is T_1 – space $\Rightarrow$ W is T_1 – space (since T_1 is hereditary property)

$\because$ W is closed in X and X is N – space $\Rightarrow$ W is N – space

(by theorem : A closed subspace of N – space is N – space)

$\therefore$ W is T_4 – space.

Remark : Let (X, τ) and (Y, τ') be a topological spaces. If the product space $X \times Y$ is T_4 – space, then each X and Y is T_4 – space.
But, the conversely is not true in general i.e.,. If each X and Y is T_4 – space, then not necessary that the product space $X \times Y$ is T_4 – space.

Remark : Every metric space is T_4 – space. Since every metric space is T_1 – space and N – space.

Theorem : Every T_4 – space is R – space.

Proof : Let (X, τ) be T_4 – space $\Rightarrow$ X is T_1 – space and N – space

Let $x \in X$ and F closed set in X ; $x \notin F$

$\Rightarrow \{x\} \in \mathscr{F}$ (since X is T_1 – space $\Leftrightarrow$ $\{x\}$ closed $\forall\, x \in X$)

$\Rightarrow \{x\} \cap F = \phi$ (since $x \notin F$)

$\because$ X is N – space $\Rightarrow \exists\, U, V \in \tau$; $U \cap V = \phi$, $(\{x\} \subseteq U \wedge F \subseteq V)$

$\Rightarrow x \in U \wedge F \subseteq V$

$\therefore$ X is R – space.

Corollary : Every T_4 – space is T_3 – space.

Proof : Every T_4 – space is R – space (by the above theorem)

Every T_4 – space is T_1 – space and N – space (by def of T_4 – space)

We have, X is T_1 – space R – space

$\therefore$ X is T_3 – space.

Remark : Every T_4 – space is T_2 – space since every T_4 – space is T_3 – space and every T_3 – space is T_2 – space so that :

$$T_4 - \text{space} \Rightarrow T_3 - \text{space} \Rightarrow T_2 - \text{space} \Rightarrow T_1 - \text{space} \Rightarrow T_0 - \text{space}$$

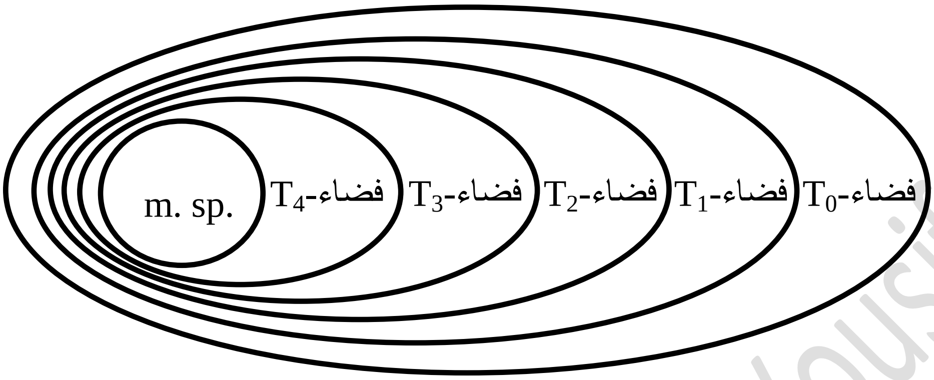
$$\not\Leftarrow \qquad\qquad \not\Leftarrow \qquad\qquad \not\Leftarrow \qquad\qquad \not\Leftarrow$$

Notes that : N – space $\not\Rightarrow$ R – space and N – space $\not\Rightarrow$ T_1 – space,

but N – space + T_1 – space $\Rightarrow$ T_3 – space

and N – space + T_1 – space $\Rightarrow$ R – space

Remark : the following figure show that the relation between separation axioms T_0, T_1, T_2, T_3, T_4, and metric space and usual topology $(\mathbb{R}, \tau_u)$:



In previous illustrate we notes that a T_2 – space not necessarily R – space. Also, if X is T_2 – space, then its not necessarily N – space, but this statements are satisfy if we add the condition that compaceness space and the following theorems show this :

Theorem : Every compact T_2 – space is R – space.

Proof : Let (X, τ) be T_2 – space and compact, to prove X is R – space.

