Chapter3: Numerical Solution of Ordinary Differential Equations

3.1 Numerical Solution of a First-Order ODE

A numerical solution of a first order ODE formulated as

$$\frac{dy}{dx} = f(x, y) \text{ with the initial condition } y(x_1) = y_1 \qquad (3.1)$$

is a set of discrete points that approximate the function y(x). When a differential equation is solved numerically, the problem statement also includes the domain of the solution. For example, a solution is required for values of the independent variable from $x = a$ to $x = b$ (the domain is [a, b]). Depending on the numerical method used to solve the equation, the number of points between a and b at which the solution is obtained can be set in advance, or it can be decided by the method. For example, the domain can be divided into N subintervals of equal width defined by N + 1 values of the independent variable from $x_1 = a$ to $x_{N+1} = b$. The solution consists of values of the dependent variable that are determined at each value of the independent variable. The solution then is a set of points $(x_1, y_1), (x_2, y_2), \ldots, (x_{N+1}, Y_{N+1})$ that define the function y(x) .

3.1.1 Overview of Numerical Methods Used/or Solving a First-Order ODE

Numerical solution is a procedure for calculating an estimate of the exact solution at a set of discrete points. The solution process is incremental, which means that it is determined in steps. It starts at the point where the initial value is given. Then, using the known solution at the first point, a solution is determined at a second nearby point. This is followed by a solution at a third point, and so on.

There are procedures with a single-step and multistep approach. In a **single-step approach**, the solution at the next point, x_{i+1}, is calculated from the already known solution at the present point, x_i. In a **multi-step approach**, the solution at x_{i+1} is calculated from the known solutions at several previous points. The idea is that the value of the function at several previous points can give a better estimate for the trend of the solution.

Also, two types of methods, explicit, and implicit, can be used for calculating the solution at each step. The difference between the methods is in the way that the solution is calculated at each step. Calculating the value of the dependent variable at the next value of the independent variable. In an **explicit formula**, the right-hand side of the equation only has known quantities. In other words, the next unknown value of the dependent variable, y_{i+1}, is calculated by evaluating an expression of the form:

$$y_{i+1} = F(x_i, x_{i+1}, y_i) \qquad (3.2)$$

where x_i, y_i, and x_{i+1} are all known quantities. In **implicit methods**, the equation used for calculating $y_{i+1}$from the known x_i, y_i, and x_{i+1} has the form:

$$y_{i+1} = F(x_i, x_{i+1}, y_{i+1}) \qquad (3.3)$$

Here, the unknown y_{i+1} appears on both sides of the equation.

3.1.2 Errors in Numerical Solution of ODEs

Two types of errors, round-off errors and truncation errors, occur when ODEs are solved numerically. Round-off errors are due to the way that computers carry out calculations. **Truncation errors** are due to the approximate nature of the method used to calculate the solution. Since the numerical solution of a differential equation is calculated in increments (steps), the truncation error at each step of the solution consists of two parts. One, called **local truncation error**, is due to the application of the numerical method in a single step. The second part, called **propagated, or accumulated, truncation error**, is due to the accumulation of local truncation errors from previous steps. Together, the two parts are the **global (total) truncation error** in the solution.

3.1.3 Single-step explicit methods

In a single-step explicit method, illustrated in Fig. 3-1,

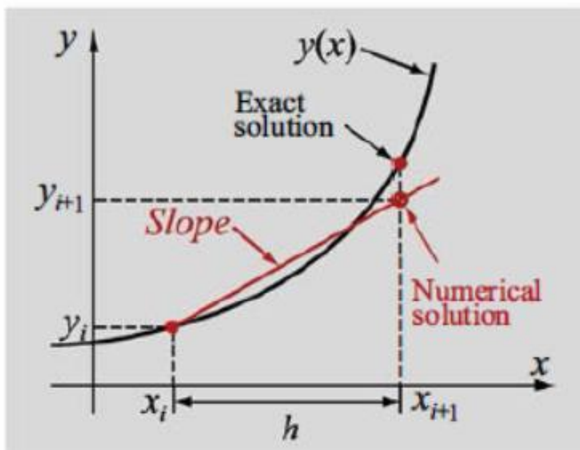

Figure 3-1: Single-step explicit methods.

The approximate numerical solution (x_{i+1}, y_{i+1}) is calculated from the known solution at point (x_i, y_i) by:

$$x_{i+1} = x_i + h \qquad (3.4)$$

$$y_{i+1} = y_{i+1} + \text{Slope}\cdot h \qquad (3.5)$$

where h is the step size, and the Slope is a constant that estimates the value of $\frac{dy}{dx}$ in the interval from $x_i$to x_{i+1}. The numerical solution starts at the point where the initial value is known. This corresponds to $i = 1$ and point (x_1, y_1). Then i is increased to $i = 2$, and the solution at the next point, (x_2, y_2), is calculated by using Eqs. (3.4) and (3.5). The procedure continues with $i = 3$ and so on until the points cover the whole domain of the solution.

