Bivariate Discrete Random Variables

Def: A discrete bivariate random variable (X, Y) is an ordered pair of discrete random variables

Def: Let (X, Y) be a bivariate random variable and Let R_X and R_Y be the range spaces of X and Y, respectively. A real-valued function $f : R_X \times R_Y \longrightarrow \mathbb{R}$ is called a joint probability density function for X and Y if and only if

$$f(x, y) = P(X = x, Y = y)$$

for all $(x, y) \in R_X \times R_Y$.

Here, the event $(X = x, Y = y)$ means the intersection of the events $(X = x)$ and $(Y = y)$ that is

$$(X = x) \cap (Y = y)$$

ex: Roll a pair of unbiased dice. If X denotes the smaller and Y denotes the larger outcome on the dice then what is the joint probability density function of X and Y?

sol: The sample space S is

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

The probability density function $f(x,y)$ can be computed for $X=2$ and $Y=3$ as follows:

There are two outcomes namely $(2,3)$ and $(3,2)$ in the sample space S of 36 outcomes which contribute to the joint $(X=2, Y=3)$.

$$\therefore f(2,3) = P(X=2, Y=3) = \frac{2}{36}$$

	1	2	3	4	5	6
6	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
5	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0
4	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0
3	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0
2	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0
1	$\frac{1}{36}$	0	0	0	0	0
	1	2	3	4	5	6

$$\therefore f(x,y) = \begin{cases} \frac{1}{36} & \text{if } 1 \leq x = y \leq 6 \\ \frac{2}{36} & \text{if } 1 \leq X < Y \leq 6 \\ 0 & \text{otherwise.} \end{cases}$$

The Joint probability mass functⁿ for Two Discrete random Variables

Def: Let X and Y be two discrete r.v's defined on the sample space S of an experiment.
The joint probability mass functⁿ $p(x,y)$ is defined for each pair of numbers (x,y) by

$$p(x,y) = P(X=x \text{ and } Y=y)$$
$$= P(X=x) \cap P(Y=y).$$

Let $(x,y) \in A$, then the probability that the random pair (X,Y) lies in A is obtained by summing the joint p.m.f over pairs in A:

$$P[(X,Y)\in A] = \sum_{(x,y)\in A}\sum p(x,y)$$

The joint p.m.f satisfy the following properties.

① $0 \leq p(x,y) \leq 1$ for all pairs (x,y)

② $\sum_{\text{all } x}\sum_{\text{all } y} p(x,y) = 1$

A joint p.m.f of X and Y can be represented by a joint prob. table

X \ Y	y_1	y_2	- - -	- - -	y_n	
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$			$f(x_1,y_n)$	$f_1(x_1)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$			$f(x_2,y_n)$	$f_1(x_2)$
⋮					⋮	⋮
⋮					⋮	⋮
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$			$f(x_m,y_n)$	$f_1(x_m)$
Totals →	$f_2(y_1)$	$f_2(y_2)$	- - -	- - -	$f_2(y_n)$	1 ← Totals

$$P(X=x_i)=f_1(x_i)=\sum_{j=1}^{n} f(x_i,y_j) \quad \text{for } j=1,2,\dots,n$$

$$P(Y=y_j)=f_2(y_j)=\sum_{i=1}^{m} f(x_i,y_j) \quad \text{for } i=1,2,\dots,m$$

t the table below represent the joint p.m.f
c r.v's X and Y

	0	1	2 (Y)	
	0.12	0.42	0.06	0.6
	0.21	0.06	0.03	0.3
	0.07	0.02	0.01	0.1
	0.4	0.5	0.1	1

$P(X=1, y=2)=0.03$

$P(X=2, y=0)=0.07$

$, y=1) = 0.42$

Marginal p.m.f for r.v.s X and Y

Def: Let (X,Y) be a discrete joint random variable.

Let R_X and R_Y be the range of X and Y, respectively.

Let $f(x,y)$ be the joint p.m.f of X and Y.

The function $f_1(x) = \sum_{y \in R_Y} f(x,y)$

is called the marginal probability density function of X.

Similarly, the function

$$f_2(y) = \sum_{x \in R_X} f(x,y)$$

is called the marginal probability density function of Y.

ex: If the joint probability density function of the discrete random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{36} & \text{if } 1 \le x = y \le 6 \\ \frac{2}{36} & \text{if } 1 \le x < y \le 6 \\ 0 & \text{otherwise} \end{cases}$$

then what are marginals of X and Y?

sol:

$$f_1(x) = \sum_{y \in R_Y} f(x,y)$$

$$= \sum_{y=1}^{6} f(x,y) = f(x,x) + \sum_{y>x} f(x,y) + \sum_{y<x} f(x,y)$$

$$= \frac{1}{36} + (6-x)\frac{2}{36} + 0 = \frac{1}{36}[13-2x]$$

$x = 1, 2, \ldots, 6$

$$f_2(y) = \sum_{x \in R_x} f(x,y) = \sum_{x=1}^{6} f(x,y)$$

$$= f(y,y) + \sum_{x<y} f(x,y) + \sum_{x>y} f(x,y)$$

$$= \frac{1}{36} + (y-1)\frac{2}{36} + 0 = \frac{1}{36}[2y-1] \quad y=1,2,\dots,6$$

ex) Let X and Y be discrete random variables with joint probability density funct:

$$f(x,y) = \begin{cases} \frac{1}{21}(x+y) & \text{if } x=1,2;\ y=1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

What are the marginal probability density funct of X and Y?

