



# LINEAR ALGEBRA (1)

Second Class  
Department of Mathematics

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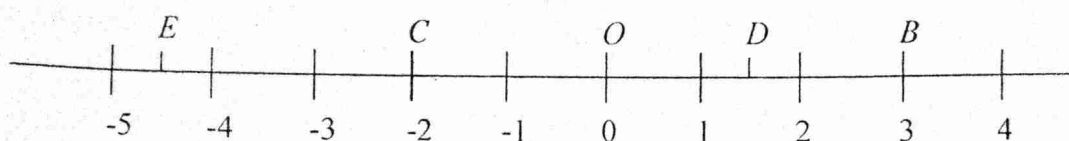


# Linear Algebra

## Vectors in the plane

### Coordinate systems:

We recall that the real numbers system may be visualized as a straight line  $L$ , which is usually taken as a horizontal position. A point  $O$  called the **origin**, on  $L$ ;  $O$  corresponds to the number 0. A point  $A$  is chosen to the right of  $O$  and fixing the length of  $OA$  as 1 and specifying a positive direction. Thus the positive real numbers lie to right of  $O$ ; the negative real numbers lie to the left of  $O$  (see figure 1).



The absolute value  $|x|$  of the real number  $x$  is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus  $|3| = 3$ ,  $|-2| = 2$  and  $|0| = 0$

If  $a, b$  are two points on the line  $L$ , then the distance between the point  $a$  and  $b$  is  $|b-a|$ .

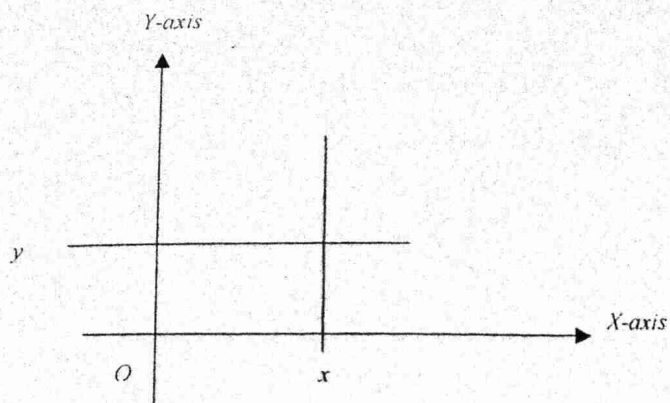
**Example:** If  $a=3$ ,  $b=1.5$ , the distance between  $a$  and  $b$  is

$$|b-a| = |1.5 - (-3)| = 4.5.$$

### In the plane:

We draw a pair of perpendicular lines intersecting at a point  $O$ , called the origin. One of the lines, the  $x$ -axis, is usually taken in a horizontal position, the other line, the  $y$ -axis, is taken in a vertical position.

We now choose a point on the  $x$ -axis to right of  $O$  and a point of the  $y$ -axis above  $O$  to fix the units of length is used for both axis.



Thus with every point in the plane we associate an order pair  $(x,y)$  of real numbers its coordinates. The point  $p$  with coordinates  $x$  and  $y$  is denoted by  $p(x,y)$ .

Conversely, it is easy to see how we can associate a point in the plane with each order pair  $(x,y)$  of real number.

The correspondence given above between points in the plane and ordered pairs of real number is called rectangular coordinate system or the Cartesian coordinate system. The set of all points in the plane is denoted by  $R^2$ . It is also called 2-space.

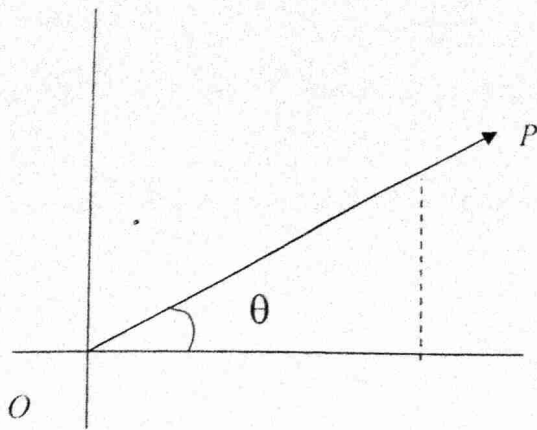
## Vectors:

Consider the  $2 \times 1$  matrix

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

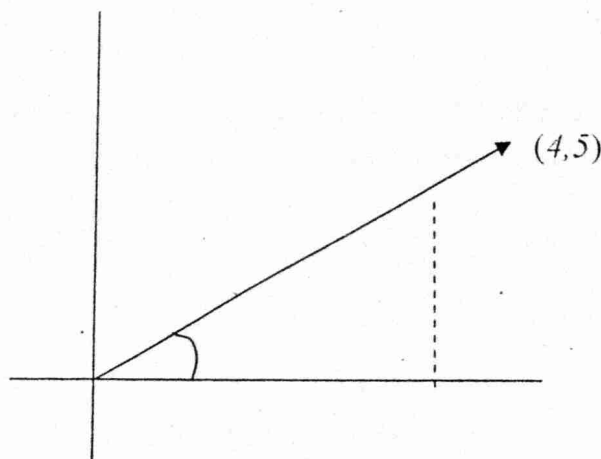
Where  $x$  and  $y$  are real numbers. With  $X$  we associate the directed line segment with initial point at the origin  $O (0, 0)$  and terminal point at  $p(x,y)$ . It is denoted by  $\overrightarrow{OP}$ ;  $O$  is called its **tail** and  $P$  its **head**.

A directed line segment has a direction, which is the angle made with the positive  $x$ -axis, indicated by the arrow at its head



The magnitude of a directed line segment is its length.

**Example:** Let  $X = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$  be a vector with the directed line  $\overline{OP}$  with head  $p(4,5)$ ,



**Definition:** A vector in the plane is a  $2 \times 1$  matrix  $X = \begin{bmatrix} x \\ y \end{bmatrix}$  where  $x$  and  $y$  are real numbers, called the **component** of  $X$ .

**Note:** Since a vector is a matrix, the vectors

$$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Are said to be **equal** if  $x_1 = x_2$  and  $y_1 = y_2$ . That is two vectors are equal if their respective components are equal.

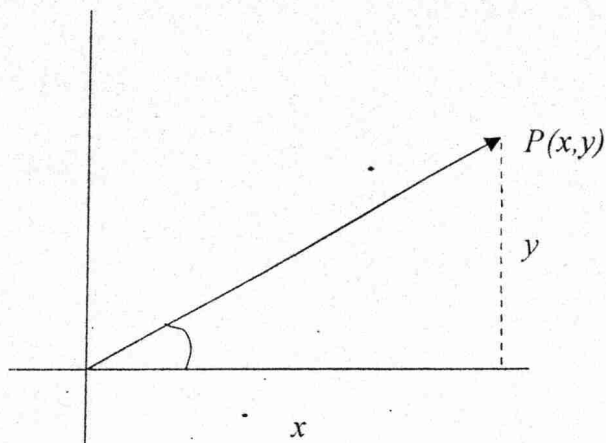


**Example:** The vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are not equal.

**Length:**

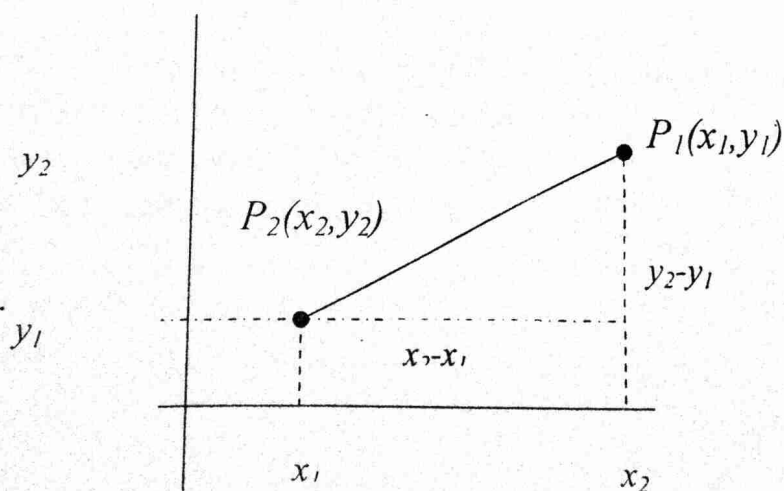
By the Pythagorean theorem the length or magnitude of the vector  $X=(x,y)$  is

$$\|X\| = \sqrt{x^2 + y^2} \quad \dots \quad (1)$$



If  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  are two points on the plan then by the Pythagorean theorem, the length of the directed segment with initial point  $P_1(x_1, y_1)$  and terminal point  $P_2(x_2, y_2)$  is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \dots \quad (2)$$



**Example:** If  $X = (2, -5)$ , then



$$\|X\| = \sqrt{(2)^2 + (-5)^2} = \sqrt{4+25} = \sqrt{29}$$

**Example:** The distance between  $P(3, 2)$  and  $Q(-1, 5)$  is

$$\|\overline{PQ}\| = \sqrt{(-1-3)^2 + (5-2)^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

**Note:** Two vectors  $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$  and  $Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$  is said to be **parallel** if  $x_1 y_2 = x_2 y_1$  or have the same **slopes**.

## **Véctor operations:**

**Definition:** Let  $X=(x_1, y_1)$  and  $Y=(x_2, y_2)$  be two vectors in the plane. The **sum** of the vectors  $X$  and  $Y$  is the vector

$$(x_1 + x_2, y_1 + y_2)$$

And is denoted by  $X+Y$ .

