Bernuolli trial:

A Bernuolli trial is a random experiment in which there are only two possible outcomes, which we conveninently call "Success" (S) and "failure" (F).

. then this experiment have the sample space $\{S, F\}$ into the set of real numbers as follows:

$$X(S) = 1 \quad \text{and} \quad X(F) = 0$$

Bernoulli Distribution:

Def: The random variable X is called the Bernoulli random variable if its probability mass function is of the form

$$P(X=x) = P^{x}(1-P)^{1-x} \qquad x = 0, 1$$

where P is the probability of success and $1-P = q$ is the probability of failure.

we denote the Bernoulli r.v by writing $X \sim Ber(P)$

ex) What is the probability of getting a score of not less than 5 a throw of a six-sided die?

sol)

$S = \{1, 2, 3, 4, 5, 6\}$

Any score in $\{1, 2, 3, 4\}$ is a failure and any score in $\{5, 6\}$ is success. Thus, this is a Bernoulli trial with

$P(X=0) = P(\text{failure}) = \frac{4}{6} \Rightarrow (1-P)$

$P(X=1) = P(\text{success}) = \frac{2}{6} \Rightarrow P$

$$\therefore P(X \geq 5) = \left(\frac{2}{6}\right)^1 \left(\frac{4}{6}\right)^0 = \frac{2}{6}$$

Theorem: If X is a Bernoulli random variable with parameter P, then the mean, variance and moment generating funct are respectively given by

$$\mu_X = P \quad , \quad V(X) = \sigma_X^2 = P(1-P)$$

$$M_X(t) = (1-P) + Pe^t$$

proof:

The mean of the Bernoulli r.v is

$$E(X) = \mu_X = \sum_{x=0}^{1} x\, f(x)$$

$$= \sum_{x=0}^{1} x\, P^x (1-P)^{1-x} = (0) + (1)(P)^1 (1-P)^{1-1}$$

$$= P$$

$$E(X^2) = \sum_{x=0}^{1} x^2 f(x)$$

$$= \sum_{x=0}^{1} x^2 P^x (1-P)^{1-x}$$

$$= 0 + (1)^2 P^1 (1-P)^{1-1}$$

$$= P$$

$$\therefore V(X) = E(X^2) - (E(X))^2$$

$$= P - P^2 = P(1-P)$$

$$\therefore M_X(t) = E(e^{tX}) = \sum_{x=0}^{1} e^{tx} f(x)$$

$$= \sum_{x=0}^{1} e^{tx} P^x (1-P)^{1-x}$$

$$= (1-P) + e^{t} P$$

Note: where $p = 0.5$ for the Bernoulli dist. all its moments about zero are same and equal to p.

Binomial Distribution:

Consider a fixed number n of mutually independent Bernoulli trials. Suppose these trials have same probability of success say p. A r.v X is called a Binomial r.v if it represents the total number of successes in n independent Bernoulli trials.

Def: The random variable X is called the Binomial random variable with parameters p and n if its probability mass function is of the form

$$P(X=x) = \cancel{\sum_{x=0}^{n}} C_x^n \, p^x (1-p)^{n-x}$$

The values of n and p are called the parameters of the distribution

ex: Is the real valued function $f(x) = P(X=x) = C_x^n p^x (1-p)^{n-x}$ is a probability mass function $\sum_{x=0}^{n} C_x^n p^x (1-p)^{n-x} = (p+1-p)^n = (1)^n = 1$ $\therefore f(x)$ is a p.m.f

ex: Consider an exam that contains 10 multiple-choice questions with 4 possible choices for each question, only one of which is correct.

1. What is the probability for the student to get no answer correct
2. What is the probability for the student to get two answer correct?

sol ① $p = 0.25$, $1-p = q = 0.75$

$$P(X=0) = C_0^{10}\, p^x (1-p)^{1-x}$$

$$= \frac{10!}{0!(10-0)!} (0.25)^0 (0.75)^{10-0}$$

$$= (0.75)^{10} = 0.0563$$

sol ②

$$P(X=2) = C_2^{10}\, p^2 (1-p)^{10-2}$$

$$= \frac{10!}{2!(10-2)!} (0.25)^2 (1-0.25)^8$$

$$= 45 \cdot (0.25)^2 (0.75)^8$$

$$= 0.2816$$

what is the probability for the student to fail the test
i.e to have less than 6 correct answer ?

$$P(X<5) = \sum_{x=0}^{4} C_x^{10}\, p^x (1-p)^{10-x}$$

$$= C_0^{10} p^0 (1-p)^{10-0} + C_1^{10} p^1 (1-p)^9 + C_2^{10} p^2 (1-p)^8 +$$

$$C_3^{10} p^3 (1-p)^7 + C_4^{10} p^4 (1-p)^6$$

$$= 0.0563 + 0.1877 + 0.2816 + 0.2503 + 0.1460$$

$$= 0.9219$$

~~what is the mean~~

ex: In ~~∞~~ rolling a die 5 times

① What is the probability of two sixes?

② What is the probability at most two sixes?

Sol: ① Let X denote number of sixes in 5 independent casts of a fair die.

then $n=5$, $P=\frac{1}{6}$, $1-P=1-\frac{1}{6}=\frac{5}{6}$

$$P(X=2)=C_2^5\,P^2(1-P)^{5-2}$$

$$=C_2^5\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3=10\left(\frac{1}{36}\right)\left(\frac{125}{216}\right)$$

$$=\frac{1225}{7776}=0.1607$$

Sol ②

$$P(X\leq 2)=C_0^5\,P^0(1-P)^{5-0}+C_1^5\,P^1(1-P)^{5-1}+C_2^5\,P^2(1-P)^{5-2}$$

$$=\frac{5!}{0!(5-0)!}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5+\frac{5!}{1!(5-1)!}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4+\frac{5!}{2!(5-2)!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

$$=\frac{1}{2}(0.9421+0.9734)=0.9577$$

Theorem: If X is binomial random variable with parameters P, n then the mean, variance and moment generating function are respectively given by

$$\mu_x=nP\quad,\quad V(X)=\sigma_x^2=nP(1-P)$$

$$M_x(t)=\left[(1-P)+Pe^t\right]^n$$

Proof:

By definition of expectation

$$E(X)=\sum_x x\,p(x)=\sum_{x=0}^{n}x\,C_x^n\,P^x(1-P)^{n-x}$$

$$E(X) = 0 + \sum_{x=1}^{n} x \frac{n!}{(n-x)!\,x!} p^{x} (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \cancel{x} \frac{n!}{\cancel{x}(x-1)!\,(n-x)!} p^{x} (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x)!} p\,p^{x-1} (1-p)^{n-x}$$

let $z = x-1$

$$= np \sum_{z=0}^{n-1} \frac{(n-1)!}{z!\,(n-1-z)!} p^{z} (1-p)^{n-1-z}$$

$$= np \sum_{z=0}^{n-1} C^{n}_{z}\, p^{z} (1-p)^{n-1-z}$$

$\because \sum_{z=0}^{n-1} C^{n}_{z}\, p^{z} (1-p)^{n-1-z}$ is a probability mass funct $= 1$