3.2 EULER'S METHODS

Euler's method is the simplest technique for solving a first-order ODE
of the form of Eq. (3.1):

$$\frac{dy}{dx} = f(x, y)\ with\ the\ initial\ condition\ y(x_1) = y_1$$

The method can be formulated as an explicit or an implicit method.

3.2.1 Euler's Explicit Method

Euler's explicit method (also called the forward Euler method) is a single-step, numerical technique for solving a first-order ODE. The method uses Eqs. (3.4) and (3.5), where the value of the constant Slope in Eq. (3.5) is the slope of *y(x)* at point $(x_i,\ y_i)$. This slope is actually calculated from the differential equation:

$$Slope = \frac{dy}{dx}\Big|_{x=x_i} = f(x_i, y_i) \qquad (3.6)$$

Euler's method assumes that for a short distance h near (x_i, y_i), the function *y(x)* has a constant slope equal to the slope at (x_i, y_i). With this assumption, the next point of the numerical solution (x_{i+1}, y_{i+1}) is calculated by:

$$x_{i+1} = x_i + h \qquad (3.7)$$

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad (3.8)$$

Equation (3.8) of Euler's method can be derived in several ways. Starting with the given differential equation:

$$\frac{dy}{dx} = f(x, y) \qquad (3.9)$$

An approximate solution of Eq. (3.9) can be obtained either by numerically integrating the equation or by using a finite difference approximation for the derivative.

3.2.1.1 Deriving Euler's method by using finite difference approximation for the derivative

Euler's formula, Eq. (3.8), can be derived by using an approximation for the derivative in the differential equation. The derivative $\frac{dy}{dx}$ in Eq. (3.8) can be approximated with the forward difference formula by evaluating the ODE at the point $x = x_i$:

$$\frac{dy}{dx}\Big|_{x=x_i} \approx \frac{y_{i+1}-y_i}{x_{i+1}-x_i} = f(x_i, y_i) \qquad (3.10)$$

Solving Eq. (3.10) for y_{i+1} gives Eq. (3.8) of Euler's method. (Because the equation can be derived in this way, the method is also known as the **forward Euler method**.)

Example 3-1: Use Euler's explicit method to solve the ODE

$$\frac{dy}{dx} = -1.2y + 7e^{-0.3x}$$

from $x = 0$ to $x = 2.5$ with the initial condition $y = 3$ at $x = 0$.

(a) Solve by hand using $h = 0.5$.

(b) Write a MATLAB program in a script file that solves the equation using $h = 0.5$.

(c) Use the program from part (b) to solve the equation using h = 0.1.

In each part compare the results with the exact (analytical) solution:

$$y(x) = \frac{70}{9}e^{-0.3x} - \frac{43}{9}e^{-1.2x}$$

Solution:

(a) Solution by hand: The first point of the solution is (0, 3), which is the point where the initial condition is given. For the first point $i = 1$. The values of x and y are $x_1 = 0$ and $y_1 = 3$. The rest of the solution is determined by using Eqs. (3.7) and (3.8). In the present problem these equations have the form:

$$x_{i+1} = x_i + h = x_i + 0.5 \qquad (3.11)$$
$$y_{i+1} = y_i + f(x_i, y_i)h = y_i + (-1.2y_i + 7e^{-0.3x_i})0.5 \qquad (3.12)$$

Equations (3.11) and (3.12) are applied five times with i = 1, 2, 3, 4, and 5.

First step: For the first step $i = 1$. Equations (3.11) and (3.12) give:

$$x_2 = x_1 + h = 0 + 0.5 = 0.5$$
$$y_2 = y_1 + (-1.2y_1 + 7e^{-0.3x_1})0.5 = 4.7$$

The second point is (0.5, 4.7).

Second step: For the second step $i = 2$. Equations (3.11) and (3.12) give:

$$x_3 = x_2 + h = 0.5 + 0.5 = 1$$
$$y_3 = y_2 + (-1.2y_2 + 7e^{-0.3x_2})0.5 = 4.8924779$$

The third point is (1, 4.8924779).

Third step: For the third step $i = 3$. Equations (3.11) and (3.12) give:

$$x_4 = x_3 + h = 1 + 0.5 = 1.5$$
$$y_4 = y_3 + (-1.2y_3 + 7e^{-0.3x_3})0.5 = 4.5498549$$

The fourth point is (1.5, 4.5498549).