**Example:** Let  $X=(2, 3)$  and  $Y=(-5, 6)$ . Then

$$Q(x_2, y_2) \quad X+Y = (2+(-5), 3+6) = (-3, 9)$$

**Definition:** If  $X=(x, y)$  is a vector and  $c$  is a scalar (real number), then the **scalar multiple**  $cX$  of  $X$  by  $c$  is the vector  $(cx, cy)$

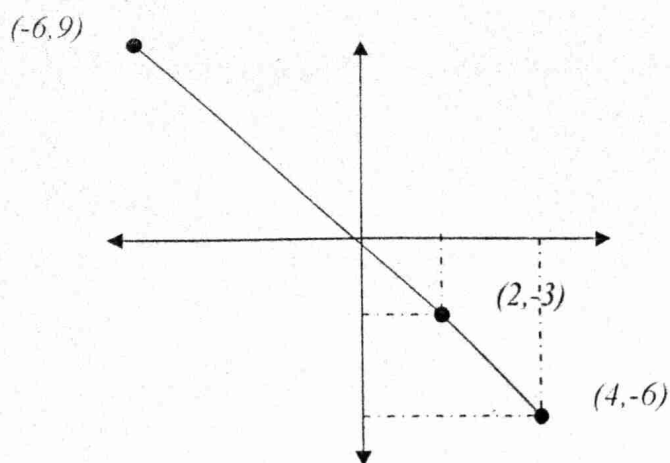
If  $c>0$ , then  $cX$  in the same direction of  $X$

If  $c<0$ , then  $cX$  in the opposite direction of  $X$

**Example:** If  $c=2$ ,  $d=-3$  and  $X=(2, -3)$ , then

$$cX=2(2, -3)=(4, -6) \text{ and } dX=-3(2, -3)=(-6, 9)$$





**Definition:** The vector  $(0,0)$  is called the **zero vector** and is denoted by  $O$ .  
If  $X$  is any vector, then

$$X+O=X.$$

We can also show that

$$X+(-1)X=O.$$

And we can write  $(-1)X$  as  $-X$  and call it the **negative** of  $X$ .

And we write  $X+(-1)Y$  as and call it the difference between  $X$  and  $Y$ . The vector  $X-Y$  is

### The angle between two vectors:

The angle between the nonzero vector  $X=(x_1, y_1)$  and  $Y=(x_2, y_2)$  is the angle  $\theta$ , where  $0 \leq \theta \leq \pi$

Applying the law of cosines to the triangle we have

$$\|X-Y\|^2 = \|X\|^2 + \|Y\|^2 - 2\|X\|\|Y\|\cos\theta \quad \dots (2)$$

From (2)

$$\begin{aligned} \|X-Y\|^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2(x_1x_2 + y_1y_2) \\ &= \|X\|^2 + \|Y\|^2 - 2(x_1x_2 + y_1y_2) \end{aligned}$$

$$\|X\|^2 + \|Y\|^2 - 2(x_1x_2 + y_1y_2) = \|X\|^2 + \|Y\|^2 - 2\|X\|\|Y\|\cos\theta$$

$$\|X\| \neq 0 \text{ and } \|Y\| \neq 0$$

Then

$$\cos\theta = \frac{x_1x_2 + y_1y_2}{\|X\|\|Y\|} \quad \dots(3)$$



**Definition:** The **inner product** or **dot product** of the vectors  $X=(x_1, y_1)$  and  $Y=(x_2, y_2)$  is defined to be :

$$X \cdot Y = x_1 x_2 + y_1 y_2 \quad \dots(4)$$

Thus we can rewrite (3) as

$$\cos \theta = \frac{X \cdot Y}{\|X\| \|Y\|} \quad (0 \leq \theta \leq \pi) \dots(5)$$

**Example:** If  $X=(2,4)$ ,  $Y=(-1,2)$ , then

$$X \cdot Y = (2)(-1) + (4)(2) = 6$$

Also

$$\|X\| = \sqrt{(2)^2 + (4)^2} = \sqrt{20} \text{ And } \|Y\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

Hence

$$\cos \theta = \frac{6}{\sqrt{20} \cdot \sqrt{5}} = 0.6 \text{ Then } \theta = 53^\circ 8' \text{ from table.}$$

**Note:** If the nonzero vectors  $X$  and  $Y$  at right angles, then the cosine of the angle  $\theta$ ,  $\cos \theta = 0$ . Hence  $X \cdot Y = 0$ , conversely if  $X \cdot Y = 0$  then  $\cos \theta = 0$  then  $\theta = \frac{\pi}{2} = 90^\circ$

Thus the nonzero vectors  $X$  and  $Y$  are perpendicular or orthogonal if and only if  $X \cdot Y = 0$ .

**Example:**

$$\text{If } X=(2,-4), Y=(4,2)$$

$$X \cdot Y = 4 \cdot 2 + 2(-4) = 0 \text{ then } X \perp Y$$

## The properties of the dot product

**Theorem:** If  $X, Y$  and  $Z$  are vectors and  $c$  is a scalar, then:

$$(1) X \cdot X = \|X\|^2 \geq 0, \text{ with equality if and only if } X=0,$$

$$(2) X \cdot Y = Y \cdot X,$$

$$(3) (X+Y) \cdot Z = X \cdot Z + Y \cdot Z,$$

$$(4) (cX) \cdot Y = X \cdot (cY) = c(X \cdot Y).$$

**Proof:** H.W.

**Unit vectors:**



A **unit vector** is a vector whose length is 1 and denoted by  $U$ . If  $X$  is any nonzero vector, then the vector

$$U = \frac{1}{\|X\|} \cdot X$$

Is a unit vector in the direction of  $X$ .

**H.W.** prove that for any unit vector  $U$ , then  $\|U\| = 1$ .

**Example:** Let  $X=(-3,4)$ , then

$$\|X\| = \sqrt{(-3)^2 + 4^2} = 5$$

Then the vector  $U$  is

$$U = \frac{1}{5}(-3,4) = \left(-\frac{3}{5}, \frac{4}{5}\right) \text{ is the unit vector.}$$

$$\|U\| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9+16}{25}} = 1$$

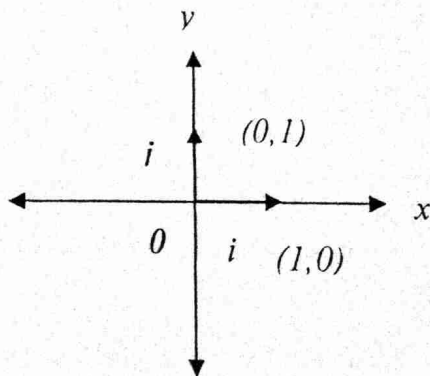
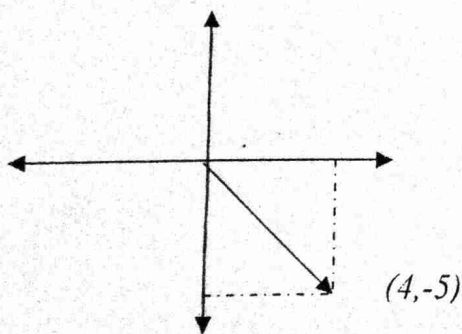
Then  $U$  lies in direction of  $X$ .

Now, there are two unit vectors in  $R^2$ . They are  $i=(1,0)$  and  $j=(0,1)$ .

If  $X=(x,y)$  is any vector in  $R^2$ , then we write  $X$  in the term of  $i$  and  $j$  as

$$X=xi+yj$$

**Example:** If  $X=(4,-5)$  then we can say that  $X=4i-5j$ .

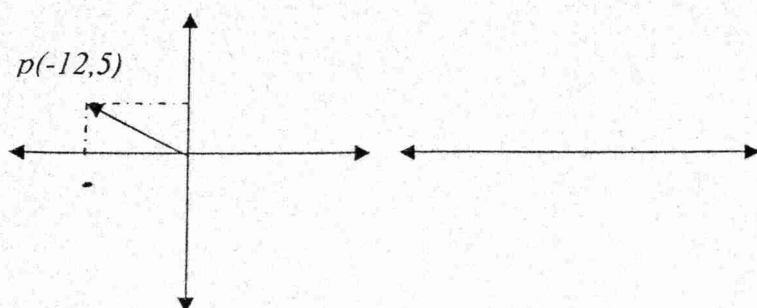


$$X=4(1,0)+(-5)(0,1)=(4,0)+(0,-5)=(4,-5)$$

**Applications:**

Suppose that a force/power of 12 pounds act on a solid in the direction of a negative x-axis and force/power of 5 pounds act on the same solid in the direction of a positive y-axis, find the value and the direction of the magnitude.

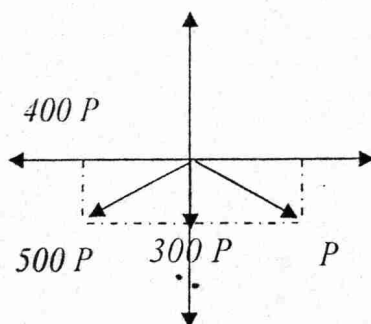
$$\|\vec{OP}\| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$$



**Example:** A ship is being pushed by a tugboat with a force of 300 pounds along the negative y-axis while another tugboat is pushed along the negative x-axis with a force of 400 pounds. Find the magnitude and sketch the direction of the resultant force.

**Solution:**

$$\|\vec{OP}\| = \sqrt{16000 + 90000} = \sqrt{250000} = 500 \text{ pounds.}$$



## Exercises :

- 1- Find  $X+Y$ ,  $X-Y$ ,  $2X$  and  $3X-2Y$  if  $X=(2,3)$ ,  $Y=(-2,5)$ .
- 2- Let  $X=(1,2)$ ,  $Y=(-3,4)$ ,  $Z=(x,4)$  and  $U=(-2,y)$  find  $x$  and  $y$  so that
  - (a)  $Z=2X$
  - (b)  $\frac{3}{2}U=Y$
  - (c)  $Z+U=X$
- 3- Find the length of  $X=(-4,-5)$ .
- 4- Find the distance between  $(0,3)$ ,  $(2,0)$ .



5-Find  $X \cdot Y$ ,  $X=(-2,-3)$ ,  $Y=(2,-1)$ , find the cosine of the angle between  $X, Y$ .