$\therefore \mu_x = np$

$\because V(X) = E(x^2) - (E(x))^2$

then

$$E(x^2) = \sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} x^2 C^{n}_{x}\, p^{x} (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x^2 \frac{n!}{x!\,(n-x)!} p^{x} (1-p)^{n-x}$$

$\because$ x^2 dose not appear as a factor of $x!$

$$E(x) \Rightarrow E[x(x-1)] = E(x^2 - x) = E(x^2) - E(x)$$

$$\therefore E(x^2) = E[X(X-1)] + E(X) = E[X(X-1)] + \mu$$

$$\therefore E[X(X-1)] = \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} p^{x} (1-p)^{n-x}$$

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the first and second terms of this sum equal zero (when $x=0$ and $x=1$).

then $E(X(X-1)) = \sum_{x=2} \frac{\cancel{x(x-1)}\, n!}{\cancel{x(x-1)}(x-2)!\,(n-x)!} p^{x}(1-p)^{n-x}$

let $z = x-2$

$$\therefore E[X(X-1)] = \sum_{x=2}^{n-2} \frac{n(n-1)(n-2)!}{(x-2)!\,(n-x)!} p^{2} p^{x-2} (1-p)^{n-x}$$

let $z = x-2$

$$= n(n-1)p^{2} \sum_{z=0}^{n-2} \frac{(n-2)!}{z!\,(n-2-z)!} p^{z}(1-p)^{n-2-z}$$

$\because \sum_{z=0}^{n-2} \frac{(n-2)!}{z!\,(n-2-z)!} p^{z}(1-p)^{n-2-z} = 1$ is a probability mass funct

$$\therefore E[X(X-1)] = n(n-1)p^{2}$$

$$\because E(X^{2}) = E[X(X-1)] + \mu = n(n-1)p^{2} + np$$

$$\because V(X) = \sigma_{X}^{2} = E(X^{2}) - (E(X))^{2}$$

then $V(X) = n(n-1)p^{2} + np - (np)^{2}$

$$= np[(n-1)p + 1 - np] = np(1-p)$$

$$\because M_{X}(t) = E(e^{tX})$$

$$= \sum_{x=0}^{n} e^{tx} p(x) = \sum_{x=0}^{n} e^{tx} C_{x}^{n} p^{x}(1-p)^{n-x}$$

$$= \sum_{x=0}^{n} C_{x}^{n} (e^{t}p)^{x} (1-p)^{n-x}$$

$$\therefore \boxed{M_{X}(t) = (e^{t}p + 1 - p)^{n}}$$

ex) Suppose that 2000 points are selected independent and at random from the unit squares $S = \{(x,y) \mid 0 \le x, y \le 1\}$ let X equal the number of points that fall in A such that

$A = \{(x,y) \mid x^2 + y^2 < 1\}$

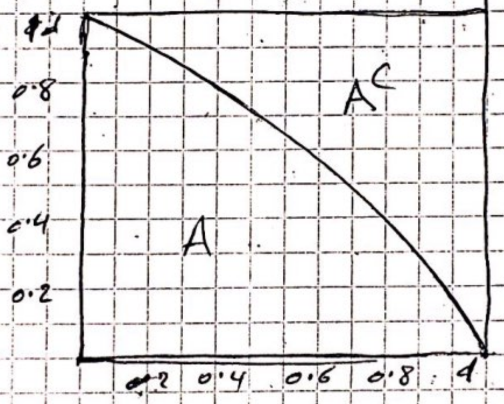
① How is X distributed?

② What are the mean, variance and standard deviation of X?

sol ①) If a point in A, then it is a success. If a point falls in A^c then it is a failure.

The probability of success is

$$P = \frac{\text{area of } A}{\text{area of } S} = \frac{1}{4}\pi$$



$\therefore$ X has a binomial distribution with $n = 2000$, $P = \frac{\pi}{4}$.

i.e $X \sim BIN(2000, \frac{\pi}{4})$

Sol ②)

$$\mu_x = nP \Rightarrow \mu_x = 2000 \cdot \frac{\pi}{4} = 500\pi = 1570.8$$

$$V(X) = \sigma_x^2 = n(1-P)P = 2000\left(1 - \frac{\pi}{4}\right)\frac{\pi}{4} = 337.1$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{337.1} = 18.36$$

Note In general if $X_i \sim BER(P)$ then $\sum_{i=1}^{n} X_i \sim BIN(n,P)$

and hence $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \underbrace{P + P + \cdots + P}_{n \text{ time}}$

$$= nP$$

and $V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i) = \underbrace{P(1-P) + P(1-P) + \cdots + P(1-P)}_{n \text{ times}}$

$$= nP(1-P)$$

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proof Geometric p.m.f $= 1$

ex Is the real valued function $f(x)$ defined by

$$f(x) = P(X=x) = (1-P)^{x-1} P \qquad x = 1, 2, \cdots, \infty$$

where $0 < P < 1$ is a parameter, a probability mass function?

① $f(x) = \sum_{x=1}^{\infty} (1-P)^{x-1} P \geq 0$

where $y = x-1$

② $\sum_{x=1}^{\infty} (1-P)^{x-1} P = P \sum_{y=0}^{\infty} (1-P)^{y} = P \frac{1}{1-(1-P)} = \frac{P}{P} = 1$

$\therefore f(x)$ is a probability density function

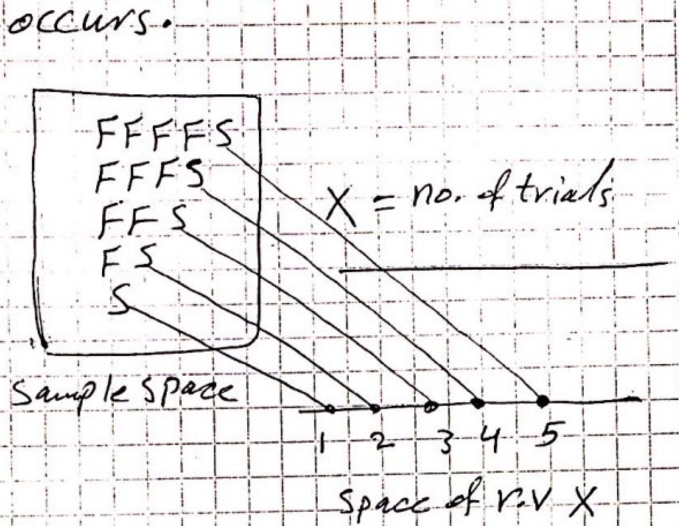
proof Bernoulli p.m.f $= 1$

$$\sum_{x=0}^{1} P^{x} (1-P)^{1-x} = P^{0} (1-P)^{1-0} + P^{1} (1-P)^{1-1}$$

$$= 1 - \cancel{P} + \cancel{P} = 1$$

Geometric Distribution:

Let X denote the trial number on which the first success occurs.