Fourth step: For the fourth step $i = 4$. Equations (3.11) and (3.12) give:

$$x_5 = x_4 + h = 1.5 + 0.5 = 2$$
$$y_5 = y_4 + (-1.2y_4 + 7e^{-0.3x_4})0.5 = 4.0516405$$

The fifth point is (2, 4.0516405).

Fifth step: For the fourth step $i = 5$. Equations (3.11) and (3.12) give:

$$x_6 = x_5 + h = 2 + 0.5 = 2.5$$
$$y_6 = y_5 + (-1.2y_5 + 7e^{-0.3x_5})0.5 = 3.5414969$$

The sixth point is (2.5, 3.5414969).

The values of the exact and numerical solutions, and the error, which is the difference between the two, are:

i	x_i	y_i numerical	$y(x_i)$ exact	Error
1	0	3.0000000	3.0000000	0
2	0.5000	4.7000000	4.0722953	0.6277047
3	1.0000	4.8924779	4.3228804	0.5695975
4	1.5000	4.5498549	4.1695687	0.3802862
5	2.0000	4.0516405	3.8351047	0.2165358
6	2.5000	3.5414969	3.4360905	0.1054064

(b) To solve the ODE with MATLAB:

```
function d=euler(f,y1,a,b,n)
h=(b-a)/n;x(1)=a;y(1)=y1;
for k=1:n
    x(k+1)=x(k)+h;
    y(k+1)=y(k)+h*f(x(k),y(k));
end
d=[x' y']
```

3.2.2 Analysis of Truncation Error in Euler's Explicit Method

As mentioned in Section 3.1.2, when ODEs are solved numerically there are two sources of error, round-off and truncation. The round-off errors are due to the way that computers carry out calculations. The truncation error is due to the approximate nature of the method used for calculating the solution in each increment (step). In addition, since the numerical solution of a differential equation is calculated in increments (steps), the truncation error consists of a local truncation error and propagated truncation error. The truncation errors in Euler's explicit method are discussed in this section.

The discussion is divided into two parts. First, the **local truncation error** is analyzed, and then the results are used for determining an estimate of the **global truncation error**.

Definition 3.1: Assume that $\{(x_k,y_k),k=1,\dots,N\}$ is the set of discrete approximations and that $y=y(x)$ is the unique solution to the initial value problem. The ***global discretization error*** $\boldsymbol{e_k}$ is defined by:

$$e_k=y(x_k)-y_k \quad \text{for } k=1,\dots,N \tag{3.13}$$

The local discretization error $\in_{k+1}$ is defined by:

$$\in_{k+1}=y(x_{k+1})-y_k-h\emptyset(x_k,y_k) \quad \text{for } k=1,\dots,N-1 \tag{3.14}$$

for some function $\emptyset$ called an increment function.

Theorem 3.1: (Precision of Euler's Method)

Assume that $y(x)$ is the solution to the IVP given in (3.1).If $y(x)\in C^2[t_0,b]$ and $\{(x_k,y_k),k=1,\dots,N\}$ is the sequence of approximations generated by Euler's method, then:

$$|e_k|=|y(x_k)-y_k|=O(h) \tag{3.15}$$

$$|\in_{k+1}|=|y(x_{k+1})-y_k-hf(x_k,y_k)|=O(h^2) \tag{3.16}$$

The error at the end of the interval is called the ***final global error (FGE)***:

$$E(y(b),h)=|y(b)-y_M|=O(h) \tag{3.17}$$

3.2.3 Euler's Implicit Method

The form of Euler's implicit method is the same as the explicit scheme, except, for a short distance, h, near (x_i,y_i) the slope of the function $y(x)$ is taken to be a constant equal to the slope at the endpoint of the interval (x_{i+1},y_{i+1}). With this assumption, the next point of the numerical solution (x_{i+1},y_{i+1}) is calculated by:

$$x_{i+1}=x_i+h \tag{3.18}$$

$$y_{i+1}=y_i+f(x_{i+1},y_{i+1})h \tag{3.19}$$

Now, the unknown y_{i+1} appears on both sides of Eq. (3.19), and unless $f(x_{i+1}, y_{i+1})$depends on y_{i+1} in a simple linear or quadratic form, it is not easy or even possible to solve the equation for y_{i+1} explicitly.