Sol:

$$f_1(x) = \sum_{y=1}^{3} \frac{1}{21}(x+y) =$$

$$= f(x,x) + \sum_{y>x} f(x,y) + \sum_{y<x} f(x,y)$$

$$= \frac{1}{21}(3x) + \frac{1}{21}[1+2+3] + 0$$

$$= \frac{x+2}{7} \quad x=1,2.$$

$$f_2(y) = \sum_{x=1}^{2} \frac{1}{21}(x+y) = f(y,y) + \sum_{x<y} f(x,y) + \sum_{x>y} f(x,y)$$

$$= \frac{2y}{21} + \frac{3}{21} = \frac{3+2y}{21} \quad y=1,2,3$$

ex: For What value of the constant K the function given by

$$f(x,y)=\begin{cases} Kxy & \text{if } x=1,2,3 \;;\; y=1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

is a joint probability density function of some r.vs X and Y?

sol

$$\sum_{x=1}^{3}\sum_{y=1}^{3} f(x,y)=1$$

$$=\sum_{x=1}^{3}\sum_{y=1}^{3} Kxy=1$$

$$=K[1+2+3+2+4+6+3+6+9]=1$$

$$36K=1 \Rightarrow K=\frac{1}{36}$$

$$\therefore f(x,y)=\begin{cases} \frac{1}{36}xy & x=1,2,3 \;;\; y=1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Def: Let X and Y be two discrete random variables. The real-valued function $F:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is called the joint cumulative probability distribution function of X and Y if and only if

$$F(x,y)=P(X\leq x, Y\leq y)$$

for all $(x,y)\in\mathbb{R}^2$.

$(X\leq x, Y\leq y)$ means $(X\leq x)\cap(Y\leq y)$

$$\therefore \quad F(x,y)=\sum_{t\leq x}\sum_{s\leq y} f(t,s)$$

Note:

$$F(a \leq X \leq b, c \leq Y \leq d) =$$

$$= F(b,d) + F(a,c) - F(a,d) - F(b,c$$

ex: In the experiment rolling two dice find the cumulative distribution function for $X=2$

sol:

$$F(2,3) = P(X \leq 2, Y \leq 3)$$

$$= \sum_{t \leq x} \sum_{s \leq y} P(x,y) = P(1,1) + P(1,2) + P(1,3) + P($$
$$+ P(2,1)$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{6}{36}$$

$$\therefore F(2,3) = \frac{6}{36} = \frac{1}{6}$$

The joint Probability Density Function For Two Continuous Random variables

Def: Let X and Y be continuous r.v.s. Then $f(x,y)$ is the joint probability density function for X and Y if for any two-dimensional set A

$$P[(X,Y)\in A] = \iint_A f(x,y)\,dx\,dy$$

In particular, if A is the two-dimensional rectangle $\{(x,y):\ a \leq x \leq b,\ c \leq y \leq d\}$. then

$$P[(X,Y)\in A] = P(a \leq X \leq b,\ c \leq Y \leq d)$$
$$= \int_a^b \int_c^d f(x,y)\,dx\,dy$$

Note: The joint p.d.f $f(x,y)$ must satisfy the following properties.

1. $f(x,y) \geq 0$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

ex: Let X and Y be two continuous r.v.s having the joint function

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Show that $f(x,y)$ is a joint probability density function.

② Find $P(0 \leq X \leq \frac{1}{4}, 0 \leq Y \leq \frac{1}{4})$

sol ①

① $f(x,y) \geq 0$ because $x, y \geq 0$

② To prove $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \int_0^1\int_0^1 \frac{6}{5}x\,dx\,dy + \int_0^1\int_0^1 \frac{6}{5}y^2\,dx\,dy$$

$$= \int_0^1 \frac{6}{5}x\,dx + \int_0^1 \frac{6}{5}y^2\,dy = \frac{6}{10} + \frac{6}{15} = 1$$

$\therefore f(x,y)$ is a joint p.d.f for the r.v.s X and Y

sol ②

$$P(0 \leq X \leq \tfrac{1}{4}, 0 \leq Y \leq \tfrac{1}{4}) = \int_0^{1/4}\int_0^{1/4} \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \frac{6}{5}\int_0^{1/4}\int_0^{1/4} x\,dx\,dy + \frac{6}{5}\int_0^{1/4}\int_0^{1/4} y^2\,dx\,dy$$

$$= \frac{6}{20}\cdot\frac{x^2}{2}\Big|_{x=0}^{x=1/4} + \frac{6}{20}\cdot\frac{y^3}{3}\Big|_{y=0}^{y=1/4} = \frac{7}{640}$$

ex

Let the joint probability density function of X and Y be given by

$$f(x,y) = \begin{cases} kxy^2 & 0 \leq x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of the constant k?

sol,

$\because f(x,y)$ is a joint p.d.f we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$= \int_0^1\int_0^y K x y^2\,dx\,dy = 1 \Rightarrow \int_0^1 K y^2 \int_0^y x\,dx\,dy$$

$$\Rightarrow \frac{K}{2}\int_0^1 y^4\,dy = 1$$

$$\Rightarrow \frac{K}{2}\left[\frac{y^5}{5}\right]_0^1 = 1 \Rightarrow \frac{K}{10} = 1 \Rightarrow K = 10$$

$$\therefore f(x,y) = \begin{cases} 10xy^2 & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

ex) let the joint prob. density of the r.v.s X and Y is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+2xy) & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

what is the probability of the event $(X \leq Y)$?