6- which of the vectors  $X=(1,2)$ ,  $Y=(0,1)$ ,  $Z=(-2,-4)$ ,  $W=(-2,1)$ ,  $U=(-6,3)$  are orthogonal, in same direction, in opposite direction

7- Show that if  $Z$  orthogonal to  $X$  and  $Y$  then  $Z$  orthogonal to  $rX + sY$ , where  $r, s$  are scalars.

## n-vectors:

**Definition:** An n-vectors is an  $n \times 1$  matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Where  $x_1, x_2, \dots, x_n$  and real numbers, which are called the **component** of  $X$ .

Since an n-vector is a matrix, the n-vectors

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Are said to be **equal** if  $x_i = y_i$  where  $1 \leq i \leq n$ .

**Example:** The 4-vectors  $\begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$  are not equal.

**Note:** The set of all n-vectors is denoted by  $R^n$  and called **n-space**.

## Vector operations:

**Definition:** Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Be two vectors in  $R^n$ . The **sum** of the vectors X and Y is the vector

$$\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

And it is denoted by  $X+Y$ .

**Example:** If

$$X = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$$

Are vectors in  $R^3$ , then

$$X + Y = \begin{bmatrix} 1+2 \\ -2+3 \\ 3+(-3) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

**Definition:** If  $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  is a vector in  $R^n$  and c a scalar, then the scalar

multiple  $cX$  of X and c is the vector  $\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$ .

**Example:** if  $X = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$  is a vector in  $R^3$  and  $c=-2$ , then  $cX = (-2) \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \end{bmatrix}$ .

**Theorem:** Let X, Y and Z be any vectors in  $R^n$ ; let c and d be any scalars.  
Then:



(a)  $X+Y$  is a vector in  $R^n$  (that is,  $R^n$  is closed under the operation of addition)

(1)  $X+Y=Y+X$ ,

(2)  $X+(Y+Z)=(X+Y)+Z$ ,

(3) There is a unique vector  $0$  in  $R^n$ , where  $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  such that  $X+0=0+X=X$ ,  $0$  is

called **zero vector**.

(4) There is a unique vector  $-X$ , where  $-X = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$  such that  $X+(-X)=0$ .

(b)  $cX$  is a vector in  $R^n$

(1)  $c(X+Y)=cX+cY$ ,

(2)  $(c+d)X=cX+dX$ ,

(3)  $c(dX)=(cd)X$ ,

(4)  $1X=X$ .

**Proof:** (a) and (b) are immediately from the definitions for vector sum and scalar multiple.

We prove that  $(c+d)X=cX+dX$

$$(c+d)X = (c+d) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (c+d)x_1 \\ (c+d)x_2 \\ \vdots \\ (c+d)x_n \end{bmatrix} = \begin{bmatrix} cx_1 + dx_1 \\ cx_2 + dx_2 \\ \vdots \\ cx_n + dx_n \end{bmatrix}$$

$$(4) \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 \\ 1 \cdot x_2 \\ \vdots \\ 1 \cdot x_n \end{bmatrix} = \begin{bmatrix} x_1 \cdot 1 \\ x_2 \cdot 1 \\ \vdots \\ x_n \cdot 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = X$$

**Example:** If  $X$  and  $Y$  are vectors such that  $X = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$  and  $Y = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$  then

$$X - Y = \begin{bmatrix} 1-2 \\ -2-3 \\ 3-(-3) \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 6 \end{bmatrix}$$

**Application:** The vector in  $R^n$  can be used to handle a large amounts of data. Inded a number of computer programming languages.

**Example:** Suppose that a store handles 100 different items. The inventory on hand at the beginning of the week can be described by the inventory vector  $A$  in  $R^{100}$ . The number of item sold at the end of the week can be described by the vector  $S$  and the vector  $A-S$  represents the inventory at the end of the week.

$$A = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{100} \end{bmatrix} \in R^{100}$$

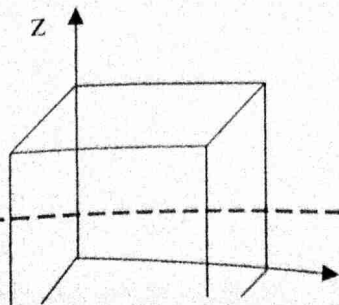
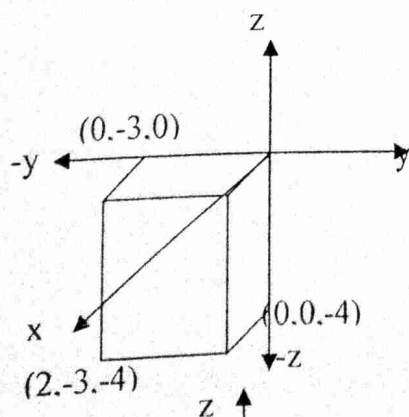
If the store receives a new shipment of goods represented by the vector  $B$ . Then its new inventory would be

$$A-S+B$$

### The space $R^3$ :

We draw the three dimension system by fixing the point called **origin** point then we draw the three coordinate axis

**Example:** Find  $(2,-3,-4)$  and  $(3,5,7)$  on the coordinate system.



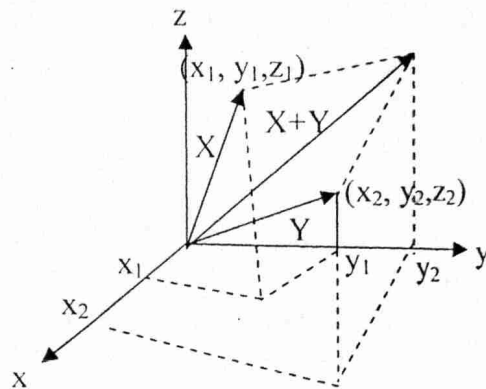


(0,0,7)

(3,5,7)

**Note:** The sum  $X+Y$  of the vectors in  $\mathbb{R}^3$  is the diagonal of the parallelogram determined by  $X$  and  $Y$ .

To illustrate the above note, let  $X=(x_1, y_1, z_1)$  and  $Y=(x_2, y_2, z_2)$  then:



**Definition:** The length or norm of the vector  $X=(x_1, x_2, \dots, x_n)$  in  $\mathbb{R}^n$  is

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

We also define the distance between the points  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  by

$$\|X - Y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

**Example<sup>(\*)</sup>:** Let  $X=(2,3,2,-1)$  and  $Y=(4,2,1,3)$ . Then

$$\|X\| = \sqrt{2^2 + 3^2 + 2^2 + (-1)^2} = \sqrt{18}$$

$$\|Y\| = \sqrt{4^2 + 2^2 + 1^2 + 3^2} = \sqrt{30}$$

$$\|X - Y\| = \sqrt{(2-4)^2 + (3-2)^2 + (2-1)^2 + (-1-3)^2} = \sqrt{22}$$

**Definition:** If  $X=(x_1, x_2, \dots, x_n)$  and  $Y=(y_1, y_2, \dots, y_n)$  are vectors in  $R^n$ , then their **inner product** is defined by:

$$X.Y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Also called **dot product**.

**Example:** If  $X=(2, 3, 2, -1)$  and  $Y=(4, 2, 1, 3)$ , then

$$\begin{aligned} X.Y &= (2)(4) + (3)(2) + (-1)(3) \\ &= 13 \end{aligned}$$

**Example:(Revenue Monitoring):** Consider the store in the above example with (\*), if the vector  $P$  denoted the price of each of the 100 items, then the dot product  $S.P$  given the total revenue received at the end of the week

$$S.P = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{100} \end{bmatrix} \begin{bmatrix} price \\ price \\ \vdots \\ price \end{bmatrix}$$

**Theorem(Properties of the Inner product):**

If  $X, Y$  and  $Z$  are vectors in  $R^n$  and  $c$  is a scalar then:

$$1. X.X = \|X\|^2 \geq 0, \text{ with equality if and only if } X=0,$$

$$2. X.Y = Y.X$$

$$3. (X+Y).Z = X.Z + Y.Z$$

$$4. (cX).Y = X.(cY) = c(X.Y)$$

**Theorem(Cauchy-Schwarz Inequality):**

If  $X$  and  $Y$  are vectors in  $R^n$ , then

$$|X.Y| \leq \|X\| \|Y\| \dots (3)$$

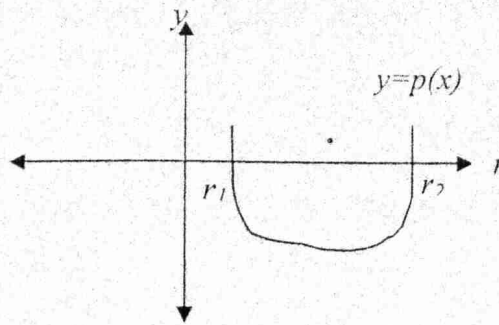
**Proof:** If  $X=0$ , then  $\|X\|=0$  and  $X.Y=0$ , so hold.

Now, suppose that  $X$  and  $Y$  are nonzero. Let  $r$  be a scalar and consider the vector  $rX+Y$ . Then

$$\begin{aligned} 0 \leq (rX+Y).(rX+Y) &= r^2 X.X + 2rX.Y + Y.Y \\ &= r^2 a + 2rb + c \end{aligned}$$



Where  $a=X.X$ ,  $b=X.Y$  and  $c=Y.Y$



Now ,  $p(r)=ar^2+2br+c$

$P(r)$  is a quadratic polynomial in  $r$  (whose graph is a parabola opening upward, since  $a>0$ )

That is nonzero for all values of  $r$ ? Why?

This means that either this polynomial has no real roots, or it has real roots then both roots are equal(why?)

**(The answer) :** if  $p(r)$  had two distinct root  $r_1$  and  $r_2$ , then it would be negative for some value of  $r$ .