Hence the probability that the first success occurs on x^{th} trial is given by

$$f(x) = P(X = x) = (1-p)^{x-1}\, p \qquad x = 1, 2, 3, \ldots, \infty$$

where p denotes the probability of success in a single Bernoulli trial.

Def: A random variable X has a geometric distribution if its probability mass function is given by

$$f(x) = P(X = x) = (1-p)^{x-1}\, p \quad , x = 1, 2, 3, \ldots, \infty$$

where p denotes the probability of success in a single Bernoulli trial.

We denote it as $X \sim GEO(p)$.

ex: Is ~~the probability mass funct~~ the funct

$$f(x) = (1-p)^{x-1}\, p \qquad x = 1, 2, 3, \ldots, \infty$$

a probability mass funct?

sol)

must $\sum_{x=1}^{\infty} f(x) = 1 \Rightarrow \sum_{x=1}^{\infty} (1-p)^{x-1} p$

$$= p \sum_{y=0}^{\infty} (1-p)^{y} \quad \text{where } y = x-1.$$

$$= p \frac{1}{1-(1-p)} = \frac{p}{p} = 1$$

$\therefore f(x)$ is a probability mass functi

Note) The sum of Geometric series $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$, $\sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$, $\sum_{i=0}^{m} r^i = \frac{1-r^{m+1}}{1-r}$

ex) The probability a machine produces a defective item is 0.02 Each item is checked as it is produced. Assuming that these are independent trials. What is the probability that at least 100 items must be checked to find one that is defective?

sol) let X denote the trial number on which the first defective item is observed

$$P(X \geq 100) = \sum_{x=100}^{\infty} f(x)$$

$$= \sum_{x=100}^{\infty} (1-p)^{x-1} p = (1-p)^{99} \sum_{y=0}^{\infty} (1-p)^{y} p$$

$$= (1-p)^{99} (1) = \cancel{0.98}$$

$$= (1-0.02)^{99} = (0.98)^{99} = 0.1353.$$

ex) suppose that the probability of engine malfunctin during any one-hour period is $p = 0.02$. Find the probability that a given engine will survive two hours.

sol: Let X be represent the survive of engine

$P(X=2) = P(X \geq 3) = \sum_{x=3}^{\infty} P(x) = 1 - P(X \leq 2) = 1 - \sum_{x=1}^{2} (1-p)^{x-1} p$

ex) A gambler plays roulette at Monte Carlo and continues gambling, wagering the same amount each on "Red", until he wins for the first time. If the probability of "Red" is $\frac{18}{38}$ and the gambler has only enough money for 5 trials, then ① what is the probability he will win before he exhausts his funds?
② what is the probability that he wins on the second trial.

Sol)
$$P = P(Red) = \frac{18}{38}$$

$$\therefore P(X \leq 5) = 1 - P(X \geq 6)$$
$$= 1 - (1-P)^5$$
$$= 1 - \left(1 - \frac{18}{38}\right)^5$$
$$= 1 - (0.5263)^5 = 1 - 0.044 = 0.956$$

Sol ②

$$P(X=2) = (1-P)^{2-1} P$$
$$= \left(1 - \frac{18}{38}\right)^1 \left(\frac{18}{38}\right)$$
$$= \frac{360}{1444} = 0.2493$$

Theorem If X is a geometric random variable with parameter p, then the mean, variance and ~~generating~~ moment generating functions are respectively given by

$$\mu_x = \frac{1}{p}, \quad V(X) = \sigma_x^2 = \frac{1-p}{p^2}, \quad M_x(t) = \frac{pe^t}{1-(1-p)e^t}$$

if $t < -\ln(1-p)$

proof

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1} p$$

$$= p \sum_{y=0}^{\infty} e^{t(y+1)} (1-p)^y \qquad \text{where } y = x-1$$

$$= pe^t \sum_{y=0}^{\infty} \left(e^t(1-p)\right)^y$$

$$= pe^t \frac{1}{1-(1-p)e^t} \qquad \text{if } t < -\ln(1-p)$$

$$\boxed{M_x(t) = \frac{pe^t}{1-(1-p)e^t} \qquad \text{if } t < -\ln(1-p)}$$

$$\therefore E(X) = \mu_x = M_x'(0)$$

$$\because M_x'(t) = \frac{(1-(1-p)e^t)\,pe^t + pe^t(1-p)e^t}{\left[1-(1-p)e^t\right]^2}$$

$$= \frac{pe^t\left[1-(1-p)e^t + (1-p)e^t\right]}{\left[1-(1-p)e^t\right]^2}$$

$$= \frac{pe^t}{\left[1-(1-p)e^t\right]^2}$$

$$\therefore M'_x(0) = \frac{p}{p^2} = \frac{1}{p}$$

$$M''_x(t) = \frac{\left[1-(1-p)e^t\right]^2 pe^t + pe^t\, 2\left[1-(1-p)e^t\right](1-p)e^t}{\left[1-(1-p)e^t\right]^4}$$

$$M''_x(0) = \frac{p^3 + 2p^2(1-p)}{p^4} = \frac{2-p}{p^2}$$

$$\therefore V(X) = \sigma_x^2 = E(X^2) - (E(X))^2$$

$$= M''_x(0) - (M'_x(0))^2$$

$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$\boxed{V(X) = \sigma_x^2 = \frac{1-p}{p^2}}$$

Negative Binomial Distribution:

Def: A random variable X is said to have a negative binomial probability distribution if and only if

$$P(X) = C^{x-1}_{r-1}\, p^{r}(1-p)^{x-r} \quad , \; x = r, r+1, r+2, \ldots \quad 0 \leq p \leq 1$$

We denote the r.v X has distribution negative binomial by writing
$X \sim NBIN(r, p)$

ex: A geological study indicates that an exploratory oil well drilled in a particular region should strike oil with probability 0.2.
Find the probability that the third oil strike comes on the fifth well drilled

sol:

$\therefore p = 0.2 \quad , 1-p = 1-0.2 = 0.8$

$$P(X=5) = C^{5-1}_{3-1}\, p^{3}(1-p)^{5-3}$$

$$= C^{4}_{2}\,(0.2)^{3}(0.8)^{2}$$

$$= 6(0.008)(0.64) = 0.307$$

Theorem: If Y is a random variable with negative binomial distribution then

$$\mu_Y = E(Y) = \frac{r}{p} \text{ and } \sigma^2_Y = V(Y) = \frac{r(1-p)}{p^2}$$