3.3 MODIFIED EULER'S METHOD

The modified Euler method is a single-step, explicit, numerical technique for solving a first-order ODE. The method is a modification of Euler's explicit method. (This method is sometimes called **Heun's method**). As discussed in Section 3.2.1, the main assumption in Euler's explicit method is that in each subinterval (step) the derivative (slope) between points (x_i, y_i) and $(x_{i+1}, y_{i+1})$is constant and equal to the derivative (slope) of *y(x)* at point (x_i, y_i). This assumption is the main source of error. In the modified Euler method the slope used for calculating the value of y_{i+1} is modified to include the effect that the slope changes within the subinterval. The slope used in the modified Euler method is the average of the slope at the beginning of the interval and an estimate of the slope at the end of the interval. The slope at the beginning is given by:

$$\frac{dy}{dx}|_{x=x_i} = f(x_i, y_i) \qquad (3.20)$$

The estimate of the slope at the end of the interval is determined by first calculating an approximate value for y_{i+1} written as y_{i+1}^{Eu}using Euler's explicit method:

$$y_{i+1}^{Eu} = y_i + f(x_i, y_i) \qquad (3.21)$$

and then estimating the slope at the end of the interval by substituting the point $\left(x_{i+1}, y_{i+1}^{Eu}\right)$ in the equation for $\frac{dy}{dx}$:

$$\frac{dy}{dx}\Big|_{\substack{x=x_{i+1} \\ y=y_{i+1}^{Eu}}} = f(x_{i+1}, y_{i+1}^{Eu}) \qquad (3.22)$$

The modified Euler method is summarized in the following algorithm.

<u>Algorithm for the modified Euler method</u>

1. Given a solution at point (x_i, y_i), calculate the next value of the independent variable:

$$x_{i+1} = x_i + h$$

2. Calculate$f(x_i, y_i)$.
3. Estimate y_{i+1} using Euler's method:

$$y_{i+1}^{Eu} = y_i + f(x_i, y_i)$$

4. Calculate (x_{i+1}, y_{i+1}^{Eu}) .
5. Calculate the numerical solution at $x = x_{i+1}$:

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{Eu})\right]$$

Example 10-2: Use the modified Euler method to solve the ODE

$$\frac{dy}{dx} = -1.2y + 7e^{-0.3x}$$

from *x=0* to *x* = 2.5 with the initial condition *y(0) = 3*. Using *h = 0.5*. Compare the results with the exact (analytical) solution:

$y(x) = \frac{70}{9}e^{-0.3x} - \frac{43}{9}e^{-1.2x}$.

Solution:

The first point of the solution is (0, 3), which is the point where the initial condition is given. For the first point $i = 1$. The values of x and y are $x_1 = 0$ and $y_1 = 3$.

In the present problem these equations have the form:

$$x_{i+1} = x_i + h = x_i + 0.5$$
$$y_{i+1}^{Eu} = y_i + f(x_i, y_i)h = y_i + (-1.2y_i + 7e^{-0.3x_i})0.5$$
$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{Eu})\right] = y_i + \frac{0.5}{2}\left[(-1.2y_i + 7e^{-0.3x_i}) + \left(-1.2y_{i+1}^{Eu} + 7e^{-0.3x_{i+1}}\right)\right]$$

First step: For the first step $i = 1$:

$$x_2 = x_1 + h = 0 + 0.5 = 0.5$$
$$y_2^{Eu} = y_1 + (-1.2y_1 + 7e^{-0.3x_1})0.5 = 4.7$$
$$y_i + \frac{0.5}{2}[(-1.2y_1 + 7e^{-0.3x_1}) + (-1.2y_2^{Eu} + 7e^{-0.3x_2})] = 3.946238958743852$$

The second point is (0.5, 3.946238958743852).

The values of the exact and numerical solutions, and the error, which is the difference between the two, are:

i	x_i	y_i numerical	$y(x_i)$ *exact*	Error
1	0	3.0000000	3.0000000	0
2	0.5000	3.946238958743852	4.0722953	0.126056374335137
3	1.0000	4.187746065761980	4.3228804	0.135134415959749
4	1.5000	4.063314737957255	4.1695687	0.106253975375624
5	2.0000	3.763482617314995	3.8351047	0.071622108811351
6	2.5000	3.393629530605291	3.4360905	0.042460997400584

Comparing the error values here with those in Example 3-1, where the problem was solved with Euler's explicit method using the same size subintervals, shows that the error with the modified Euler method is much smaller.

3.4 RUNGE-KUTTA METHODS

Runge-Kutta methods are a family of single-step, explicit, numerical techniques for solving a first-order ODE. As was stated in Section 3.1, for a subinterval (step) defined by $[x_i, x_{i+1}]$, where h = x_{i+1}-x_i, the value of $y_{i+1}$is calculated by:

$$y_{i+1} = y_i + slop.h \qquad (3.23)$$

where Slope is a constant. The value of Slope in Eq. (3.23) is obtained by considering the slope at several points within the subinterval. Various types of Runge-Kutta methods are classified according to their order. The order identifies the number of points within the sub interval that are used for determining the value of Slope in Eq. (3.23). Second order Runge-Kutta methods use the slope at two points, third-order methods use three points, and so on. The so-called classical Runge-Kutta method is of fourth order and uses four points. The order of the method is also related to the global truncation error of each method. For example, the second-order Runge-Kutta method is second-order accurate globally; that is, it has a local truncation error of $O(h^3)$ and a global truncation error of $O(h^2)$.