Recall that the roots of  $p(r)$  are given by quadratic formula as  $\frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a}$

and  $\frac{-2b - \sqrt{4b^2 - 4ac}}{2a}$  (where  $a \neq 0$  since  $X \neq 0$ ). Thus we must have

$$4b^2 - 4ac \leq 0$$

$$4b^2 \leq 4ac$$

Which means that :

$$b^2 \leq ac$$

Taking square roots of both side and observing that  $b \leq \sqrt{a}\sqrt{c}$  Where

$$\sqrt{a} = \sqrt{X.X} = \|X\| \text{ and } \sqrt{c} = \sqrt{Y.Y} = \|Y\|$$

Thus

$$|X.Y| \leq \|X\| \|Y\|.$$

**Example:** If  $X=(1,2,-1,2)$  and  $Y=(3,1,-1,2)$  then:

$$\|X\| = \sqrt{10}, \|Y\| = \sqrt{15} \text{ and } |X.Y| = 10 \leq \sqrt{10}\sqrt{15}.$$

**Definition:** The angle between two nonzero vectors  $X$  and  $Y$  is defined as the unique number  $\theta$ ,  $0 \leq \theta \leq \pi$  such that :

$$\cos \theta = \frac{X \cdot Y}{\|X\| \|Y\|}$$

It follows from the Cauchy-Schwarz inequality that :

$$\left| \frac{X \cdot Y}{\|X\| \|Y\|} \right| \leq 1.$$

**Example:** Let  $X=(1,0,0,1)$  and  $Y=(0,1,0,1)$  then we have that

$$\|X\| = \sqrt{2}, \|Y\| = \sqrt{2} \text{ and } X \cdot Y = 1$$

Thus

$$\cos \theta = \frac{1}{2} \text{ and } \theta = 60^\circ$$

**Definition:** Two nonzero vector  $X$  and  $Y$  in  $\mathbb{R}^n$  are said to be **Orthogonal** if  $X \cdot Y = 0$ . They are said **Parallel** if  $|X \cdot Y| = \|X\| \|Y\|$ . They are in the **Same direction** if  $X \cdot Y = \|X\| \|Y\|$ . That is, they are orthogonal if  $\cos \theta = 0$ , parallel if  $\cos \theta = \pm 1$ , and in the same direction if  $\cos \theta = 1$ .

**Example:** Consider the vectors  $X=(1,0,0,1)$ ,  $Y=(0,1,0,1)$  and  $Z=(3,0,0,3)$  then  $X \cdot Y = 0$  and  $Y \cdot Z = 0$  (check).

Which implies that  $X$  and  $Y$  are orthogonal and  $Y$  and  $Z$  are orthogonal too.

Also  $X \cdot Z = 6$ ,  $\|X\| = \sqrt{2}$ ,  $\|Z\| = \sqrt{18}$ , and  $X \cdot Z = \|X\| \|Z\|$

Hence  $X$  and  $Z$  are in the same direction.

**Theorem:**(Triangle Inequality) If  $X$  and  $Y$  are vectors in  $\mathbb{R}^n$ , then

$$\|X + Y\| \leq \|X\| + \|Y\|$$

**Proof:**By theorem (\*)

$$\begin{aligned} \|X + Y\|^2 &= (X + Y) \cdot (X + Y) \\ &= X \cdot X + 2(X \cdot Y) + Y \cdot Y \\ &= \|X\|^2 + 2(X \cdot Y) + \|Y\|^2 \end{aligned}$$

By the Cauchy-Schwarz inequality we have:

$$\|X\|^2 + 2(X \cdot Y) + \|Y\|^2 \leq \|X\|^2 + 2\|X\| \|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2.$$

**Example:** Let  $X=(1,0,0,1)$  and  $Y=(0,1,0,1)$  then

$$\|X + Y\| = \sqrt{4} = 2 < \sqrt{2} + \sqrt{2} = \|X\| + \|Y\|$$

**Note:**If  $X$  and  $Y$  are vectors in  $\mathbb{R}^n$ , then

$$\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$$

If and only if  $X$  and  $Y$  are orthogonal.

**Definition:** A **Unit vector**  $U$  in  $R^n$  is a vector of unit length. Furthermore, if  $X$  is a nonzero vector, then the vector

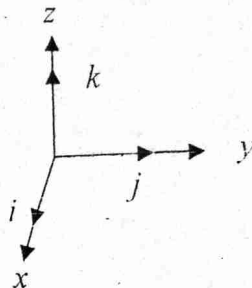
$$U = \left[ \frac{1}{\|X\|} \right] \cdot X$$

Is a unit vector in the direction of  $X$ .

**Example:** If  $X = (1, 0, 0, 1)$ , then  $\|X\| = \sqrt{2}$  and  $U = \frac{1}{\sqrt{2}} (1, 0, 0, 1)$  is a unit vector in the direction of  $X$ .

In the case of  $R^3$  the unit vector in the position direction are  $i = (1, 0, 0)$ ,  $j = (0, 1, 0)$  and  $k = (0, 0, 1)$ .

If  $X = (x, y, z)$ , then  $X = xi + yj + zk$ .



**Example:** if  $X = (2, -1, 3)$ , then  
 $X = 2i - j + 3k$

In  $R^n$  the unit vector are

$$E_1 = (1, 0, 0, \dots, 0), E_2 = (0, 1, 0, \dots, 0), \dots, E_n = (0, 0, \dots, 1)$$

And if  $X = (x_1, x_2, \dots, x_n)$  is a vector in  $R^n$ , we have

$$X = x_1 E_1 + x_2 E_2 + \dots + x_n E_n$$

## Exercises :

1- Find  $X+Y$ ,  $X-Y$ ,  $2X$  and  $3X-2Y$  if  $X = (2, 3, 5)$ ,  $Y = (-2, 5, 3)$ .

2- Let  $X = (1, 2, 2, 1)$ ,  $Y = (-3, 4, -2, -1)$ ,  $Z = (x, 4, 0, y)$  and  $U = (-2, u, v, 4)$  find  $x, u, v$  and  $y$  so that

(a)  $Z = 3X$

(b)  $Z - Y = Y$

(c)  $Z + U = X$

3- Find the length of  $X = (1, 6, -4, -5)$ .

4- Find the distance between  $(0, 3, 2)$ ,  $(2, 0, 4)$ .



5- Find  $X \cdot Y$ ,  $X=(-2,-3,-4)$ ,  $Y=(2,-1,2)$ , find the cosine of the angle between  $X, Y$ .

6- which of the vectors  $X=(4,2,6,-8)$ ,  $Y=(-2,3,-1,-1)$ ,  $Z=(-2,-1,-3,4)$ ,  $W=(1,0,0,2)$  are orthogonal, in same direction, parallel.

7- Prove the parallelogram law.  $\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2$

## Cross product in $R^3$

**Definition:** if  $X = x_1i + x_2j + x_3k$  and  $Y = y_1i + y_2j + y_3k$  are two vectors in  $R^3$ , then their cross product is the vector  $X \times Y$  defined by:

$$X \times Y = (x_2y_3 - x_3y_2)i + (x_3y_1 - x_1y_3)j + (x_1y_2 - x_2y_1)k$$

The cross product  $X \times Y$  can also be written as:

$$X \times Y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$X \times Y$  is a vector (from definition)

So

$$X \times Y = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$

**Example:** Let  $X=2i+j+2k$  and  $Y=3i-j-3k$ . Then

$$X \times Y = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix} = -i + 12j - 5k.$$

## Properties of cross product:

**Theorem:** If  $X$ ,  $Y$  and  $Z$  are vectors and  $c$  is a scalar, then :

- 1-  $X \times Y = -(Y \times X)$
- 2-  $X \times (Y + Z) = X \times Y + X \times Z$
- 3-  $(X + Y) \times Z = X \times Z + Y \times Z$
- 4-  $c(X \times Y) = (cX) \times Y = X \times (cY)$
- 5-  $X \times X = 0$
- 6-  $X \times 0 = 0 \times X = 0$

$$7- (X \times Y) \times Z = (Z \cdot X)Y - (Z \cdot Y)X$$

$$\text{Also } X \times (Y \times Z) = (X \cdot Z)Y - (X \cdot Y)Z$$

$$8- (X \times Y) \cdot Z = X \cdot (Y \times Z)$$

**Proof:**

$$X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \text{ and } Z = (z_1, z_2, z_3)$$

$$X \times Y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \underbrace{(x_2 y_3 - x_3 y_2)}_{u_1} i + \underbrace{(x_3 y_1 - x_1 y_3)}_{u_2} j + \underbrace{(x_1 y_2 - x_2 y_1)}_{u_2} k$$

$$(X \times Y) \times Z = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = (u_2 z_3 - u_3 z_2) i + (u_3 z_1 - u_1 z_3) j + (u_1 z_2 - u_2 z_1) k$$

Substituted instead of u's then we get:

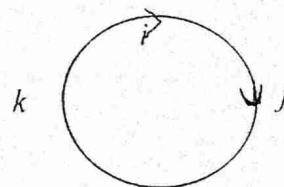
$$= [(x_3 y_1 - x_1 y_3) z_3 - (x_1 y_2 - x_2 y_1) z_2] i + [(x_1 y_2 - x_2 y_1) z_1 - (x_2 y_3 - x_3 y_2) z_3] j + [(x_2 y_3 - x_3 y_2) z_2 - (x_3 y_1 - x_1 y_3) z_1] k$$

**Example:** From definition we have that:

$$i \times i = j \times j = k \times k = 0 \text{ and } i \times j = k, j \times k = i, k \times i = j,$$

Also

$$j \times i = -k, k \times j = -i, i \times k = -j$$



**Example:** Let  $X = 2i + j + 2k$ ,  $Y = 3i - j - 3k$  and  $Z = i + 2j + 3k$

$$X \times Y = -i + 12j - 5k \quad (X \times Y) \cdot Z = 8$$

$$Y \times Z = 3i - 12j + 7k \quad X \cdot (Y \times Z) = 8$$

Which illustrate equation (8). Also to illustrate (\*)

$$X \times (Y \times Z) = 31i - 8j - 27k, \quad X \cdot Z = 10, \quad X \cdot Y = -1$$

$$(X \cdot Z)Y = 30i - 10j - 30k, \quad (X \cdot Y)Z = -i - 2j - 3k$$

Hence

$$(X \cdot Z)Y - (X \cdot Y)Z = 31i - 8j - 27k$$

Also from (8), (1) and (5) we get:

$$(X \times Y) \cdot Y = X \cdot (Y \times Y) = X \cdot 0 = 0$$

And

$$(X \times Y) \cdot X = -(Y \times X) \cdot X = -Y \cdot (X \times X) = -Y \cdot 0 = 0$$

**Note:** If  $X \times Y \neq 0$ , then  $X \times Y$  is orthogonal to both  $X$  and  $Y$  and to the plane determined by them.