The Hypergeometric Probability Distribution

Def: A random variable X is said to have a hypergeometric probability distribution if and only if

$$P(X) = \frac{C_x^r \, C_{n-x}^{N-r}}{C_n^N}$$

where x is any integer $0, 1, 2, \ldots, n$, subject to the restrictions $x \leq r$ and $n-x \leq N-r$

We denote $X \sim HYP(r, N-r, n)$

ex: Suppose there are 3 defective items in a lot of 50 items. A sample of size 10 is taken at random and without replacement. Let X denote the number of defective itemes in the sample. What is the probability that the sample contains at most one defective item?

sol: $X \sim HYP(3, 47, 10)$

$$\therefore P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{C_0^3 \, C_{10}^{47}}{C_{10}^{50}} + \frac{C_1^3 \, C_9^{47}}{C_{10}^{50}} = 0.504 + 0.4$$

$$= 0.904$$

ex: A random sample of 5 students is drawn without replacem at from among 300 seniors, and each of these 5 seniors is asked if she/he has tried a certain drug. Suppose 50% of the seniors actually have tried the druge. What is the probability that two of the students interviewed tried the drug?

Sol)

$$P(X=2) = \frac{C_2^{150}\, C_3^{300-150}}{C_5^{300}} = 0.3146.$$

Theorem) If X is a random variable with a hypergeometric distribution then

$$\mu_X = E(X) = \frac{nr}{N} \quad \text{and} \quad V(X) = n\left(\frac{r}{n}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$$

if we define $p = \frac{r}{n}$ and $1-p = \frac{N-r}{N}$

$$\therefore \mu_X = np \quad \text{and} \quad V(X) = np(1-p)\left(\frac{N-n}{N-1}\right)$$

when $N \to \infty$ for a fixed n

then $\frac{N-n}{N-1} \longrightarrow 1$

Poisson Distribution:

Def) A random variable X is said to have a Poisson distribution if its probability density (mass) function is given by

$$f(x) = P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots, \infty$$

where $0 < \lambda < \infty$ is a parameter.

We denote such a random variable by $X \sim POI(\lambda)$

ex) Is the real valued function defined by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots, \infty$$

where $0 < \lambda < \infty$, a probability density function?

sol:

① $f(x) \geq 0$

② To prove that $\sum_{x=0}^{\infty} f(x) = 1$

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = e^0 = 1$$

ex: The number of traffic accidents per week in a small city has a Poisson distribution with mean equal to 3. What is the probability of exactly 2 accidents occur in 2 weeks?

sol:

The mean traffic accident is 3 in a week

$\therefore$ the mean in 2 weeks is

$$\lambda = (3)(2) = 6$$

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!} \Rightarrow f(2) = P(X=2) = \frac{e^{-6}(6)^2}{2!}$$

$$= 18 e^{-6}$$

Theorem: If X is a random variable possessing a Poisson distribution with parameter λ, then

$$\mu = E(X) = \lambda \quad \text{and} \quad V(X) = \sigma_x^2 = \lambda$$

proof:

$$\therefore E(X) = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} \cancel{x} \frac{\lambda \lambda^{x-1} e^{-\lambda}}{\cancel{x}(x-1)!}$$

let $x-1 = z$

$$\mu_x = \lambda \sum_{z=0}^{\infty} \frac{\lambda^z e^{-\lambda}}{z!} = \lambda(1) = \lambda$$

$$\therefore \boxed{\mu_x = \lambda}$$

$$E(x^2) = E[x(x-1)] + E(x)$$

$$E(x^2) = E[x(x-1)] = \sum_{x=2}^{\infty} \cancel{x(x-1)} \frac{\lambda^2 \lambda^{x-2} e^{-\lambda}}{\cancel{x(x-1)}(x-2)!}$$

let $x-2 = y$

$$= \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = \lambda^2$$

$$\therefore E(x^2) = \lambda^2 + \lambda$$

$$\therefore V(x) = E(x^2) - (E(x))^2$$

$$\boxed{V(x) = \lambda^2 + \lambda - (\lambda)^2 = \lambda}$$

the Momentgenerating funct for the poisson distributi is

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$$

$$\boxed{M_x(t) = e^{\lambda(e^t - 1)}}$$

The relation between the binomial dist. and Poisson distributi

When the number of trial became verey large and the probability of success became verey small then the limiting of the binomial dist gose to the poisson distributi

i.e $n \to \infty$ and $p \to 0$ in the binomial distributi

i.e letting $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$$\therefore \lim_{n \to \infty} C^n_x p^x (1-p)^{n-x} = \lim_{n \to \infty} \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n \frac{n(n-1)\cdots(n-x+1)}{n^x}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\left(1-\frac{1}{n}\right)\times\left(1-\frac{2}{n}\right)\times\cdots$$

$$\times\left(1-\frac{x-1}{n}\right)$$

$$\because \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \quad \text{and} \quad \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}\left(1-\frac{1}{n}\right)\times\cdots\times\left(1-\frac{x-1}{n}\right) = 1$$

$$\therefore P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda} \quad \text{is a poisson dist.}$$

i.e: The binomial probability density func. converges to the poisson probability density function.

ex1 A certain type of tree has seedlings randomly dispersed in a large area, with the mean density of seedlings being approximately five per square yard. If a forester randomly locates ten 1-square yard sampling regions in the area, find the probability that none of the regions will contain seedlings.

sol: Let X represent the number of seedlings per region then $\lambda = 5$ (the average density is five per square yard)

$$\therefore P(X=0) = P(0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-5} = 0.006738$$

ex) If a publisher of nontechnical books takes great pains to ensure that its books are free of typographical errors, so that the probability of any given page containing at least one such error is 0.005 and errors are independent from page to page, ① what is the probability that one of its 400-page novels will contain exactly one page with errors?
② What is the prob. At most three pages with errors?

Sol: ① $\because \lambda = nP \Rightarrow \lambda = 400(0.005) = 2$

$$P(X=1) \simeq BIN(400, 0.005) \simeq POI(2)$$

$$P(X=1) = \frac{(2)^1 e^{-2}}{1!} = 0.270671$$

Sol ②

$$P(X\leq 3) \simeq \sum_{x=0}^{3} POI(2) = \sum_{x=0}^{3} \frac{2^x}{x!} e^{-2}$$

$$= 0.13533 + 0.270671 + 0.270671 + 0.180447$$

$$= 0.8571$$

and this result $\simeq$ the binomial value $P(X\leq 3) = 0.8576$

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Uniform Distribution

Def: let X be a random variable has the probability density (mass) function is in the form

$$P(X = x_i) = \frac{1}{n} \quad , i = 1, 2, 3, \dots, n$$

we denote $X \sim UNI(n)$

then $E(X) = \sum_{i=1}^{n} x_i \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i$

and

$$V(X) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)^2$$

Some Continuous Distributions

Uniform probability Dist.