sol, let $A = (X \leq Y)$.

$$\therefore P(A) = \iint_A f(x,y)\,dx\,dy = \int\left[\int_0^y \frac{6}{5}(x^2+2xy)\,dx\right]dy$$

$$= \frac{6}{5}\int_0^1 \left.\frac{x^3}{3} + x^2 y\right|_{x=0}^{x=y} dy = \frac{6}{5}\int_0^1 \frac{4}{3}y^3\,dy$$

$$= \frac{6}{5}\left[\frac{4}{3}\cdot\frac{y^4}{4}\right]_0^1 = \frac{2}{5}y^4\Big|_0^1 = \frac{2}{5}$$

$$\therefore P(X \leq Y) = \frac{2}{5}$$

The marginal Probability density function for the joint Continuous random variables X and Y

Def: The marginal probability density functions of X and Y denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{for } -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad \text{for } -\infty < y < \infty$$

ex: Let the joint prob. density function for the two r.vs X and Y is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

① Find the marginal prob. density function of X ?
② Find the " " " " of Y ?
③ Find $P(\frac{1}{4} \le Y \le \frac{3}{4})$?

sol ①

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{6}{5}(x+y^2)\,dy = \frac{6}{5}x + \frac{2}{5}$$

$$\therefore f_X(x) = \begin{cases} \frac{6}{5}x + \frac{2}{5} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

sol ②:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{6}{5}(x+y^2)\,dx = \frac{6}{5}y^2 + \frac{3}{5}$$

$$\therefore f_Y(y) = \begin{cases} \frac{6}{5}y^2 + \frac{3}{5} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

sol③: $P(\frac{1}{4} \leq Y \leq \frac{3}{4}) = \int_{1/4}^{3/4} f_Y(y)\, dy$

$$= \int_{1/4}^{3/4} (\frac{6}{5} y^2 + \frac{3}{5})\, dy = \frac{2}{5} y^3 + \frac{3}{5} y \Big|_{1/4}^{3/4} = \frac{37}{80} = 0.462$$

ex: If the joint prob. density functn for the r.vs X and Y is given by

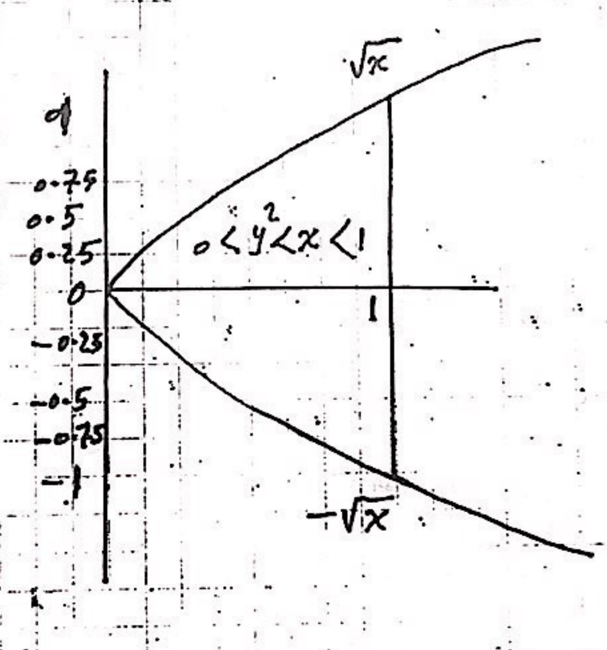
$$f(x,y) = \begin{cases} \frac{3}{4} & \text{for } 0 < y^2 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the marginal density functn of X, for $0 < x < 1$

sol: The domain of $f(x,y)$ consists of the region bounded by the curve $x = y^2$ and vartical line $x = 1$

$$\Rightarrow y = \pm\sqrt{x}$$

$$\therefore f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4}\, dy = \frac{3}{4} y \Big|_{-\sqrt{x}}^{\sqrt{x}} = \frac{3}{2}\sqrt{x}$$



joint Cumulative distribution function

Def: Let X and Y be the continuous random variables with joint probability density function $f(x,y)$.
The joint cumulative distribution function $F(x,y)$ of X and Y is defined as

$$F(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv$$

for all $(x,y) \in \mathbb{R}^2$

Note: From the fundamental theorem of calculus, we again obtain

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

ex: If the joint cumulative distribution function of X and Y is given by

$$F(x,y) = \begin{cases} \frac{1}{5}(2x^3y + 3x^2y^2) & 0 < x,y < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the joint prob. density function of X and Y?

Sol:

$$f(x,y) = \frac{1}{5}\frac{\partial}{\partial x}\frac{\partial}{\partial y}(2x^3y + 3x^2y^2)$$

$$= \frac{1}{5}\frac{\partial}{\partial x}(2x^3 + 6x^2y)$$

$$= \frac{1}{5}(6x^2 + 12xy) = \frac{6}{5}(x^2 + 2xy).$$

ex) Let (X,Y) be distributed uniformaly on the circular disk centered at $(0,0)$ with radius $\frac{2}{\sqrt{\pi}}$. What is the marginal density function of X where nonzero?