**To find the angle between  $X$  and  $Y$  :**

$$\begin{aligned}\|X \times Y\|^2 &= (X \times Y) \cdot (X \times Y) \\ &= X \cdot [Y \times (X \times Y)] \text{ by (8)} \\ &= X \cdot [(Y \cdot Y)X - (Y \cdot X)Y] \text{ by (7)} \\ &= (X \cdot X)(Y \cdot Y) - (Y \cdot X)(Y \cdot X) \text{ by (2) and (4)} \\ &= \|X\|^2 \|Y\|^2 - (X \cdot Y)^2 \text{ by (1)}\end{aligned}$$

$$\begin{aligned}X \times Y &= \|X\| \|Y\| \sin \theta \\ \|X \times Y\|^2 &= \|X\|^2 \|Y\|^2 \sin^2 \theta \\ &= \|X\|^2 \|Y\|^2 (1 - \cos^2 \theta) \\ \therefore \|X \times Y\|^2 &= \|X\|^2 \|Y\|^2 \sin^2 \theta \\ \therefore \|X \times Y\| &= \|X\| \|Y\| \sin \theta \quad \dots (9)\end{aligned}$$

**Note:** (1) We do not have  $|\sin \theta|$ , since  $\sin \theta$  is nonnegative for  $0 \leq \theta \leq \pi$   
 (2)  $X$  and  $Y$  are not parallel if and only if  $X \times Y \neq 0$

## Applications:

### Area of triangle:

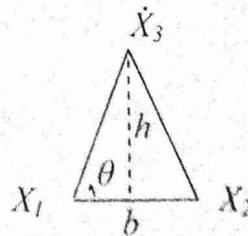
Area of triangle consider a triangle with vertices  $X_1, X_2$ , and  $X_3$ . The area of this triangle is

$$A_T = \frac{1}{2}bh$$

Where  $b$  is the base and  $h$  is the height.

$$b = \|X_2 - X_1\| \text{ and } h = \|X_3 - X_1\| \sin \theta$$

Where





$$\sin \theta = \frac{h}{\|X_3 - X_1\|} \text{ so } A_T = \frac{1}{2} \|X_2 - X_1\| \|X_3 - X_1\| \sin \theta$$

From (9)

$$A_T = \frac{1}{2} \|(X_2 - X_1) \times (X_3 - X_1)\|$$

**Example:** Find the area of the triangle with vertices  $X_1=(2,2,4)$ ,  $X_2=(-1,0,5)$  and  $X_3=(3,4,3)$

**Solution:**

$$X_2 - X_1 = -3i - 2j + k$$

$$X_3 - X_1 = i + 2j - k$$

$$A_T = \frac{1}{2} \|(-3i - 2j + k) \times (i + 2j - k)\|$$

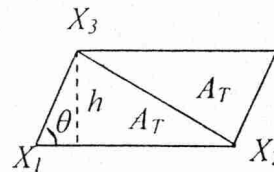
$$= \frac{1}{2} \|-2j - 4k\|$$

$$= \|-j - 2k\| = \sqrt{5}$$

**Area of a parallelogram:**

The area  $A_P$  of parallelogram with sides  $X_2 - X_1$  and  $X_3 - X_1$  is  $2A_T$ .

$$A_P = \|(X_2 - X_1) \times (X_3 - X_1)\|$$



**Example:** If  $X_1=(2,2,4)$ ,  $X_2=(-1,0,5)$  and  $X_3=(3,4,3)$ .

Then

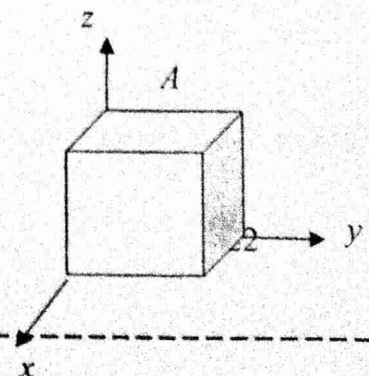
$$A_P = 2\sqrt{5} \text{ (check).}$$

Consider the parallelepiped with vertex at the origin and edges  $X$ ,  $Y$  and  $Z$ . The volume of the parallelepiped is the product of the area of the face containing  $Y$ ,  $Z$  and the distance  $h$  from the face parallel to it.

$$h = \|X\| \cos \theta$$

$\theta$  between  $X$  and  $Y \times Z$ , the area of the face determined by  $Y$  and  $Z$  is  $\|Y \times Z\|$

$$\therefore V = \|Y \times Z\| \|X\| \cos \theta = |X \cdot (Y \times Z)|$$



**Example:**  $X=i-2j+3k$ ,  $Y=i+3j+k$  and  $Z=2i+j+2k$  (H.W.)

Exercises :

1-Show that  $X$  and  $Y$  are parallel iff  $X \times Y = 0$  .

2-Show that  $\|X \times Y\|^2 + (X \cdot Y)^2 = \|X\|^2 \|Y\|^2$

3-Prove the Jacobi identity :

$$(X \times Y) \times Z + (Y \times Z) \times X + (Z \times X) \times Y = 0$$

### Vector space

In the following lectures we study the vector space , subspace , study the linear Independence , basis and the rank of a matrix .

#### Definition :

A real vector space is a set  $V$  of elements with two operations  $\oplus$  and  $\odot$  defined with the following properties .

(a) If  $X$  and  $Y$  are any elements in  $V$  . then  $X \oplus Y$  is in  $V$  (that is closed under the operation  $\oplus$  ) .

1-  $X \oplus Y = Y \oplus X$  for all  $X, Y$  in  $V$  .

2-  $X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$  for all  $X, Y, Z$  in  $V$

3- There is a unique element  $0$  in  $V$  such that  $X \oplus 0 = 0 \oplus X = X$  for every  $X$  in  $V$  .

4- For each  $X$  in  $V$  there exists a unique  $-X$  in  $V$  such that  $X \oplus -X = 0$

(b) If  $X$  is any element in  $V$  and  $c$  is any real number then  $c \odot X$  is in  $V$  .

5-  $c \odot (X \oplus Y) = c \odot X \oplus c \odot Y$  for any  $X, Y$  in  $V$  , and any real number  $c$ .

6-  $(c+d) \odot X = c \odot X \oplus d \odot X$  for any  $X$  in  $V$  and any real numbers  $c$  and  $d$  .

7-  $c \odot (d \odot X) = (cd) \odot X$  for any  $X$  in  $V$  and real numbers  $c$  and  $d$  .

$8-1 \oplus X = X$  for any  $X$  in  $v$ .

---

$(V, \oplus, \odot)$  is vector space. The operation  $\oplus$  is called vector addition.

The operation  $\odot$  is called scalar multiplication.

The vector  $0$  is called Zero vector.

### Example 1:

Let  $R^n$  be the set of ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  where we define

$\oplus$  by  $(a_1, a_2, \dots, a_n) \oplus (b_1, b_2, \dots, b_n)$

$= (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$  and  $\odot$  by  $c \odot (a_1, a_2, \dots, a_n)$

$= (ca_1, ca_2, \dots, ca_n)$

$R^n$  is a vector space.

### Example 2:

Let  $V$  be the set of ordered triples of real number  $(a_1, a_2, 0)$  where we define  $\oplus$  by  $(a_1, a_2, 0) \oplus (b_1, b_2, 0)$

$= (a_1 + b_1, a_2 + b_2, 0)$  and  $\odot$  by  $c \odot (a_1, a_2, 0) = (ca_1, ca_2, 0)$

$V$  is a vector space.

### Example 3:

Let  $V$  be the set of ordered triples of real number  $(x, y, z)$  where we define

$\oplus$  by  $(x, y, z) \oplus (x', y', z')$

$= (x + x', y + y', z + z')$  and  $\odot$  by  $c \odot (x, y, z) = (cx, y, z)$

$V$  is not vector space the property  $(c+d) \odot X = c \odot X \oplus d \odot X$  fails to hold

thus  $(c+d) \odot (x, y, z) = ((c+d)x, y, z),$

On other hand  $c \odot (x, y, z) \oplus d \odot (x, y, z) = (cx, y, z) \oplus (dx, y, z)$

$= (cx + dx, y + y, z + z) = ((c+d)x, 2y, 2z).$

### Example 3:

Let  $V$  be the set of  $2 \times 3$  matrices under usual operation of matrix addition and scalar multiplication

$V$  is vector space c.h.



**Example 4:**

Let  $V$  be the set of all real-valued function on  $\mathbb{R}$ . if  $f$  and  $g$  are in  $V$  we define  $f \oplus g$  by  $(f \oplus g)(t) = f(t) \oplus g(t)$  and if  $f$  and  $c$  is a scalar define  $c \odot f$  by  $c \odot f = c f(t)$ .  
 $V$  is vector space c.h.