Def: A random variable X is said to be uniform on the interval $[a,b]$ if its probability density function is of the form

$$f(x) = \frac{1}{b-a} \qquad a \leq x \leq b$$

where a and b are constants. We denote a random variable X with the Uniform distribution on the interval $[a,b]$ as

$$X \sim UNIF(a,b)$$

Theorem: If X is uniform on the interval $[a,b]$ then the mean, variance and moment generating function of X are given by

$$E(X) = \frac{b+a}{2}, \quad V(X) = \sigma_X^2 = \frac{(b-a)^2}{12}$$

and

$$M_X(t) = \begin{cases} 1 & \text{if } t=0 \\ \dfrac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \end{cases}$$

ex: If X has a uniform distribution on the interval $[0,10]$. What is $P(X+\frac{10}{X} \geq 7)$

sol: Since $X \sim UNIF(0,10)$, the pdf of X is

$$f(x) = \frac{1}{10-0} = \frac{1}{10}, \quad 0 \leq x \leq 10$$

$$\begin{aligned} P(X+\tfrac{10}{X} \geq 7) &= P(X^2+10 \geq 7X) \\ &= P(X^2-7X+10 \geq 0) \\ &= P((X-5)(X-2) \geq 0) \\ &= P(X \leq 2 \text{ or } X \geq 5) \end{aligned}$$

$$= 1 - P(2 \le X \le 5)$$

$$= 1 - \int_2^5 f(x)\,dx = 1 - \int_2^5 \frac{1}{10}\,dx = 1 - \left[\frac{1}{10}(5-2)\right]$$

$$= 1 - \frac{3}{10} = \frac{7}{10} = 0.7$$

ex: A box to be constructed so that its height is 10 inches and its base is X inches by X inches. If X has uniform dist. over the interval (2, 8), then what is the expected volume of the box in cubic inches?

sol: Since $X \sim UNIF(2,8)$

$$f(x) = \frac{1}{8-2} = \frac{1}{6} \quad \text{on } (2,8)$$

The volume V of the box is $V = 10X^2$

$$\therefore E(V) = E(10X^2) = 10\,E(X^2)$$

$$= 10 \int_2^8 x^2 \frac{1}{6}\,dx = \frac{10}{6}\left[\frac{x^3}{3}\right]_2^8$$

$$= \frac{10}{18}\left[8^3 - 2^3\right] = 280 \text{ cubic inches.}$$

proof theorem *:

① $E(X) = \mu_X = \int_a^b x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$

$$= \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{b+a}{2}$$

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a} \left.\frac{x^3}{3}\right|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ba+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ba+a^2}{3}$$

$$\therefore V(X) = E(X^2) - (E(X))^2 \Rightarrow V(X) = \frac{b^2+ba+a^2}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4b^2+4ba+4a^2-3b^2-6ba-3a^2}{12} = \frac{b^2-2ba+a^2}{12}$$

$$\therefore V(X) = \frac{(b-a)^2}{12}$$

The Normal Distribution:

Def: A continuous random variable X is said to have a normal distribution with parameters μ and σ (or μ and σ^2), where $-\infty<\mu<\infty$ and $0<\sigma$, if the pdf of X is

$$f(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-(x-\mu)^2/(2\sigma^2)} \qquad -\infty<x<\infty$$

we denote that X has a normal distribution as

$X \sim N(\mu,\sigma^2)$

The Normal Distribution

Def: A random variable X is said to have a normal distribution if its probability density function is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty<x<\infty$$

where $-\infty<\mu<\infty$ and $0<\sigma^2<\infty$ are arbitrary parameter.
If X has a normal distribution with parameters μ and σ^2 then we write $X\sim N(\mu,\sigma^2)$.

ex: Is the real-valued function $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty<x<\infty$

a probability density function of some r.v X?

sol: ① $f(x)\geq 0$ because the exponential function always positive.

② To prove $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= 2\int_{\mu}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad \text{let } z = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} e^{-z}\, \frac{\sigma}{\sqrt{2z}}\, dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{z}}\, e^{-z}\, dz = \frac{1}{\sqrt{\pi}}\, \Gamma\left(\tfrac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$$

$\therefore f(x)$ is a p.d.f

Note the parameter μ is the mean and σ^2 is the variance of the normal distribution.

Theorem: If $X \sim N(\mu, \sigma^2)$ then

$$E(X) = \mu \;,\; V(X) = \sigma^2 \text{ and}$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

proof:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2 - 2\sigma^2 t x) + \sigma^2 t^2 - \sigma^2 t^2}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2 t)^2} e^{\mu t + \frac{1}{2}\sigma^2 t^2}\, dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2 t)^2}\, dx$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

because $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2 t)^2}\, dx = 1$

is a p.d.f of $X \sim N(\mu + \sigma^2 t, \sigma^2)$

$$\therefore \boxed{M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}}$$

ex1 IF X is any random variable with mean μ and variance $\sigma^2 > 0$, then what are the mean and the variance of the random variable of the r.v $Y = \frac{X-\mu}{\sigma}$?

Def: A normal random variable is said to be standard normal if its mean is zero and variance is one.

We denote a standard normal random variable X by $X \sim N(0,1)$, and the probability density function is in the form

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

ex: If $X \sim N(0,1)$. What is the probability of the r.v X less than or equal to -1.72?

sol:

$$P(X \leq -1.72) = 1 - P(X \leq 1.72)$$
$$= 1 - \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(1.72)^2}{2}}\right)$$
$$= 1 - 0.9573 = 0.0427$$

ex: If $X \sim N(3,16)$, then what is $P(4 \leq X \leq 8)$?

sol:

$$P(4 \leq X \leq 8) = P\left(\frac{4-3}{4} \leq \frac{X-3}{4} \leq \frac{8-3}{4}\right)$$
$$= P\left(\frac{1}{4} \leq Z \leq \frac{5}{4}\right)$$
$$= P(Z \leq 1.25) - P(Z \leq 0.25)$$
$$= 0.8944 - 0.5987$$
$$= 0.2957$$

ex: If $X \sim N(25,36)$, then what is the value of the constant c such that $P(|X-25| \leq c) = 0.9544$?

sol:

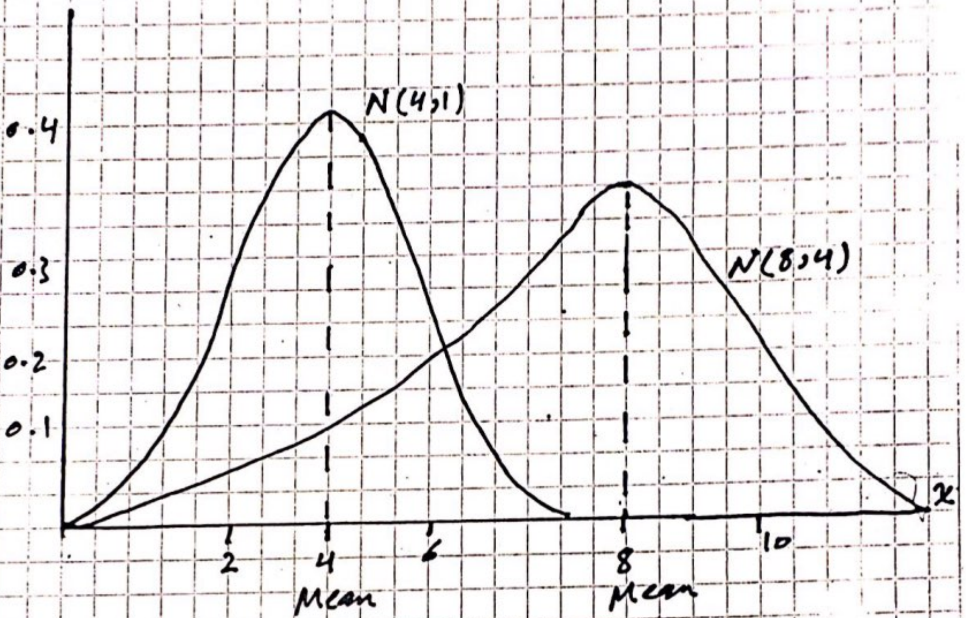
$$0.9544 = P(|X-25| \leq c)$$
$$= P(-c \leq X \leq c)$$
$$= P\left(-\frac{c}{6} \leq \frac{X-25}{6} \leq \frac{c}{6}\right)$$
$$= P\left(-\frac{c}{6} \leq Z \leq \frac{c}{6}\right)$$
$$= P\left(Z \leq \frac{c}{6}\right) - P\left(Z \leq -\frac{c}{6}\right)$$
$$= 2P\left(Z \leq \frac{c}{6}\right) - 1$$

$$\therefore 0.9544 = 2P(Z \leq \tfrac{c}{6}) - 1$$

$$P(Z \leq \tfrac{c}{6}) = \frac{0.9544+1}{2} = 0.9772$$

and from the table of normal dist. we get

$$\frac{c}{6} = 2 \Rightarrow c = 12$$



PDF of Normal Random Variables

ex: If $X \sim N(7,4)$, what is $P(15.364 \leq (X-7)^2 \leq 20.095)$?

sol: $\because X \sim N(7,4)$ then $\mu = 7$ and $\sigma = 2$

$$\therefore P(15.364 \leq (X-7)^2 \leq 20.095) = P\left(\frac{15.364}{4} \leq \left(\frac{X-7}{2}\right)^2 \leq \frac{20.098}{4}\right)$$

$$= P(3.841 \leq Z^2 \leq 5.024)$$

$$= P(0 \leq Z^2 \leq 5.024) - P(0 \leq Z^2 \leq 3.841)$$

$$= 0.975 - 0.949 = 0.026$$

Gamma Distribution:

The gamma dist. involves the notation of gamma function

Def: The gamma function is defined as.

$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x}\, dx$$

where z is positive real number, $z>0$

① $\Gamma(1) = \int_0^\infty x^0 e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1$

② $\Gamma(z) = (z-1)\,\Gamma(z-1)$ for all real number $z>1$

Proof: let z be a real number such that $z>1$ and consider

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

by using integral by part.
let $u = x^{z-1} \Rightarrow du = (z-1)x^{z-2}$
$dv = e^{-x} \Rightarrow v = -e^{-x}$

$$= -x^{z-1}e^{-x}\Big|_0^\infty + \int_0^\infty (z-1)x^{z-2}e^{-x}\, dx$$

$$= (z-1)\int_0^\infty x^{z-2}e^{-x}\, dx$$

$$= (z-1)\,\Gamma(z-1)$$

③ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

④ $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$

ex: If n is a natural number, then $\Gamma(n+1) = n!$

sol: $\because \Gamma(z) = (z-1)\Gamma(z-1)$

$$\therefore \Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$\vdots$$
$$= n(n-1)(n-2)\cdots(1)\Gamma(1)$$

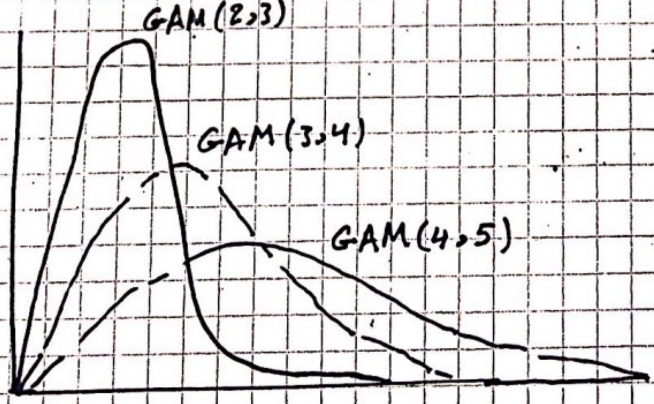
$\because \Gamma(1) = 1$

$$= n(n-1)(n-2)\cdots 1 = n!$$

Def: A continuous random variable X is said to have a gamma distribution if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

where $\alpha > 0$ and $\theta > 0$. We denote a r.v with gamma dist. as $X \sim GAM(\theta, \alpha)$



P.d.f of GAMMA Dist.

Theorem: If $X \sim GAM(\theta, \alpha)$, then

$\mu_X = E(X) = \theta\alpha$, $V(X) = \theta^2\alpha$ and

$$M_X(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha} \quad \text{if } t < \frac{1}{\theta}$$

Proof:

$$M_X(t) = E(e^{tX})$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} e^{tx} \, dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{1}{\theta}(1-\theta t)x} \, dx$$

let $y = \frac{1}{\theta}(1-\theta t)x$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\cancel{\theta^\alpha}} \frac{\cancel{\theta^\alpha}}{(1-\theta t)^\alpha} y^{\alpha-1} e^{-y} \, dy \qquad x = \frac{\theta y}{1-\theta t} \Rightarrow x^{\alpha-1} = \left(\frac{\theta y}{1-\theta t}\right)^{\alpha-1}$$

$dx = \frac{\theta}{1-\theta t} dy$

$$= \frac{1}{(1-\theta t)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} \, dy$$