3.4.1 Second-Order Runge-Kutta Methods

The general form of second-order Runge-Kutta methods is:

$$\left.\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(k_1 + k_2) \\ k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + h, y_i + k_1 h) \end{aligned}\right\} \qquad (3.24)$$

Example 3-3: Solving by hand a first-order ODE using the second-order Runge-Kutta method to solve the ODE

$$\frac{dy}{dx} = -1.2y + 7e^{-0.3x}$$

from $x=0$ to $x = 2.5$ with the initial condition $y(0) = 3$. Using $h = 0.5$. Compare the results with the exact (analytical) solution:

$y(x) = \frac{70}{9}e^{-0.3x} - \frac{43}{9}e^{-1.2x}$.

Solution:

The first point of the solution is (0, 3), which is the point where the initial condition is given. For the first point $i = 1$. The values of x and y are $x_1 = 0$ and $y_1 = 3$.

The rest of the solution is done by steps. In each step the next value of the independent variable is given by:

$$x_{i+1} = x_i + h = x_i + 0.5 \qquad (3.25)$$

The value of the dependent variable y_{i+1} is calculated by first calculating k_1 and k_2 using :

$$\left.\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + h, y_i + k_1 h) \end{aligned}\right\} \qquad (3.26)$$

and then substituting the k's in :

$$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2) \qquad (3.27)$$

First step: In the first step $i = 1$. Equations (3. 25)-(3. 27) give:

$x_2 = x_1 + 0.5 = 0.5$

$k_1 = f(x_1, y_1) = f(0,3) = -1.2(3) + 7e^{-0.3(0)} = 3.4$

$k_2 = f(x_1 + h, y_1 + k_1 h) = f\big(0 + 0.5, 3 + 3.4(0.5)\big) = f(0.5, 1.7)$

$= -1.2(1.7) + 7e^{-0.3(0.5)} = 0.384955834975405$

$y_2 = y_1 + \frac{h}{2}(k_1 + k_2) = 3 + \frac{0.5}{2}(3.4 + 0.384955834975405) = 3.946238958743852$

Second step: In the first step $i = 2$. Equations (3. 25)-(3. 27) give:

$x_3 = x_2 + 0.5 = 1.0$

$k_1 = f(x_2, y_2) = f(0.5, 3.946238958743852)$

$= -1.2(3.946238958743852) + 7e^{-0.3(0.5)} = 1.289469084482783$

$k_2 = f(x_2 + h, y_2 + k_1 h)$

$= f\big(0.5 + 0.5, 3.946238958743852 + 1.289469084482783(0.5)\big)$

$= -0.323440656410266$

$y_3 = y_2 + \frac{h}{2}(k_1 + k_2)$ =4.187746065761980

Third step:

k1 = 0.160432265857648

k2 = -0.658157577076552

y_4 =4.063314737957255

Fourth step:

k1 = -0.412580624196292

k2 = -0.786747858372744

$y_5 =$ 3.763482617314995

Fifth step:

k1 = -0.674497688119808

k2 = -0.804914658719007

$y_6 =$ 3.393629530605291

The values of the exact and numerical solutions, and the error, which is the difference between the two, are:

i	x_i	y_i numerical	*y*(x_i) *exact*	Error
1	0	3.0000000	3.0000000	0
2	0.5000	3.946238958743852	4.0722953	0.126056374335137
3	1.0000	4.187746065761980	4.3228804	0.135134415959749
4	1.5000	4.063314737957255	4.1695687	0.106253975375624
5	2.0000	3.763482617314995	3.8351047	0.071622108811351
6	2.5000	3.393629530605291	3.4360905	0.042460997400584

The solution obtained is obviously identical (except for rounding errors) to the solution in example 3-2.

3.4.2 Fourth-Order Runge-Kutta Methods

The general form of classical fourth-order Runge-Kutta method is:

$$\left.\begin{aligned} & y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ & with \\ & \qquad k_1 = f(x_i, y_i) \\ & \qquad k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2}\right) \\ & \qquad k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{hk_2}{2}\right) \\ & \qquad k_4 = f(x_i + h, y_i + hk_3) \end{aligned}\right\} \qquad (3.28)$$

Example 3-4: Solving by hand a first-order ODE using the fourth-order Runge-Kutta method to solve the ODE

$$\frac{dy}{dx} = -1.2y + 7e^{-0.3x}$$

from *x=0* to *x* = 2.5 with the initial condition *y(0)* = *3*. Using *h* = *0.5*. Compare the results with the exact (analytical) solution:

$y(x) = \frac{70}{9}e^{-0.3x} - \frac{43}{9}e^{-1.2x}$.