**Example 5:**

Let  $p_n$  be the set of all real polynomials of degree  $\leq n$  with zero polynomial. if  $p(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n$  and  $q(t) = b_0 t^n + b_1 t^{n-1} + \dots + b_{n-1} t + b_n$  are in  $V$  we define  $p(t) \oplus q(t)$  by  $p(t) \oplus q(t) = (a_0 + b_0)t^n + (a_1 + b_1)t^{n-1} + \dots + (a_{n-1} + b_{n-1})t + (a_n + b_n)$  and if  $c$  is a scalar define  $c \odot p(t)$  by

$$c \odot p(t) = (ca_0)t^n + (ca_1)t^{n-1} + \dots + ca_{n-1}t + ca_n$$

the above definition show that the degree of  $p(t) \oplus q(t)$  and  $c \odot p(t) \leq n$

$-p(t) = -a_0 t^n - a_1 t^{n-1} + \dots - a_{n-1} t - a_n$  is negative of  $p(t)$  and since  $a_i + b_i = b_i + a_i$  then  $p(t) \oplus q(t) = q(t) + p(t)$

And

$$\begin{aligned} (c+d) \odot p(t) &= (c+d)a_0 t^n + (c+d)a_1 t^{n-1} + \dots + (c+d)a_{n-1} t + (c+d)a_n \\ &= c(a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n) + d(a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n) \\ &= c \odot p(t) \oplus d \odot p(t) \end{aligned}$$

$V$  is vector space c.h.

**Theorem :**

If  $V$  is a vector space then .

- 1-  $0 \odot X = 0$  for any vector  $X$  in  $V$
- 2-  $c \odot 0 = 0$  for any scalar  $c$
- 3- If  $c \odot X = 0$  then either  $c = 0$  or  $X = 0$
- 4-  $(-1) \odot X = -X$  for any  $X$  in  $V$ .

Proof:

$$1) 0X = (0+0)X = 0X + 0X \text{ by (6) of def. adding } -0X$$

$$0 = 0X + (-0X) = (0X + 0X) + (-0X)$$

$$= 0X + [0X + (-0X)]$$

$$=0X+0=0X.$$

$$\begin{aligned} 2) \quad c.0 &= c.(0+0) = c.0 + c.0 \\ c.0 - c.0 &= c.0 + c.0 - c.0 \\ 0 &= c.0 \end{aligned}$$

3) suppose  $cX=0$  and  $c \neq 0$  then

$$0 = \left(\frac{1}{c}\right).0 = \left(\frac{1}{c}\right)(cX) = \left[\left(\frac{1}{c}\right)c\right]X = 1.X$$

$$4) \quad (-1)X + X = (-1)X + (1)X = (-1+1)X = 0X = 0 \text{ so that } (-1)X = -X$$

### Definition :

Let  $V$  be a vector space and  $W$  a nonempty subset of  $V$  if  $W$  is a vector space with respect to the same operations as these in  $V$ , then  $W$  is called a **subspace** of  $V$ .

Example : If  $(V, \oplus, \odot)$  is vector space then  $\{0\} \subseteq V, V \subseteq V$  are two subspaces.

### Example :

Let  $W$  be the set of ordered triples of real number  $(a_1, a_2, 0)$  where we define  $\oplus$  by  $(a_1, a_2, 0) \oplus (b_1, b_2, 0)$

$$= (a_1 + b_1, a_2 + b_2, 0) \text{ and } \odot \text{ by } c \odot (a_1, a_2, 0) = (ca_1, ca_2, 0)$$

Then  $(W, \oplus, \odot)$  is subspace of  $(R^3, \oplus, \odot)$ .

### Theorem:

Let  $(V, \oplus, \odot)$  be a vector space and let  $W$  be a nonempty subset of  $V$ .  $W$  is a subspace of  $V$  if and only if the following condition hold

1- If  $X, Y$  are any vectors in  $W$  then  $X \oplus Y$  is in  $W$

2- If  $c$  is any real number and  $X$  is any vector in  $W$  then  $c \odot X$  is in  $W$ .

Proof: H.W.

**Example:**

Let  $W$  be the set of all  $2 \times 3$  matrices of form

$$W = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}, a, b, c \in R \right\}, W \text{ is subset of vector space } V \text{ of all}$$

$2 \times 3$  matrices under usual operations of matrices addition and scalar multiplication then  $W$  is subspace of  $V$ .

Solution:

$$\text{Consider } X = \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix} \text{ in } W \text{ then}$$

$$X+Y = \begin{bmatrix} a_1+a_2 & b_1+b_2 & 0 \\ 0 & c_1+c_2 & d_1+d_2 \end{bmatrix} \text{ is in } W \text{ also let } r \in R$$

$$rX = \begin{bmatrix} ra_1 & rb_1 & 0 \\ 0 & rc_1 & rd_1 \end{bmatrix} \text{ is in } W, W \text{ is subspace of } V.$$

**Example:**

Let  $W$  be the sub set of  $(R^3, \oplus, \odot)$ .

$W$  is ordered triples of real number  $(a, b, 1)$ ,

let  $X = (a_1, a_2, 1), Y = (b_1, b_2, 1)$

$X+Y = (a_1+b_1, a_2+b_2, 2)$  Then  $W$  is not subspace of  $(R^3, \oplus, \odot)$ .

**Example 5:**

Let  $W$  be the set of all real polynomials of degree exactly=2

$W$  is subset of  $p_2$  but not subspace of  $p_2$  since

$2t^2+3t+1$  and  $-2t^2+t+2$  is polynomial of degree 1 is not in  $W$ .



**Exercises:**

- 1- Let  $W = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, a = 2c + 1 \right\}$   $W$  is subset of vector space  $V$  of all  $2 \times 3$  matrices under usual operations of matrices addition and scalar multiplication is  $W$  is subspace of  $V$ .
- 2- Let  $W = \{(a, b, c), b = 2a + 1\}$  subset of vector space  $R^3$  is  $W$  is subspace?

Definition( 1 - 7 )

Let  $X_1, X_2, \dots, X_n$  be vectors in a vectors space  $V$ . A vector  $X$  in  $V$  is called linear combination of this vectors if it can written as  $X = c_1 X_1 + c_2 X_2 + \dots + c_n X_n$  for some real number where  $c_1, c_2, \dots, c_n$  are scalars.

**Example:** Consider the vector space  $R^4$ . let  $X_1 = (1, 2, 1, -1)$ ,  $X_2 = (1, 0, 2, -3)$ ,  $X_3 = (1, 1, 0, -2)$  the vector  $X = (2, 1, 5, -5)$  is linear combination of  $X_1, X_2, X_3$  if we find  $c_1, c_2, c_3$  s.t..

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$(2, 1, 5, -5) = c_1(1, 2, 1, -1) + c_2(1, 0, 2, -3) + c_3(1, 1, 0, -2)$$

$$(2, 1, 5, -5) = (c_1, 2c_1, c_1, -c_1) + (c_2, 0, 2c_2, -3c_2) + (c_3, c_3, 0, -2c_3)$$

$$c_1 + c_2 + c_3 = 2$$

$$2c_1 + c_3 = 1$$

$$c_1 + 2c_2 = 5$$

$$-c_1 - 3c_2 - 2c_3 = -5$$

solving this linear system by Gauss-Jordan we obtain  $c_1=1, c_2=2, c_3=-1$

then  $X$  is linear combination of  $X_1, X_2, X_3$

**Example:** Consider the vector space  $\mathbb{R}^3$ . let  $X_1=(1, 2, -1)$ ,  $X_2=(1, 0, -1)$ , is the vector  $X=(1, 0, 2)$  is linear combination of  $X_1, X_2$  if we find  $c_1, c_2$  s.t..

$$X = c_1 X_1 + c_2 X_2$$

$$(1, 0, 2) = c_1(1, 2, -1) + c_2(1, 0, -1)$$

$$c_1 + c_2 = 1$$

$$2c_1 = 0$$

$$-c_1 - 2c_2 = 2$$

Which has no solution then  $X$  is not linear combination of  $X_1, X_2$ .

**Example:** Consider the vector space  $\mathbb{R}^3$ . let  $X_1=(1, 0, 1)$ ,  $X_2=(-1, 1, 0)$ ,  $X_3=(0, 0, 1)$  is the vector  $X=(1, 1, 1)$  is linear combination of  $X_1, X_2, X_3$  if we find  $c_1, c_2, c_3$  s.t..

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$(1, 1, 1) = c_1(1, 0, 1) + c_2(-1, 1, 0) + c_3(0, 0, 1)$$

$$c_1 - c_2 = 1$$

$$c_2 = 1$$

$$c_1 + c_3 = 1$$

solving this linear system by Gauss-Jordan we obtain  $c_1=2, c_2=1, c_3=-1$   
then  $X$  is linear combination of  $X_1, X_2, X_3$

Definition :

Let  $S = \{X_1, X_2, \dots, X_n\}$  be the set of vectors in a vector space  $V$ . the set spans  $V$ , or  $V$  is spanned by  $S$ , if every vector in  $V$  is a linear combination of vector in  $S$ .

Example :

Let  $V$  be the vector space  $R^3$ . let  $S = \{X_1, X_2, X_3\}$  set of vectors where  $X_1=(1,2,1), X_2=(1,0,2), X_3=(1,1,0)$  is the set  $S$  spans  $V$ ?

SOL.:

Let  $X=(a,b,c)$  be any vector in  $R^3$ . and  
$$X = c_1X_1 + c_2X_2 + c_3X_3$$

$$(a,b,c) = c_1(1,2,1) + c_2(1,0,2) + c_3(1,1,0)$$

$$c_1 + c_2 + c_3 = a$$

$$2c_1 + c_3 = b$$

$$c_1 + 2c_2 = c$$

solving this linear system by Gauss-Jordan we obtain

$$c_1 = \frac{-2a+2b+c}{3}, c_2 = \frac{a-b+c}{3}, c_3 = \frac{4a-b-2c}{3}$$

since we obtained a solution for every choice of  $a, b$  and  $c$  then  
 $S = \{X_1, X_2, X_3\}$  spans  $V$ .