$\because \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} \, dy = 1$ is a p.d.f of $GAM(1, \alpha)$

$$\boxed{\therefore M_X(t) = \frac{1}{(1-\theta t)^\alpha}}$$

$$M_X'(t) = \frac{d}{dt}(1-\theta t)^{-\alpha}$$

$$= (-\alpha)(1-\theta t)^{-\alpha-1}(-\theta)$$

$$= \alpha\theta(1-\theta t)^{-\alpha-1}$$

$$\therefore E(X) = M_X'(t)\Big|_{t=0} = M_X'(0)$$

$$= \alpha\theta(1-0)^{-\alpha-1}$$

$$= \alpha\theta$$

$$M_X''(t) = \frac{d}{dt}\left(\alpha\theta(1-\theta t)^{-(\alpha+1)}\right)$$

$$= \alpha\theta(\alpha+1)\theta(1-\theta t)^{-(\alpha+2)}$$

$$= \alpha(\alpha+1)\theta^2(1-\theta t)^{-(\alpha+2)}$$

$$\therefore V(X) = M_X''(0) - \left(M_X'(0)\right)^2$$

$$= \alpha(\alpha+1)\theta^2 - \alpha^2\theta^2$$

$$\boxed{V(X) = \alpha\theta^2}$$

ex/ If the random variable X has a gamma distribution with parameters $\alpha = 1$, $\theta = 1$, then what is the probability that X is between its mean and median?

Sol/ $\because X \sim GAM(1,1)$ then the p.d.f of X is

$$f(x) = \begin{cases} \frac{1}{\Gamma(1)(1)^1} x^{1-1} e^{-\frac{x}{1}} & 0<x<\infty \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore f(x) = \begin{cases} e^{-x} & 0<x<\infty \\ 0 & \text{otherwise} \end{cases}$$

Let q is the median of X can be calculated from

$$\frac{1}{2} = \int_0^q e^{-x}\,dx$$

$$\frac{1}{2} = -e^{-x}\Big|_0^q$$

$$\frac{1}{2} = 1 - e^{-q} \Rightarrow -\frac{1}{2} = -e^{-q} \Rightarrow e^{-q} = \frac{1}{2}$$

$$e^{-q} = \frac{1}{2} \Rightarrow q = \ln 2$$

The mean of X is $E(X) = \alpha\theta = (1)(1) = 1$

$\therefore \mu_X = 1$ and median of $X = \ln 2$

$$\therefore P(\ln 2 \leq X \leq 1) = \int_{\ln 2}^{1} e^{-x}\,dx = -e^{-x}\Big|_{\ln 2}^{1}$$

$$= e^{-\ln 2} - \frac{1}{e} = \frac{1}{2} - \frac{1}{e}$$

$$= \frac{e-2}{2e}$$

<u>ex:</u> If the random variable X has a gamma distribution with parameters $\alpha = 1$, $\theta = 2$, then what is the probability density function of the random variable $Y = e^{X}$?

<u>sol:</u>

$$F(y) = P(Y \leq y)$$

$$= P(e^{X} \leq y)$$

$$= P(X \leq \ln y)$$

$$= \int_0^{\ln y} \frac{1}{2} e^{-\frac{x}{2}}\,dx$$

$$e^{\frac{1}{2}\ln y} = e^{(\ln y)^{1/2}} = y^{\frac{1}{2}} = \sqrt{y}$$

$$= \frac{1}{2}\left[-2e^{-\frac{x}{2}}\right]_0^{\ln y} = 1 - \frac{1}{e^{\frac{\ln y}{2}}} = 1 - \frac{1}{\sqrt{y}}$$

$\therefore$ p.d.f of $Y = \frac{d}{dy} F(y)$

$$= \frac{d}{dy}\left(1 - \frac{1}{\sqrt{y}}\right) = \frac{1}{2y\sqrt{y}}$$

$\therefore$ If $X \sim GAM(1,2)$, then p.d.f of e^{X} is

$$f(x) = \begin{cases} \frac{1}{2x\sqrt{x}} & \text{if } 1 \leq x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

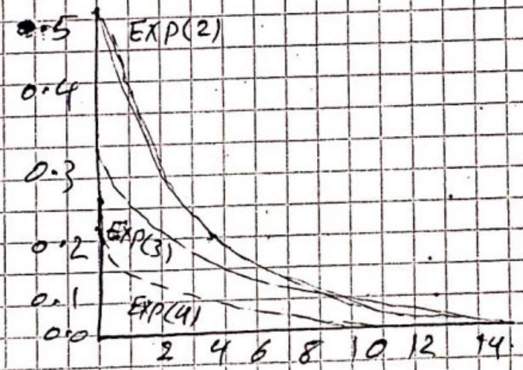
Exponential Distribution:

Def: A continuous random variable is said to be an exponential random variable with parameter θ if its probability density function is of the form

$$f(x)=\begin{cases}\frac{1}{\theta}e^{-\frac{x}{\theta}} & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}$$

where $\theta>0$. If X has an exponential distribution with parameter θ, then we denote it by writing

$X \sim EXP(\theta)$.



Note: ① The Exponential distribution is a spical case of the gamma distribution. If $\alpha=1$, then the gamma distribution reduces to exponential distribution

② $\mu_X=E(X)=\theta$ and $\sigma_X^2=V(X)=\theta^2$ from substituting $\alpha=1$ in gamma mean and variance.

ex: What is the cumulative density function of a r.v which has an exponential distribution with variance 25?

Sol: $\because$ an exponential distribution is a special case of the gamma dist. with $\alpha=1$

$\therefore V(X) = \alpha\theta^2 = 1\theta^2 = \theta^2$

but $\theta^2 = 25 \Rightarrow \theta = 5$

$\therefore$ the p.d.f of is $f(x) = \begin{cases} \frac{1}{5} e^{-\frac{x}{5}} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$

$\therefore$ C.d.f of X is $F(x) = \int_0^x f(t)\,dt$

$$= \int_0^x \frac{1}{5} e^{-\frac{t}{5}}\,dt = \frac{1}{5}\left[-5e^{-\frac{t}{5}}\right|_0^x$$

$$= 1 - e^{-\frac{x}{5}}$$

$\therefore$ C.d.f of x is $\boxed{F(x) = 1 - e^{-\frac{x}{5}}}$

Chi-Square Distribution

Def: A continuous random variable X is said to have a Chi-square distribution with r degrees of freedom if its probability density function is of the form

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}}\, x^{\frac{r}{2}-1} e^{-\frac{x}{2}} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Where $r > 0$. If X has a Chi-square distribution then we denote it by writing $X \sim \chi^2(r)$

$\therefore V(X) = \alpha\theta^2 = 1\theta^2 = \theta^2$

but $\theta^2 = 25 \Rightarrow \theta = 5$

$\therefore$ the p.d.f of is $f(x) = \begin{cases} \frac{1}{5} e^{-\frac{x}{5}} & 0 < x < \infty \\ 0 & \text{other-wise} \end{cases}$