Solution:

First step:

$k_1 = f(x_1, y_1) = f(0,3)$ =3.40

$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{hk_1}{2}\right)$ =1.874204404299870

$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{hk_2}{2}\right)$ =2.331943083009909

$k_4 = f(x_1 + h, y_1 + hk_3)$ = 1.025789985169459

$y_2 = y_2 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ =4.069840413315752

Second step:

k1 = 1.141147338996503
k2 = 0.363460833637786
k3 = 0.596766785245403
k4 = -0.056141022354118
y_3 =4.320295542849815

Third step:

k1 =0.001372893352247
k2 =-0.373741567888647
k3 =-0.261207229516379
k4 = -0.564233252357536
y_4 =4.167565713365203

Fourth step:

k1 = -0.537681794685830
k2 =-0.698886767064788
k3 =-0.650525275351102
k4 =-0.769082238169397
y_5 =3.833766703557953

Fifth step:

k1 =-0.758838591611358
k2 =-0.808773522533291
k3 =-0.793793043256712
k4 =-0.817678349128413
y_6 =3.435295864197971

The values of the exact and numerical solutions, and the error, which is the difference between the two, are:

i	x_i	y_i numerical	$y(x_i)$ *exact*	Error
1	0	3.000000000000000	3.0000000	0
2	0.5000	4.069840413315752	4.0722953	0.002454919763237
3	1.0000	4.320295542849815	4.3228804	0.002584938871915
4	1.5000	4.167565713365203	4.1695687	0.002002999967676
5	2.0000	3.833766703557953	3.8351047	0.001338022568394
6	2.5000	3.435295864197971	3.4360905	0.000794663807904

3.5 Predictor-Corrector Methods

Predictor-corrector methods refer to a family of schemes for solving ordinary differential equations using two formulae: ***predictor* and *corrector* formula**. In predictor-corrector methods, four prior values are required to find the value of y at x_n. Predictor-corrector methods have the advantage of giving an estimate of error from successive approximations to y_n. The predictor is an explicit formula and is used first to determine an estimate of the solution y_{n+1}. The value y_{n+1} is calculated from the known solution at the previous point (x_n, y_n) using single-step method or several previous points (multi-step methods). If x_n and x_{n+1} are two consecutive mesh points such that :

$$x_{i+1} = x_i + h$$

then in Euler's method, we have:

$$y_{i+1} = y_i + h\, f(x_i, y_i),\ i = 0, 1, 2, 3, \ldots \qquad (3.29)$$

Once an estimate of y_{i+1} is found, the corrector is applied. The corrector uses the estimated value of y_{i+1} on the right-hand side of an otherwise implicit formula for computing a new, more accurate value for y_{n+1} on the left-hand side. The modified Euler's method gives as:

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \qquad (3.30)$$

The value of y_{i+1} is first estimated by Eq.(3.29) and then utilized in the right-hand side of Eq.(3.30) resulting in a better approximation of y_{i+1}. The value of y_{i+1} thus obtained is again substituted in Eq.(3.30) to find a still better approximation of y_{i+1}. This procedure is repeated until two consecutive iterated values of y_{i+1} are very close. Here, the corrector equation (3.30) which is an implicit equation is being used in an *explicit* manner since no solution of a non-linear equation is required.

In addition, the application of corrector can be repeated several times such that the new value of y_{i+1} is substituted back on the right-hand side of the corrector formula to obtain a more refined value for y_{i+1}. The technique of refining an initially crude estimate of y_{i+1} by means of a more accurate formula is known as ***predictor-corrector* method**. Equation (2.29) is called the ***predictor*** and Eq. (3.30) is called the ***corrector*** of y_{n+1}.

Example 3.5: Use the PC method on (2, 3) with $h = 0.1$ for the initial value problem

$$\frac{dy}{dx} = -xy^2,\ y(2) = 1.$$

Exact solution is $y(x) = \dfrac{2}{x^2 - 2}$.

<u>Solution:</u>

First, we use Euler method:

$y_1 = y_0 + hf(x_0, y_0)$=1+0.1(-2(1)^2)=0.8

Then, we use modified Euler:

$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$=1+0.1/2*[-2*1^2+(-2.1)*(0.8)^2]=0.8328

Containing in the same manner, we obtain:

x_i	y_i	$Y(x_i)$
2	1.000000000000000	1.000000000000000
2.1	0.832800000000000	0.829875518672199
2.2	0.708036878443888	0.704225352112676
2.3	0.611802381778826	0.607902735562310
2.4	0.535592749372665	0.531914893617021
2.5	0.473938067466517	0.470588235294118
2.6	0.423170282558423	0.420168067226891
2.7	0.380742913556783	0.378071833648393
2.8	0.344835715939071	0.342465753424658
2.9	0.314114751637895	0.312012480499220
3.0	0.287581256501905	0.285714285714286