Sol) The equation of a circle with radius $\frac{2}{\sqrt{\pi}}$ and center at the origin is

$$x^2+y^2=\left(\frac{2}{\sqrt{\pi}}\right)^2$$

$$\therefore x^2+y^2=\frac{4}{\pi} \Rightarrow y=\pm\sqrt{\frac{4}{\pi}-x^2}$$

$$\therefore f_1(x)=\int_{-\sqrt{\frac{4}{\pi}-x^2}}^{\sqrt{\frac{4}{\pi}-x^2}} f(x,y)\,dy$$

$$=\int_{-\sqrt{\frac{4}{\pi}-x^2}}^{\sqrt{\frac{4}{\pi}-x^2}} \frac{1}{\text{area of the circle}}\,dy$$

$$=\int_{-\sqrt{\frac{4}{\pi}-x^2}}^{\sqrt{\frac{4}{\pi}-x^2}} \frac{1}{4}\,dy = \frac{1}{4}y\Big]_{-\sqrt{\frac{4}{\pi}-x^2}}^{\sqrt{\frac{4}{\pi}-x^2}}$$

$$=\frac{1}{2}\sqrt{\frac{4}{\pi}-x^2}$$

$\therefore$ the joint p.d.f of X and Y is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+2xy) & 0<x,y<1 \\ 0 & \text{otherwise.} \end{cases}$$

ex.

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let X and Y have the joint density function

$$f(x,y) = \begin{cases} 2x & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X+Y\leq 1 \mid X\leq \frac{1}{2})$?

Def: Let X and Y be any two random variables with joint density function $f(x,y)$ and marginals $f_1(x)$ and $f_2(y)$. The conditional probability density function g of X, given (the event) $Y=y$, is defined as

$$g(x|y) = \frac{f(x,y)}{f_2(y)} \qquad f_2(y) > 0$$

Similarly, the conditional probability density function h of Y, given (the event) $X=x$ is defined as

$$h(y|x) = \frac{f(x,y)}{f_1(x)} \qquad f_1(x) > 0$$

Q Let X and Y be discrete random variables with joint p.d.f

$$f(x,y) = \begin{cases} \frac{1}{21}(x+y) & \text{for } x=1,2,3;\ y=1,2 \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional probability density function of X given $Y=2$?

S We want to find $g(x|2)$.

$$g(x|2) = \frac{f(x,2)}{f_2(2)}$$

must first compute the marginal of Y, that is $f_2(2)$

$$f_2(y) = \sum_{x=1}^{3} \frac{1}{21}(x+y)$$

$$= \frac{1}{21}(6+3y)$$

$$\therefore f_Y(2) = \frac{1}{21}(6+6) = \frac{12}{21}$$

$$\therefore g(x|2) = \frac{f(x,2)}{f_2(2)} = \frac{\frac{1}{21}(x+2)}{\frac{12}{21}}$$

$$= \frac{1}{12}(x+2) \qquad x = 1,2,3$$

ex) Let X and Y be discrete random variables with joint probability density function

$$f(x,y) = \begin{cases} \frac{x+y}{32} & \text{for } x=1,2\ ;\ y=1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

what is the conditional probability of Y given $X=x$

sol)

$$f_X(x) = \sum_{y=1}^{4} f(x,y)$$

$$= \frac{1}{32}\sum_{y=1}^{4}(x+y) = \frac{1}{32}(4x+10)$$

$$\therefore f_X(x) = \frac{1}{32}(4x+10) \qquad x = 1,2$$

$$h(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{\frac{1}{32}(x+y)}{\frac{1}{32}(4x+10)}$$

$$= \frac{x+y}{4x+10}$$

$$\therefore h(y|x) = \begin{cases} \frac{x+y}{4x+10} & \text{for } x=1,2\ ;\ y=1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

ex: Let X and Y be continuous random variables with joint p.d.f

$$f(x,y) = \begin{cases} 12x & 0 < y < 2x < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the conditional density function of Y given X = x?

sol: first, we have to find the marginal of X.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^{2x} 12x\,dy = 12xy \Big|_{y=0}^{y=2x} = 24x^2$$

$$\therefore h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{12x}{24x^2} = \begin{cases} \frac{1}{2x} & \text{for } 0 < y < 2x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

ex: Let X and Y be random variables such that X has marginal density function $f_X(x) = \begin{cases} 24x^2 & \text{for } 0 < x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

and conditional density of Y given X = x is

$$h(y|x) = \begin{cases} \frac{y}{2x^2} & \text{for } 0 < y < 2x \\ 0 & \text{otherwise} \end{cases}$$

What is the conditional density of X given Y = y over appropriate domain?

Sol:

$$h(y|x)=\frac{f(x,y)}{f_1(x)}$$

$$\therefore f(x,y)=h(y|x)\,f_1(x)$$

$$=\frac{y}{2x^2}(24x^2)=12y \quad \text{for } 0<y<2x<1$$

joint p.d.f is: $f(x,y)=\begin{cases}12y & 0<y<2x<1\\ 0 & \text{otherwise}\end{cases}$

$$\therefore f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$$

$$=\int_{\frac{y}{2}}^{\frac{1}{2}} 12y\,dx = 6y(1-y) \quad \text{for } 0<y<1$$

$$\therefore g(x|y)=\frac{f(x,y)}{f_2(y)}$$

Let X and Y be discrete random va

density functi

$$f(x,y) = \begin{cases} \frac{1}{36} & if \\ \frac{2}{36} & if \\ 0 & other \end{cases}$$