Example :

Let  $V$  be the vector space  $R^3$ .  $S = \{i, j, k\}$  spans  $V$   
Since for every  $X=(a,b,c)$  vector in  $R^3$ .

$$(a,b,c) = (a,0,0) + (0,b,0) + (0,0,c) = ai + bj + ck.$$

Example :



Let  $V=P_2$  be the vector space all polynomials of degree  $\leq 2$  if and  
 Let  $S=\{P_1(t), P_2(t)\}$  where  $P_1(t)=t^2+2t+1$  and  $P_2(t)=t^2+2$  is  $S$   
 spans  $V$ ?

Sol ; let  $P(t)= at^2+bt+c$  polynomial in  $P_2$  where  $a,b,c$  are real  
 number suppose  $P(t)=c_1P_1(t)+c_2P_2(t)$  then

$$at^2+bt+c=c_1(t^2+2t+1)+c_2(t^2+2)$$

$$at^2+bt+c=(c_1+c_2)t^2+2c_1t+(c_1+2c_2)$$

$$c_1+c_2=a$$

$$2c_1=b$$

$$c_1+2c_2=c$$

we obtain

$$\begin{bmatrix} 1 & 1 & : & a \\ 2 & 0 & : & b \\ 1 & 2 & : & c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & : & 2a-b \\ 0 & 1 & : & c-a \\ 0 & 0 & : & b-4a+2c \end{bmatrix}$$

If  $b-4a+2c \neq 0$  then there is no solution to this system hence  $S$  does not  
 span  $P_2$ .

## Linear independence

**Definition** :Let  $S = \{X_1, X_2, \dots, X_n\}$  be the set of vectors in a  
 vectors space  $V$ . then  $S$  is said to be linearly dependent if there exist  
 constants  $c_1, c_2, \dots, c_n$  not all zero, such that

$c_1X_1+c_2X_2, \dots, c_nX_n=0$ , other wise .  $S$  is called linearly independent

That is  $S$  is linearly independent if the equation

$c_1X_1+c_2X_2, \dots, c_nX_n=0$  hold only if  $c_1=c_2=\dots=c_n=0$

**Example:** Consider the vector space  $R^4$ . let  $X_1=(1,0,1,2)$ ,  
 $X_2=(0,1,1,2)$ ,  $X_3=(1,1,1,3)$  is  $S = \{X_1, X_2, X_3\}$  is linearly  
 independent

Sol;

Let  $c_1X_1+c_2X_2+c_3X_3=0$  where  $c_1, c_2, c_3 \in \mathbb{R}$

$$c_1(1,0,1,2) + c_2(0,1,1,2) + c_3(1,1,1,3) = (0,0,0,0)$$

$$(c_1, 0, c_1, 2c_1) + (0, c_2, c_2, 2c_2) + (c_3, c_3, c_3, 3c_3) = (0,0,0,0)$$

$$c_1 + c_3 = 0$$

$$c_2 + c_3 = 0$$

$$c_1 + c_2 + c_3 = 0$$

$$2c_1 + 2c_2 + 3c_3 = 0$$

we obtain  $c_1=0, c_2=0, c_3=0$  then S is linearly independent.

Example :

Let V be the vector space  $\mathbb{R}^3$ . let  $S=\{X_1, X_2, X_3, X_4\}$  set of vectors where  $X_1=(1,2,-1)$ ,  $X_2=(1,-2,1)$ ,  $X_3=(-3,2,-1)$ ,  $X_4=(2,0,0)$  is the set S linearly independent ?

SOL.:

$$\text{Let } c_1X_1+c_2X_2+c_3X_3+c_4X_4=0$$

$$c_1(1,2,-1)+c_2(1,-2,1)+c_3(-3,2,-1)+c_4(2,0,0)=0$$

$$c_1 + c_2 - 3c_3 + 2c_4 = 0$$

$$2c_1 - 2c_2 + 2c_3 = 0$$

$$-c_1 + c_2 - c_3 = 0$$

There are infinitely many solution like  $c_1=1, c_2=2, c_3=1, c_4=0$ , then S is linearly dependent.

Example :

Let V be the vector space  $\mathbb{R}^3$ .  $S=\{i, j, k\}$  is linearly independent.

Since

$$(0,0,0)=(c_1,0,0)+(0,c_2,0)+(0,0,c_3)$$

$$\text{Then } c_1=0, c_2=0, c_3=0$$

In fact  $E_1, E_2, \dots, E_n$  are linearly independent in  $\mathbb{R}^n$ .

Example:

Let  $V=P_2$  be the vector space all polynomials of degree  $\leq 2$  if and  
 Let  $S=\{P_1(t), P_2(t), P_3(t)\}$  where  $P_1(t)=t^2+t+2$  and  $P_2(t)=2t^2+t$ ,  
 $P_3(t)=3t^2+2t+2$  is  $S$  is linearly independent?

$$\text{Sol : } c_1 P_1(t) + c_2 P_2(t) + c_3 P_3(t) = 0$$

$$c_1(t^2+t+2) + c_2(2t^2+2) + c_3(3t^2+2t+2) = 0$$

then

$$c_1 + 2c_2 + 3c_3 = 0$$

$$c_1 + c_2 + 2c_3 = 0$$

$$2c_1 + 2c_3 = 0$$

There are infinitely many solution like  $c_1=1, c_2=1, c_3=-1$ , then  $S$  is linearly dependent.

Remark 1: If  $S = \{X_1, X_2, \dots, X_n\}$  be the set of vectors and  $A$  matrix whose columns are these vectors.  $S$  is linearly independent set  
 Iff  $|A| \neq 0$ .

Remark 2: If  $S = \{X_1, X_2, \dots, X_n\}$  be the set of vectors and  $X$  is zero vector then  $S$  is linearly dependent set. Why?

Remark 3: Let  $S_1, S_2$  be finite subsets of vector space  $V$  and  $S_1$  subset of  $S_2$  then

1- If  $S_1$  is linearly dependent set so is  $S_2$ .

2- If  $S_2$  is linearly independent so is  $S_1$ .

Proof 1) : let  $S_1 = \{X_1, X_2, \dots, X_k\}$ ,

$$S_2 = \{X_1, X_2, \dots, X_k, X_{k+1}, \dots, X_n\} \quad k \leq n$$

c.h?

$$2) \quad S_1 = \{X_1, X_2, \dots, X_k\},$$

$$S_2 = \{X_1, X_2, \dots, X_k, X_{k+1}, \dots, X_n\} \quad k \leq n$$

$$\text{Let } i \leq k, c_1 X_1 + c_2 X_2, \dots, c_i X_i = 0$$

$$c_1 X_1 + c_2 X_2, \dots, c_i X_i + 0X_{k+1} + \dots + 0X_n = 0$$



$S_2$  is linearly independent then  $c_1 = c_2 = \dots = c_n = 0$

### Meaning of linearly independence in $R^2$ and $R^3$

Suppose that  $\{X_1, X_2\}$  are linearly dependent in  $R^2$  then there exist scalars  $c_1, c_2$  not both zero such that  $c_1 X_1 + c_2 X_2 = 0$  and

Either  $c_1 \neq 0$  or  $c_2 \neq 0$

If  $c_1 \neq 0$  then  $X_1 = \left(\frac{-c_2}{c_1}\right) X_2$ , if  $c_2 \neq 0$ , then  $X_2 = \left(\frac{-c_1}{c_2}\right) X_1$

Thus one of the vectors is a multiple of the other

Conversely, suppose that  $X_1 = c X_2$  then  $1 X_1 - c X_2 = 0$  since  $1 \neq 0$  then  $X_1$  and  $X_2$  are linearly dependent thus  $\{X_1, X_2\}$  are linearly dependent in  $R^2$  iff one of the vectors is a multiple of the other.

### In - $R^3$

Suppose that  $\{X_1, X_2, X_3\}$  are linearly dependent set of vectors in  $R^3$ . then we can write  $c_1 X_1 + c_2 X_2 + c_3 X_3 = 0$

Where  $c_1, c_2, c_3$  are not all zero say that  $c_3 \neq 0$  then

$X_3 = \left(\frac{-c_1}{c_3}\right) X_1 + \left(\frac{-c_2}{c_3}\right) X_2$  which means that  $X_3$  is span

$\{X_1, X_2\}$  then  $c_1 X_1 + c_2 X_2 = X_3$ ,  $-c_1 X_1 - c_2 X_2 + 1 X_3 = 0$

then  $\{X_1, X_2, X_3\}$  are linearly dependent

**Theorem:** Let  $S = \{X_1, X_2, \dots, X_n\}$  be the set of non zero vectors in a vectors space  $V$ . then  $S$  is linearly dependent iff the vector  $X_i$  is a linear combination of the preceding vectors in  $S$ .

**Proof:** suppose  $X_i$  is linear combination of the preceding vectors in  $S$ .

Then  $X_i = c_1 X_1 + c_2 X_2, \dots + c_{i-1} X_{i-1}$

Then  $c_1 X_1 + c_2 X_2, \dots + c_{i-1} X_{i-1} + (-1) X_i + \dots + 0 X_n = 0$

Since  $c_i = -1 \neq 0$  thus  $S$  is linearly dependent.

Conversely, suppose that  $S$  is linearly dependent then there exist scalars  $c_1, c_2, \dots, c_n$  not all zeros such that

$$c_1 X_1 + c_2 X_2, \dots + c_n X_n = 0$$

now let  $i$  be the largest subscript for which  $c_i \neq 0$

if  $i > 1$  then  $X_i = \left(\frac{-c_1}{c_i}\right) X_1 + \left(\frac{-c_2}{c_i}\right) X_2, \dots + \left(\frac{-c_{i-1}}{c_i}\right) X_{i-1}$

if  $i=1$  then  $c_1 X_1 = 0$  which implies that  $X_1 = 0$

contradiction to the hypothesis that nonzero of the vectors in  $S$  is the zero vector.