Let $x \in X$, $F \in \mathcal{F}$; $x \notin F \Rightarrow x \neq y \ \forall \ y \in F$

$\because$ X is T_2 – space $\Rightarrow \exists \ U_y, V_y \in \tau$; $U_y \cap V_y = \phi$, $(x \in U_y \wedge y \in V_y)$.

We have two family of open sets are $\{U_y\}_{y\in F}$ and $\{V_y\}_{y\in F}$ such that every element in F exists in one element of the family $\{V_y\}_{y\in F}$ and every element in the family $\{U_y\}_{y\in F}$ contains the element x and every U_y corresponding V_y such that $U_y \cap V_y = \phi$. Therefore, the family $\{V_y\}_{y\in F}$ open cover for F

$$\Rightarrow \{V_y\}_{y\in F} \text{ open cover of F, i.e., } F \subseteq \bigcup_{y\in F} V_y$$

$\because$ F closed in the compact space (by hypothesis), so that F compact space and we have : $\Rightarrow \exists \ y_1, y_2, \dots, y_n$; $F \subseteq \bigcup_{i=1}^{n} V_{y_i}$

Therefore, $\{V_{y_i}\}_{i=1}^{n}$ is a finite family of open sets cover F (let $V = \bigcup_{i=1}^{n} V_{y_i}$),

On the other hand, $\{U_{y_i}\}_{i=1}^{n}$ is a finite family of open sets and every element in this family contains x (let $U = \bigcap_{i=1}^{n} U_{y_i}$)

$\Rightarrow$ U and V are open sets (by second and third condition of def of top.)

Such that $x \in U$ and $F \subseteq V$

Notes that, $U \cap V = \phi$ (since $U = \bigcap_{i=1}^{n} U_{y_i} \Rightarrow U \subseteq U_{yi} \ \forall$ i and $U_{yi} \cap V_{yi} = \phi$)

$\therefore$ X is R – space.

Remark : There is a theorem similar the above theorem and their prove is similar too and we introduce this theorem but without prove.
((In T_2 – space we can separated any point x and compact subset not contains x by disjoint open sets))

Corollary : Every compact T_2 – space is T_3 – space.
Proof : Every T_2 – space is T_1 – space
Every T_2 – space and compact is R – space (by the above theorem)
We have, X is T_1 – space and R – space
$\therefore$ X is T_3 – space.

Theorem : Every compact T_2 – space is N – space.
Proof : Let (X, τ) be T_2 – space and compact, to prove X is N – space.
Let $F, E \in \mathscr{F}$; $F \bigcap E = \phi \Rightarrow F, E$ are compact.
(by theorem : Every closed set in compact space is compact)
Choose, $x \in F \Rightarrow x \notin E \Rightarrow \exists U_x, V_E \in \tau$; $U_x \bigcap V_E = \phi$, $(x \in U_x \wedge E \subseteq V_E)$.
(by previous remark, since X is T_2 – space and E compact set and $x \notin E$)
Now, repeated this method on every element in F, we have a family of open sets cover F as follows :
$\{U_x ; x \in F \wedge U_x \in \tau\} \Rightarrow F \subseteq \bigcup_{x\in F} U_x$
$\because$ F is compact set $\Rightarrow \exists x_1, x_2, \dots, x_n$; $F \subseteq \bigcup_{i=1}^{n} U_{x_i}$
Also, we have a family of open sets every elements in this family contain E as follows :
$\{V_i ; i = 1, 2, \dots, n \wedge E \subseteq V_i \ \forall \ i \wedge V_i \in \tau\}$
$V_i \bigcap U_{xi} = \phi \ \ \forall \ i$; $i = 1, 2, \dots, n$
Say, $U = \bigcup_{i=1}^{n} U_{x_i} \Rightarrow F \subseteq U \in \tau$ (by third condition of def. of top.)
Say, $V = \bigcap_{i=1}^{n} V_i \Rightarrow E \subseteq V \in \tau$ (by second condition of def of top.)
Notes that, $U \bigcap V = \phi$ (since $[U = \bigcup_{i=1}^{n} U_{xi}] \bigcap [V = \bigcap_{i=1}^{n} V_i] = \phi$)
$\therefore$ X is N – space.