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$$= f(y,y) + \sum_{x<y} f(x,y) + \sum_{x>y} f(x,y)$$

$$= \frac{1}{36} + (y-1)\frac{2}{36} + 0$$

$$= \frac{2y-1}{36} \quad \text{for } y = 1, 2, \ldots, 6$$

$$\because f(1,1) = \frac{1}{36} \neq f_1(1)\, f_2(1)$$

$$\text{i.e. } \frac{1}{36} \neq \frac{11}{36} \cdot \frac{1}{36}$$

$$\therefore f(x,y) \neq f_1(x)\, f_2(y)$$

$\therefore$ X and Y are not independent.

ex: Let X and Y have the joint density

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } 0 < x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y stochastically independent?

sol:

$$f_X(x) = \int_0^\infty f(x,y)\, dy = \int_0^\infty e^{-(x+y)}\, dy = e^{-x}$$

and

$$f_Y(y) = \int_0^\infty f(x,y)\, dx = \int_0^\infty e^{-(x+y)}\, dx = e^{-y}$$

$$\therefore f(x,y) = e^{-(x+y)} = e^{-x} \cdot e^{-y} = f_1(x)\, f_2(y)$$

$\therefore$ X and Y are stochastically independent.

Note If the joint density $f(x,y)$ of X and Y can be factored into two nonnegative functions one depending on x and the other depending on y, then X and Y are independent

ex Let X and Y have the joint density

$$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1 ; 0<y<1 \\ 0 & \text{other wise} \end{cases}$$

Are X and Y stochastically independent?

Sol

$$f(x,y) = x+y$$
$$= x\left(1+\frac{y}{x}\right)$$

the joint density cannot be factored into two nonnegative functions one depending on x and the other depending on y therefor X and Y are not independent.

Def Let X and Y be two random variables we said, to X and Y are independent and identically distributed (iid) if and only if they are independent and have the same distribution.

Let X and Y be two independent random variables with identical probability density function given by

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $W = \min\{X, Y\}$?

Sol:

Let $G(w)$ be the cumulative distribution function of W

then $G(w) = P(W \le w)$

$= 1 - P(W > w)$

$= 1 - P(\min\{X, Y\} > w)$

$= 1 - P(X > w)\, P(Y > w)$ (Since X and Y are independent)

$= 1 - \left(\int_w^\infty e^{-x} dx\right)\left(\int_w^\infty e^{-y} dy\right)$

$= 1 - \left(e^{-w}\right)^2 = 1 - e^{-2w}$

Joint Expectation & Marginal Expectation

Def: Let $g(X_1, X_2, ---, X_n)$ be a function of the discrete random variables, $X_1, X_2, ---, X_n$, which have probability density (mass) function $P(X_1, X_2, ---, X_n)$. Then the expected value of $g(X_1, X_2, ---, X_n)$ is defined as

$$E[g(X_1, X_2, ---, X_n)] = \sum_{\text{all } X_n} --- \sum_{\text{all } X_2} \sum_{\text{all } X_1} g(X_1, X_2, ---, X_n)\, P(X_1, X_2, ---, X_n)$$

If $X_1, X_2, ---, X_n$ are continuous random variables with joint density function $f(X_1, X_2, ---, X_n)$, then

$$E[g(X_1, X_2, ---, X_n)] = \int_{-\infty}^{\infty} --- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(X_1, X_2, ---, X_n)\, f(X_1, X_2, ---, X_n)\, dx_1\, dx_2 --- dx_n$$

ex: let X, Y be two continuous r.v.s have joint density funct. given by

$$f(X, Y) = \begin{cases} 2x & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(XY)$

sol:

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy = \int_0^1\int_0^1 xy\,(2x)\, dx\, dy$$

$$= \int_0^1\int_0^1 (2x^2y)\, dx\, dy = \int_0^1 \frac{2}{3} x^3 y \Big|_{x=0}^{x=1} dy = \frac{2}{3} \cdot \frac{y^2}{2} \Big|_{y=0}^{1}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Def: let X and Y be two random variable with joint probability density funct $f(x,y)$ then the marginal expectat of X is E(X) defined as

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx \quad \text{and} \quad E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

if X and Y are continuous

and

$$E(X) = \sum_{x=0}^{\infty} x f_X(x) \quad \text{and} \quad E(Y) = \sum_{y=0}^{\infty} y f_Y(y)$$

where $f_X(x)$ and $f_Y(y)$ are the marginal p.d.f of X and Y, respectively

ex: let X and Y have a joint prob. density funct given by

$$f(x,y) = \begin{cases} 2x & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{other wise} \end{cases}$$

① Find the expected value of XY?
② Find the marginal expected value of X?

sol: ①

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(2x)\,dx\,dy$$

$$= \int_0^1 \frac{2}{3} x^3 y \Big|_0^1 dy = \frac{2}{3}\frac{y^2}{2}\Big|_0^1 = \frac{2}{6} = \frac{1}{3}$$

sol: ② $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 2x\,dy = 2xy\Big|_0^1 = 2x$$

$$\therefore E(X) = \int_0^1 x(2x)\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}.$$

ex: Let X and Y be two r.vs with joint p.d.f is

$$f(x,y) = \begin{cases} 2(1-x) & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

① Find $E(XY)$?