Example 3.6: Approximate the *y* value at *x* = 0.4 of the following differential equation:

$$\frac{dy}{dx} = \frac{1}{2}y,\ y(0) = 1 \text{ and } 0 \le x \le 1.$$

using the PC method with h=0.1.
Solution:

x_i	y_i
0	1.000000000000000
0.1	1.051250000000000
0.2	1.105126562500000
0.3	1.161764298828125
0.4	1.221304719143066

3.6 Higher-Order Differential Equations:

Higher-order differential equations involve the higher derivatives x''(t), x'''(t), and so on. They arise in mathematical models for problems in physics and engineering. By solving for the second derivative, we can write a second-order initial value problem in the form:

$$x''(t)=f(t,x(t),x'(t)) \text{ with } x(t_0)=x_0 \text{ and } x'(t_0)=y_0 \qquad (3.31)$$

The second-order differential equation can be reformulated as a system of two first-order equations if we use the substitution:

$$x'(t)=y(t) \qquad (3.32)$$

Then x''(t)=y'(t) and the differential equation in (3.31) becomes a system:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = f(t,x,y) \qquad with \begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases} \tag{3.33}$$

A numerical procedure such as Rung-Kutta method can be used to solve (3.33) and will generate two sequences $\{x_k\}$ and $\{y_k\}$. The first sequence is the numerical solution to (3.31).

Now, consider RK2 for the system of two differential equation :

$x'(t)=f(t,x,y)$

$y'(t)=g(t,x,y)$

as follows:

$x_{k+1}=x_k+1/2(k_1+k_2)$, $y_{k+1}=y_k+1/2(p_1+p_2)$

where $k_1=hf(t_k,x_k,y_k)$, $p_1=hg(t_k,x_k,y_k)$

and $k_2=hf(t_k+h,x_k+k_1,y_k+p_1)$, $p_2=hg(t_k+h,x_k+k_1,y_k+p_1)$.

Example 3.7: Consider the second-order IVP

$$x''(t)+4x'(t)+5x(t)=0 \quad \text{with } x(0)=3 \text{ and } x'(0)=-5$$

(a) Write down the equivalent system of two first-order equation.
(b) Use The RK2 method to solve the reformulated problem over [0,1] using M=5.
(c) Compare the numerical solution with the true solution $x(t)=3e^{-2t}\cos(t)+e^{-2t}\sin(t)$.

First assume $x'(t)=y(t)$ then $x''(t)=y'(t)$ and we have:

$x'(t)=y(t)$

$y'(t)=-4y(t)-5x(t)$ with $x(0)=3$ and $y(0)=-5$, then $h=(1-0)/5=0.2$

t_k	x_k	$x(t_k)$
0	3	3
0.2		
0.4		
0.6		
0.8		
1		

Exercises:

Solve the system x'=3x-y, y'=4x-y with x(0)=0.2 and y(0)=0.5 using RK2 with h=0.5 in [0,1].

3.7 Boundary Value Problems:

Another type of differential equation has the form:

$$x''=f(t,x,x') \quad \text{for } a\leq t\leq b \tag{3.34}$$

with the boundary conditions

$$x(a)=\alpha \quad \text{and } x(b)=\beta \tag{3.35}$$

This is called ***a boundary value problem (BVP)***.

Finite-difference Method:

Methods involving difference quotient approximations for derivatives can be used for solving second-order BVP. Consider the linear equation:

$$x''=p(t)x'(t)+q(t)x(t)+r(t) \tag{3.36}$$

over [a,b] with $x(a)=\alpha$ and $x(b)=\beta$. Form a partition of [a,b] using the points $a=t_0<t_1<\ldots<t_N=b$, where $h=(b-a)/N$ and $t_j=a+jh$ for $j=0,1,\ldots N$. The central-difference formulas discussed in chapter two are used to approximate the derivatives:

$$x'(t_j)=\frac{x(t_{j+1})-x(t_{j-1})}{2h}+O(h^2) \tag{3.37}$$

$$x''(t_j)=\frac{x(t_{j+1})-2x(t_j)-x(t_{j-1})}{h^2}+O(h^2) \tag{3.38}$$

To start derivation, we replace each term $x(t_j)$ on the right side of (3.37) and (3.38) with x_j and the resulting equations are substituted into (3.36), to obtain the relation:

$$\frac{x_{j+1}-2x_j+x_{j-1}}{h^2}=p_j\left(\frac{x_{j+1}-x_{j-1}}{2h}\right)+q_jx_j+r_j \tag{3.39}$$

which is used to compute numerical approximation to the differential equation(3.36). This is carried out by multiplying each side of (3.39) by h^2 and then collecting terms involving x_{j-1}, x_j and x_{j+1} and arranging them in a system of linear equations:

$$\left(\frac{-h}{2}p_j - 1\right)x_{j-1} + \left(2 + h^2 q_j\right)x_j + \left(\frac{h}{2}p_j - 1\right)x_{j+1} = -h^2 r_j \qquad (3.40)$$

for j=1,2,…,N-1, where $x_0 = \alpha \ and \ x_N = \beta$.