$\therefore$ C.d.f of X is $F(x) = \int_0^x f(t)\,dt$

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$$= \int_0^x \frac{1}{5} e^{-\frac{t}{5}} dt = \frac{1}{5}\left[-5e^{-\frac{t}{5}}\right]_0^x$$

$$= 1 - e^{-\frac{x}{5}}$$

$\therefore$ C.d.f of X is $\boxed{F(x) = 1 - e^{-\frac{x}{5}}}$

Chi-Square Distribution,

Def: A continuous random variable X is said to have a Chi-square distribution with r degrees of freedom if its probability density function is of the form

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where $r > 0$. If X has a Chi-square distribution, then we denote it by writing $X \sim \chi^2(r)$

The Beta function

$$B(a,b) = B(b,a) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$$

The relation between Beta and Gamma function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

in Beta function let $x = \frac{t}{1-t}$ then Beta function became as:

$$B(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\,dx$$

The Beta Distribution

Let x be a r.v we say that $X \sim \underset{BETA}{B}(a,b)$ if its pdf is given by:

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The mean and the variance are given by

$$\mu = \frac{a}{a+b}, \quad V(x) = \sigma_x^2 = \frac{ab}{(a+b)^2(a+B+1)}$$

Note: Beta distributions are used in Bayesian statistics as Conjugate priors for the distributions in the Bernoulli process. In Beta(a,b), a counts the number of successes observed while b Keeps track of the failures.

Theorem: If $X \sim BETA(\alpha, \beta)$ then $E(X) = \frac{\alpha}{\alpha+\beta}$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Proof:

$$E(X) = \int_0^1 x f(x)\,dx = \int_0^1 \frac{1}{B(\alpha,\beta)}\, x \cdot x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{1}{B(\alpha,\beta)} \cdot B(\alpha+1,\beta) = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\dfrac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}}{\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}} = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{\alpha\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}} \cdot \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}} = \frac{\alpha}{\alpha+\beta}$$

Similarly we can show that

$$E(X^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\therefore V(X) = E(X^2) - (E(X))^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

② عازات

ex ~~tt~~ The proportion of time per day that all checkout counters in a supermarket are busy follows a distribution

$$f(x) = \begin{cases} K x^2 (1-x)^9 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

36 B

What is the value of the constant K so that $f(x)$ is a valid probability density function?

Sol

using the definition of the beta function, we get that

$$\int_0^1 x^2 (1-x)^9 \, dx = B(3,10)$$

$$\because B(3,10) = \frac{\Gamma(3)\Gamma(10)}{\Gamma(3+10)} = \frac{1}{660} \Rightarrow K \cdot \frac{1}{660} = 1 \Rightarrow K = 660$$

Note: The gamma dist. reduces to the chi-square dist. if $\alpha = \frac{r}{2}$ and $\theta = 2$

the chi-square dist is a special case of the gamma distribution, Further, if $r \to \infty$ then the chi-square dist. tends to the normal distribution.

ex: If $X \sim GAM(1,1)$, then what is the probability density function of the r.v $2X$?

sol:

$\because$ the moment generating function of gamma dist. is

$$M_X(t) = \frac{1}{(1-\theta t)^{\alpha}} \quad \text{if } t < \frac{1}{\theta}$$

$\because \alpha = 1$ and $\theta = 1$

$$\therefore M_X = \frac{1}{1-t} \quad t<1$$

$$\therefore M_{2X}(t) = M_X(2t)$$

$$= \frac{1}{1-2t} = \frac{1}{(1-2t)^{\frac{2}{2}}}$$

$$= \text{MGF of } \chi^2(2).$$

note: If X is an exponential with parameter 1, then $2X$ is chi-square with 2 degrees of freedom.

Theorem: If X has a chi-square distribution with r degrees of freedom, then

$$\mu_X = E(X) = r \quad \text{and} \quad \sigma_X^2 = V(X) = 2r$$

proof: by substituting $\alpha = \frac{r}{2}$ and $\theta = 2$ in the mean of gamma distribution and the variance. we get the mean of chi-square and the variance.

Logistic Distribution:

Def: A random variable X is said to have a logistic distribution if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{\pi}{\sigma\sqrt{3}} \dfrac{e^{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu}{\sigma}\right)}}{\left[1+e^{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu}{\sigma}\right)}\right]^2} & -\infty < x < \infty \end{cases}$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are parameters

If X has a logistic distribution with parameter μ and σ, then we write $X \sim LOG(\mu, \sigma)$

Theorem: If $X \sim LOG(\mu, \sigma)$, then

$$\mu_X = E(X) = \mu \quad , \quad V(X) = \sigma^2 \quad \text{and}$$

$$M_X(t) = e^{\mu t}\, \Gamma\left(1 + \frac{\sqrt{3}}{\pi}\sigma t\right) \Gamma\left(1 - \frac{\sqrt{3}}{\pi}\sigma t\right) , \quad |t| < \frac{\pi}{\sigma\sqrt{3}}$$

Bivariate Discrete Random Variables

Def: A discrete bivariate random variable (X, Y) is an ordered pair of discrete random variables

Def: Let (X, Y) be a bivariate random variable and Let R_X and R_Y be the range spaces of X and Y, respectively. A real-valued function $f: R_X \times R_Y \longrightarrow \mathbb{R}$ is called a joint probability density function for X and Y if and only if

$$f(x, y) = P(X = x, Y = y)$$

for all $(x, y) \in R_X \times R_Y$.

Here, the event $(X = x, Y = y)$ means the intersection of the events $(X = x)$ and $(Y = y)$ that is

$$(X = x) \cap (Y = y)$$

ex: Roll a pair of unbiased dice. If X denotes the smaller and Y denotes the larger outcome on the dice then what is the joint probability density function of X and Y?

sol: The sample space S is

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

The probability density function $f(x,y)$ can be computed for $X=2$ and $Y=3$ as follows:

There are two outcomes namely $(2,3)$ and $(3,2)$ in the sample space S of 36 outcomes which contribute to the joint $(X=2, Y=3)$.

$$\therefore f(2,3) = P(X=2, Y=3) = \frac{2}{36}$$

	1	2	3	4	5	6
6	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
5	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0
4	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0
3	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0
2	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0
1	$\frac{1}{36}$	0	0	0	0	0
	1	2	3	4	5	6

$$\therefore f(x,y) = \begin{cases} \frac{1}{36} & \text{if } 1 \leq x = y \leq 6 \\ \frac{2}{36} & \text{if } 1 \leq X < Y \leq 6 \\ 0 & \text{otherwise.} \end{cases}$$

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