② Find the marginal expected value of X?

sol:

$$E(XY) = \int_0^1\int_0^1 xy\,[2(1-x)]\,dx\,dy = 2\int_0^1\int_0^1 (xy - x^2y)\,dx\,dy$$

$$= 2\int_0^1 \frac{x^2}{2}y - \frac{x^3}{3}y\Big|_0^1\,dy = 2\int_0^1 y\left(\frac{1}{6}\right)dy$$

$$= \frac{1}{3}\left[\frac{y^2}{2}\Big|_0^1 = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}.$$

sol 2:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx =$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (2-2x)\,dy = 2y - 2xy\Big|_0^1 = 2(1-x)$$

$$\therefore E(X) = \int_0^1 x[2(1-x)]\,dx$$

$$= 2\int_0^1 (x - x^2)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1\right.$$

$$= 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

H.W

ex;

Prove that if X and Y are two independent r.v's with joint p.d.f is

$f(x,y)$ then ① $E(XY) = E(X)E(Y)$

② $E(X+Y) = E(X) + E(Y)$

ex;

Let X and Y be random variables with joint p.d.f is

$$f(x,y) = \begin{cases} 2x & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the variance of X ?

sol;

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 2x\,dy = 2xy\Big|_{y=0}^{y=1} = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore E(X) = \int_{-\infty}^{\infty} x f_X(x) = \int_0^1 x(2x)\,dx = \frac{2}{3}x^3\Big|_{x=0}^{x=1} = \frac{2}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2(2x)\,dx = \frac{2}{4}x^4\Big|_{x=0}^{x=1} = \frac{1}{2}$$

$$\because V(X) = E(X^2) - (E(X))^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

Def: The Covariance between two r.vs X and Y is defined as

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= \begin{cases} \sum\limits_{x}\sum\limits_{y}(x-\mu_x)(y-\mu_y)\,P(x,y) & \text{if } X,Y \text{ discrete} \\ \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(x-\mu_x)(y-\mu_y)\,f(x,y)\,dx\,dy & \text{if } X,Y \text{ Cont.} \end{cases}$$

ex: let X and Y be two discrete random variables with joint p.m.f

$f(x,y) = P(x,y) =$

$x \backslash y$	0	100	200
100	0.20	0.10	0.20
250	0.05	0.15	0.30

Find the Covariance between X and Y?

sol:

$$f_X(x) = P_X(x) = \sum_{y=0}^{200} P(x,y) = P(x,0) + P(x,100) + P(x,200)$$

$$f_Y(y) = P_Y(y) = \sum_{x=100}^{250} P(x,y) = P(100,y) + P(250,y)$$

$\therefore$

x	100	250
$P_X(x)$	0.5	0.5

y	0	100	200
$P_Y(y)$	0.25	0.25	0.5

$$\mu_X = E(X) = \sum x P_X(x) = 100(0.5) + 250(0.5) = 175$$

$$\mu_Y = E(Y) = \sum y P_Y(y) = 0(0.25) + 100(0.25) + 200(0.5)$$
$$= 125$$

$$\therefore Cov(X,Y) = \sum_{(x,y)}\sum (x-175)(y-125)\, P(x,y)$$

$$= (100-175)(0-125)(0.20) + (100-175)(100-125)(0.10) +$$
$$(100-175)(200-125)(0.20) + (250-175)(0-125)(0.05) +$$
$$(250-175)(100-125)(0.15) + (250-175)(200-125)(0.30)$$

$$= 1875$$

Proposition: $Cov(X,Y) = E(XY) - \mu_X \mu_Y$
$$= E(XY) - E(X)E(Y)$$

Note: $Cov(X,X) = E(X^2) - (E(X))^2 = V(X)$

ex: let X and Y be discrete random variables with joint density func:

$$f(x,y) = \begin{cases} \dfrac{x+2y}{18} & \text{for } x=1,2 \;;\; y=1,2 \\ 0 & \text{otherwise} \end{cases}$$

Find the covariance σ_{XY} between X and Y?

sol:
$$f_X(x) = \sum_{y=1}^{2} \frac{1}{18}(x+2y) = \frac{1}{18}(2x+6)$$

$$E(X) = \sum_{x=1}^{2} x f_X(x) = 1\left(\frac{2(1)+6}{18}\right) + 2\left(\frac{2(2)+6}{18}\right) = \frac{8}{18} + \frac{20}{18}$$
$$= \frac{28}{18}$$

$$f_Y(y) = \sum_{x=1}^{2} \frac{x+2y}{18} = \frac{3+4y}{18}$$

$$E(Y) = \sum_{y=1}^{2} y\, f_Y(y) = = 1\, f_Y(1) + 2 f_Y(2)$$

$$= \frac{7}{18} + 2\,\frac{11}{18} = \frac{29}{18}$$

$$E(XY) = \sum_{x=1}^{2}\sum_{y=1}^{2} xy\, f(x,y)$$

$$= f(1,1) + 2f(1,2) + 2f(2,1) + 4f(2,2)$$

$$= \frac{3}{18} + 2\,\frac{5}{18} + 2\,\frac{4}{18} + 4\,\frac{6}{18}$$

$$= \frac{3+10+8+24}{18} = \frac{45}{18}$$

$\therefore$ $$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{45}{18} - \left(\frac{28}{18}\right)\left(\frac{29}{18}\right) = \frac{810-812}{324} = -\frac{2}{324}$$

$$= -0.00617$$

ex)

let X and Y have the joint density function

$$f(x,y) = \begin{cases} x+y & 0<x,y<1 \\ 0 & \text{otherwise} \end{cases}$$

Find the Coveriance between X and Y?