Example 3.8 Solve the boundary value problem

$$x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1$$

with x(0)=1.25 and x(4)=-0.95 over the interval [0,4] with h=1.

since h=1 we get N=4 and $t_0=0$, $t_1=1$, $t_2=2$, $t_3=3$ and $t_4=4$

In the same way:

$$\frac{x_{j+1} - 2x_j + x_{j-1}}{h^2} = \frac{2t_j}{1+t_j^2}\left(\frac{x_{j+1} - x_{j-1}}{2h}\right) - \frac{2}{1+t_j^2}x_j + 1$$

then, we get:

$$\left(-\frac{h}{2}\frac{2t_j}{1+t_j^2} - 1\right)x_{j-1} + \left(2 - \frac{2h^2}{1+t_j^2}\right)x_j + \left(\frac{h}{2}\frac{2t_j}{1+t_j^2} - 1\right)x_{j+1} = -h^2$$

$$\left(-\frac{ht_j}{1+t_j^2} - 1\right)x_{j-1} + \left(2 - \frac{2h^2}{1+t_j^2}\right)x_j + \left(\frac{ht_j}{1+t_j^2} - 1\right)x_{j+1} = -h^2$$

for j=1,2,3 and $x_0=1.25$, $x_4=-0.95$

so for j=1, we get

$$\left(-\frac{ht_1}{1+t_1^2} - 1\right)x_0 + \left(2 - \frac{2h^2}{1+t_1^2}\right)x_1 + \left(\frac{ht_1}{1+t_1^2} - 1\right)x_2 = -h^2$$

for j=2

$$\left(-\frac{ht_2}{1+t_2^2} - 1\right)x_1 + \left(2 - \frac{2h^2}{1+t_2^2}\right)x_2 + \left(\frac{ht_2}{1+t_2^2} - 1\right)x_3 = -h^2$$

and for j=3

$$\left(-\frac{ht_3}{1+t_3^2} - 1\right)x_2 + \left(2 - \frac{2h^2}{1+t_3^2}\right)x_3 + \left(\frac{ht_3}{1+t_3^2} - 1\right)x_4 = -h^2$$

therefore, we hence the algebraic system of three equations

$$\left.\begin{array}{c}\left(2-\frac{2}{1+1}\right)x_1+\left(\frac{1}{1+1}-1\right)x_2=-1-\left(-\frac{1}{1+1}-1\right)(1.25)\\ \left(-\frac{2}{1+4}-1\right)x_1+\left(2-\frac{2}{1+4}\right)x_2+\left(\frac{2}{1+4}-1\right)x_3=-1\\ \left(-\frac{3}{1+9}-1\right)x_2+\left(2-\frac{2}{1+9}\right)x_3=-1-\left(\frac{3}{1+9}-1\right)(-0.95)\end{array}\right\}$$

$$\left.\begin{array}{c}x_1-\frac{1}{2}x_2=-1+\frac{3}{2}(1.25)\\ -\frac{7}{5}x_1+\frac{8}{5}x_2-\frac{3}{5}x_3=-1\\ -\frac{13}{10}x_2+\frac{18}{10}x_3=-1+\frac{7}{10}(-0.95)\end{array}\right\}$$

then after solving this system, we obtain:

x_1=0.52143, x_2-0.70714and x_3=-1.4357

Problems:

1. Consider the following first-order ODE:

$$\frac{dy}{dx}=x^2/y \; from\; x=0\; to\; x=2.1\; with\; y(0)=2$$

(a) Solve with Euler's explicit method using h = 0.7.
(b) Solve with the modified Euler method using h = 0.7.
(c) Solve with the classical fourth-order Runge-Kutta method using h = 0.7.

The analytical solution of the ODE is $y=\sqrt{\frac{2x^3}{3}+4}$. In each part, calculate the error between the true solution and the numerical solution at the points where the numerical solution is determined.

2. Write the following second-order ODE as a system of two first-order ODEs:

$$\frac{d^2y}{dt^2}+5\left(\frac{dy}{dt}\right)^2-6y+e^{sint}=0$$

3. Consider the following second-order ODE:

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=2xy \quad for\; 0\le x\le 1\,, with\; y(0)=1\; and\; y(1)=1$$

Using the difference formulas for approximating the derivatives, discretize the ODE (rewrite the equation in a form suitable for solution with the finite difference method).