sol)

$$f_X(x) = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big|_{y=0}^{y=1} = x + \frac{1}{2}$$

$$E(X) = \int_0^1 x\, f_X(x)\,dx = \int^1 x\left(x+\tfrac{1}{2}\right)dx$$

$$= \frac{x^3}{3} + \frac{x^2}{4}\Big|_{x=0}^{1} = \frac{7}{12}$$

$$f_Y(y) = \int_0^1 (x+y)\,dx = \frac{x^2}{2} + yx \Big|_0^1 = \frac{1}{2} + y$$

$$\mu_Y = E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y\left(\frac{1}{2}+y\right)dy = \frac{y^2}{4} + \frac{y^3}{3}\Big|_0^1 = \frac{7}{12}$$

$$E(XY) = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\int_0^1 (x^2y + xy^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\Big|_{x=0}^{x=1}\right] dy = \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$$

$$= \frac{y^2}{6} + \frac{y^3}{6}\Big]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{4}{12}$$

$$\therefore Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{4}{12} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = \frac{48-49}{144} = -\frac{1}{144}$$

H.W

ex: let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 2 & 0<y<1-x \,,\; 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Find Covariance between X and Y?

Theorem: If X and Y are any two random variables and a, b, c and d are real constant then

$$Cov(aX+b, cY+d) = ac\, Cov(X,Y)$$

proof:

$$Cov(aX+b, cY+d) = E[(aX+b)(cY+d)] - E(aX+b)E(cY+d)$$

$$= E(acXY + adX + bcY + bd) - (aE(X)+b)(cE(Y)+d)$$

$$= acE(XY) + adE(X) + bcE(Y) + bd - [acE(X)E(Y) + adE(X) + bcE(Y) + bd]$$

$$= ac[E(XY) - E(X)E(Y)]$$

$$= ac\, Cov(X,Y)$$

Theorem: If X and Y are independent random variables with $E(X)=E(Y)=0$, then

$$V(XY) = V(X)V(Y)$$

proof:

$$V(XY) = E((XY)^2) - (E(X)E(Y))^2$$

$$= E(X^2Y^2) = E(X^2)E(Y^2) \quad \text{by independent of } X \text{ and } Y$$

$$= E(X^2) - (E(X))^2 \cdot E(Y^2) - (E(Y))^2$$

$$= V(X)V(Y)$$

ex) Let X and Y be independent random variables, each with density function is

$$f(x,y) = \begin{cases} \frac{1}{2\theta} & -\theta < x < \theta , -\theta < y < \theta \\ 0 & \text{otherwise} \end{cases}$$

If the $V(XY) = \frac{64}{9}$, then what is the value of θ?

sol)

$$E(X) = \int_{-\theta}^{\theta} x\, f(x,y)\, dx = \int_{-\theta}^{\theta} \frac{1}{2\theta}\, x\, dx = \frac{1}{2\theta}\left[\frac{x^2}{2} \right|_{-\theta}^{\theta} = 0$$

$$E(Y) = \int_{-\theta}^{\theta} y\, f(x,y)\, dy = \int_{-\theta}^{\theta} \frac{1}{2\theta}\, y\, dy = \frac{1}{2\theta}\, \frac{y^2}{2} \Big|_{-\theta}^{\theta} = 0$$

$$\therefore V(XY) = \frac{64}{9} = V(X)V(Y)$$

$$= \left(\int_{-\theta}^{\theta} \frac{1}{2\theta} x^2 dx \right)\left(\int_{-\theta}^{\theta} \frac{1}{2\theta} y^2 dy \right)$$

$$= \left(\frac{\theta^2}{3}\right)\left(\frac{\theta^2}{3}\right) = \frac{\theta^4}{9}$$

$$\therefore \frac{64}{9} = \frac{\theta^4}{9} \Rightarrow \theta^4 = 64 \Rightarrow \theta = 2\sqrt{2}$$

Def: Let X and Y be two random variables with variances σ_X^2 and σ_Y^2, respectively. Let the Covariance of X and Y be $\text{Cov}(X,Y)$.
Then the correlation coefficient ρ between X and Y is given by

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \, \sigma_Y}.$$

Theorem: If X and Y are independent, the correlation coefficient between X and Y is zero.

Proof:

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \, \sigma_Y} = \frac{0}{\sigma_X \, \sigma_Y} = 0$$

Note: If the correlation coefficient of X and Y is zero then X and Y are said to be uncorrelated.

Theorem: For any random variables X and Y, the correlation coefficient ρ satisfies

$$-1 \leq \rho_{X,Y} \leq 1$$

and $\rho = 1$ or $\rho = -1$ implies that the random variable

$$Y = aX + b$$

where a and b are arbitrary real constants with $a \neq 0$.

Joint Moment Generating function and Marginal Moment Generating funct.

Def: Let X and Y be two random variables with joint density function $f(x,y)$. A real valued function $M: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$M(s,t) = E(e^{sX+tY})$$

is called the joint moment generating function of X and Y if this expected value exists for all s is some interval $-h<s<h$ and for all t is some interval $-k<t<k$ for some positive h and k

Note: ① $M(s,0) = E(e^{sX})$, $M(0,t) = E(e^{tY})$

from this we see that

$$E(X^k) = \left.\frac{\partial^k M(s,t)}{\partial s^k}\right|_{(0,0)} \quad \text{and} \quad E(Y^k) = \left.\frac{\partial^k M(s,t)}{\partial t^k}\right|_{(0,0)}$$

for $K = 1,2,3,\dots$ and

$$E(XY) = \left.\frac{\partial^2 M(s,t)}{\partial s \partial t}\right|_{(0,0)}$$

ex: let the random variables X and Y have the joint density

$$f(x,y) = \begin{cases} e^{-y} & 0<x<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

Find the joint moment generating function for X and Y?

sol:

$$M(s,t) = E(e^{sX+tY}) = \int_0^\infty\int_0^\infty e^{sx+ty} f(x,y)\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty e^{sx+ty} \cdot e^{-y}\,dx\,dy$$

$$E\left(\frac{X}{Y}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x}{y} D(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 \frac{x}{y} f(x)\, g(y)\, dx\, dy$$

$$= \int_0^1\int_0^1 \frac{x}{y}\, 3x^2\, 4y^3\, dx\, dy$$

$$= \int_0^1 3x^3\, dx \int_0^1 4y^2\, dy = \left(\frac{3}{4}\right)\left(\frac{4}{3}\right) = 1$$

Note: ① The independence of X and Y does not imply

$E\left(\frac{X}{Y}\right) \neq \frac{E(X)}{E(Y)}$ but only implies $E\left(\frac{X}{Y}\right) = E(X)E(Y^{-1})$

② $E(Y^{-1}) = E\left(\frac{1}{Y}\right) \neq \frac{1}{E(Y)}$

Theorem: Let X and Y be any two random variables and let a and b be any two real numbers. Then

$$V(aX+bY) = a^2V(X) + b^2V(Y) + 2ab\, Cov(X,Y)$$

proof

$$V(aX+bY) = E\left(\left(aX+bY - E(aX+bY)\right)^2\right)$$

$$= E\left[\left[aX+bY - aE(X) - bE(Y)\right]^2\right]$$

$$= E\left(\left[a(X-\mu_X) + b(Y-\mu_Y)\right]^2\right)$$

$$= E\left(a^2(X-\mu_X)^2 + b^2(Y-\mu_Y)^2 + 2ab(X-\mu_X)(Y-\mu_Y)\right)$$

$$= a^2E\left((X-\mu_X)^2\right) + b^2E\left((Y-\mu_Y)^2\right) + 2abE\left((X-\mu_X)(Y-\mu_Y)\right)$$

$$= a^2V(X) + b^2V(Y) + 2ab\, Cov(X,Y)$$

Theorem: If X and Y are independent then

$$M_{ax+by}(t) = M_X(at)\,M_Y(bt)$$; where a, b real parameter

Proof:

let $W = ax+by$

$\therefore M_{ax+by}(t) = M_W(t)$

$= E(e^{tW}) = E(e^{t(ax+by)})$

$= E(e^{tax}\,e^{tby}) = E(e^{tax})\,E(e^{tby})$

$= M_X(at)\,M_Y(bt)$

ex: let $X \sim N(2,9)$ and $Y \sim N(0,16)$. If X and Y are independent then what is the probability distribution of the random variable $X+Y$?

Sol: $\because X \sim N(2,9)$, the moment generating function of X is given by $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{2t + \frac{1}{2}9t^2}$

and $Y \sim (0,16)$ then $M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{\frac{16}{2}t^2}$

$\because$ X and Y are independent, the moment generating function of X+Y is given by

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

$$= e^{2t+\frac{1}{2}9t^2} \cdot e^{\frac{16}{2}t^2}$$

$$= e^{2t + \frac{1}{2}9t^2 + \frac{16}{2}t^2} = e^{2t + \frac{25}{2}t^2}$$

$\therefore X+Y \sim N(2,25)$ $\therefore$ X+Y has normal distribution

From this information we can find the probability density function of $W = X+Y$ as

$$f(w) = \frac{1}{\sqrt{2\sigma\pi}} e^{-\frac{1}{2}\left(\frac{w-\mu}{\sigma}\right)^2} \quad -\infty < w < \infty$$

$$= \frac{1}{\sqrt{50\pi}} e^{-\frac{1}{2}\left(\frac{w-2}{5}\right)^2} \quad -\infty < w < \infty$$

ex: let X and Y are two independent Bernoulli random variables with parameter p.
Find the distribution of $X+Y$?

Sol: $\because$ X and Y are Bernoulli with parameter p

$\therefore M_X(t) = (1-P) + Pe^t$, $M_Y(t) = (1-P) + Pe^t$

$\because$ X and Y are independent

$\therefore M_{X+Y}(t) = M_X(t)\,M_Y(t)$

$$= \left((1-P)+Pe^t\right)\left((1-P)+Pe^t\right)$$

$$= (1-P+Pe^t)(1-P+Pe^t) = (1-P+Pe^t)^2$$

$\therefore X+Y \sim BIN(2, P)$

Cauchy-Schwarz Inequality:

For any two random variables X and Y, we have

$$|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$$

where equality holds if and only if $X = aY$ for some a

Proof: let $W = (X - aY)^2$, $W \geq 0$ for any $a \in R$

let $a = \frac{E(XY)}{E(Y^2)}$

$$E(W) = E(X - aY)^2$$

$$= E(X^2 - 2aXY + a^2Y^2)$$

$$= E(X^2) - 2aE(XY) + a^2E(Y^2) \quad \text{---} (*)$$

so, that substitute $a = \frac{E(XY)}{E(Y^2)}$ in equation $(*)$

$$= E(X^2) - 2\frac{E(XY)}{E(Y^2)}E(XY) + \frac{(E(XY))^2}{(E(Y^2))^2}E(Y^2)$$

$$= E(X^2) - \frac{(E(XY))^2}{E(Y^2)}$$

$$(E(XY))^2 \leq E(X^2)E(Y^2)$$