### Exercises:

1-Is the vector  $(3,6,3,0)$  is linear combination of  $X_1=(1,2,1,0)$ ,  $X_2=(4,1,-2,3)$ ,  $X_3=(1,2,6,-5)$ ,  $X_4=(-2,3,-1,2)$

2-Do the polynomials  $t^3+2t+1$ ,  $t^2-t+2$ ,  $t^3+2$ ,  $-t^3+t^2-5t+2$  span  $P_3$ ?

3-Which of the following set of vectors span  $\mathbb{R}^3$ ?

$$X_1=(1,-1,2), X_2=(0,1,1).$$

$$X_1=(2,2,3), X_2=(-1,-2,1), X_3=(0,1,0)$$

4-Which of the following set of vectors are linearly dependent in  $\mathbb{R}^3$ ?

$$X_1=(4,2,1), X_2=(2,6,-5), X_3=(1,-2,3)$$

$$X_1=(1,2,-1), X_2=(3,2,5)$$

5- Suppose that  $S=\{X_1, X_2, X_3\}$  is a linearly independent set of vector in vector space  $V$  prove that  $T=\{Y_1, Y_2, Y_3\}$  is also linearly independent where  $Y_1=X_1+X_2+X_3$ ,  $Y_2=X_2+X_3$ ,  $Y_3=X_3$ .

## Basis and Dimension

### Definition:

A set of vectors  $S = \{X_1, X_2, \dots, X_n\}$  in a vector space  $V$  is called a basis for  $V$  if  $S$  spans  $V$  and  $S$  is linearly independent.

Example :

In  $\mathbb{R}^n$  the unit vector are

$$E_1=(1,0,0,\dots,0), E_2=(0,1,0,\dots,0), \dots, E_n=(0,0,\dots,1)$$

Form a basis for  $\mathbb{R}^n$

Example; Let  $V$  be the vector space  $\mathbb{R}^4$ . let  $S = \{X_1, X_2, X_3, X_4\}$  set of vectors where

$X_1 = (1,0,1,0)$ ,  $X_2 = (0,1,-1,2)$ ,  $X_3 = (0,2,2,1)$ ,  $X_4 = (1,0,0,1)$  is the set  $S$  basis for  $V$ ?

SOL.:

Let 
$$c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0$$

$$c_1 + c_4 = 0$$

$$c_2 + 2c_3 = 0$$

$$c_1 - c_2 + 2c_3 = 0$$

$$2c_2 + c_3 + c_4 = 0$$

Only solution  $c_1 = c_2 = c_3 = c_4 = 0$

, then  $S$  is linearly independent. to show  $S$  spans  $\mathbb{R}^4$  let  $X = (a,b,c,d)$  be

any vector in  $\mathbb{R}^4$

let  $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = X$  we can find a solution for  $c_1$

,  $c_2$ ,  $c_3$ ,  $c_4$  by  $a, b, c, d$ . then  $S$  spans  $\mathbb{R}^4$

then  $S$  is a basis for  $\mathbb{R}^4$ .



Example :

The set  $S = \{t^n, t^{n-1}, \dots, t, 1\}$  spans  $p_n$ , since every polynomial of the form  $p(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n$  which is linear combination of elements in  $S$ .

$S$  is linearly independent since

$$c_1 t^n + c_2 t^{n-1} + \dots + c_n t + c_{n+1} = 0 \dots (1)$$

holds for every real number  $t$  is root of

$$p(t) = c_1 t^n + c_2 t^{n-1} + \dots + c_n t + c_{n+1}.$$

but nonzero polynomials have only a finite number of roots that (1) only if

$$c_1 = c_2 = \dots = c_n = c_{n+1} = 0$$

then  $S$  is a basis for  $p_n$

Example: The set  $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  is a basis for  $V$  of all

$2 \times 2$  matrices to show  $S$  is linearly independent

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{then } \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ then } c_1 = c_2 = c_3 = c_4 = 0$$

hence  $S$  is linearly independent. to show  $S$  spans  $V$

$$\text{Let } c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Then } c_1 = a, c_2 = b, c_3 = c, c_4 = d$$

then  $S$  is a basis for  $V$

**Theorem 1:** If  $S = \{X_1, X_2, \dots, X_n\}$  is a basis for a vector space  $V$  then every vector in  $V$  can be written in one and only one way as a linear combination of the vector in  $S$ .

**Proof :-**

First every vector  $X$  in  $V$  can be written as a linear combination of the vectors in  $S$  because  $S$  spans  $V$ .

Now let

$$X = a_1 X_1 + a_2 X_2 + \dots + a_n X_n \text{ and}$$

$$X = b_1 X_1 + b_2 X_2 + \dots + b_n X_n \text{ we must show that } a_i = b_i$$

for  $i = 1, 2, \dots, n$  we have

$$0 = (a_1 - b_1) X_1 + (a_2 - b_2) X_2 + \dots + (a_n - b_n) X_n$$

Since  $S$  is linearly independent, we conclude that

$$a_i - b_i = 0 \text{ for } i = 1, 2, \dots, n.$$

So that  $a_i = b_i$

**Theorem 2:** Let  $S = \{X_1, X_2, \dots, X_n\}$  set of non-Zero vectors and let  $W = \text{span } S$  then some subset of  $S$  is basis for  $W$ .

Proof: Ex. (like example)

**Example:** Let  $V$  be the vector space  $\mathbb{R}^4$ . let  $S = \{X_1, X_2, X_3, X_4\}$  set of vectors where

$X_1 = (1, 2, -2, 1)$ ,  $X_2 = (-3, 0, -4, 3)$ ,  $X_3 = (2, 1, 1, -1)$ ,  $X_4 = (-3, 3, -9, 6)$  Find a subset of  $S$  that is basis for  $W$ .

SOL.:

observe that every vector  $X$  in  $W$  is of the form  
 $aX_1 + bX_2 + cX_3 + dX_4 \dots \dots \dots (1)$

to find a basis for  $W$  we first determine the set

$S = \{X_1, X_2, X_3, X_4\}$  is linearly independent or not. if  $S$  linearly independent then  $S$  is basis for  $W$ . but  $S$  is not linearly independent (ch.)

$$X_1 - X_2 - 2X_3 + 0X_4 = 0 \dots \dots \dots (2)$$

then

$$X_2 = X_1 - 2X_3 \dots \dots \dots (3)$$

Substituting (3) in (1) every vector  $X$  in  $W$  is of the form

$$(a+b)X_1 + (c-2b)X_3 + dX_4 \dots \dots \dots (4)$$

Thus  $W$  spanned by  $X_1, X_3, X_4$ . we check the set

$S = \{X_1, X_3, X_4\}$  is linearly independent or not.

We find that  $X_1, X_2, X_4$  is linearly dependent and

$$-3 X_1 + 3 X_3 + X_4 = 0$$

Then  $X_4 = 3 X_1 - 3 X_3 \dots\dots\dots (5)$

Substituting (5) in (4) every vector  $X$  in  $W$  is linear combination of  $X_1, X_3$  then  $W$  spanned by  $X_1, X_3$

we check the set

$\{X_1, X_3\}$  is linearly independent or not.

$\{X_1, X_3\}$  is linearly independent and is basis for  $W$ .

**Remark:** To find a subset of  $S = \{X_1, X_2, \dots\dots\dots, X_n\}$  that is a basis for  $W = \text{span } S$  is as follows.

Step 1: first determine the set  $S$  is linearly independent or not

If  $S$  is linearly independent then  $S$  is basis for  $W$ .

Step 2: If  $S$  is linearly dependent then there is constants

$c_1, c_2, \dots\dots\dots, c_n$  not all zero, such that

$$c_1 X_1 + c_2 X_2 + \dots\dots\dots + c_n X_n = 0$$

we can then solve for one vectors in  $S$  as linear combination of other vectors, suppose  $c_j$  is nonzero

let  $S_1 = \{X_1, X_2, \dots\dots, X_{j-1}, X_{j+1}, \dots\dots\dots, X_n\}$  then  $S_1$  spans  $W$

now return Step 1 with  $S_1$

**Remark:** If  $\{X_1, X_2, \dots\dots\dots, X_n\}$  is a basis for  $V$ , then

$\{cX_1, X_2, \dots\dots\dots, X_n\}$  is also basis if  $c \neq 0$  thus a vector space always has infinitely many basis.

**Theorem 3:** If  $S = \{X_1, X_2, \dots\dots\dots, X_n\}$  that is a basis for a vector space  $V$  and  $T = \{Y_1, Y_2, Y_3, \dots\dots\dots, Y_r\}$  is linearly independent set of vectors in  $V$  then  $r \leq n$ .

**Proof:** Let  $T_1 = \{Y_1, X_1, X_2, \dots\dots\dots, X_n\}$  since  $S$  span  $V$ , So dose  $T_1$  since  $Y_1$  is linear combination of vectors of vector in  $S$ , we find that  $T_1$  is linearly dependent then by theorem



[ Let  $S = \{X_1, X_2, \dots, X_n\}$  be the set of non zero vectors in a vector space  $V$ . then  $S$  is linearly dependent iff the vector  $X_j$  is a linear combination of the preceding vectors in  $S$ ].

Some  $X_j$  is a linear combination of the preceding vectors in  $T_1$ .

Delete  $X_j$  let  $S_1 = \{Y_1, X_1, X_2, \dots, X_{j-1}, X_{j+1}, \dots, X_n\}$  not that  $S_1$  spans  $V$

Let  $T_2 = \{Y_2, Y_1, X_1, X_2, \dots, X_{j-1}, X_{j+1}, \dots, X_n\}$  then  $T_2$  is linearly dependent and some vector in  $T_2$  is a linear combination of the preceding vectors in  $T_2$ . since  $T$  is linearly independent this vector can not be  $Y_1$ , so it is  $X_i, i \neq j$ .

repeat this process over and over

If the  $X$  vectors are all eliminated before we ran out  $Y$  vector then the resulting set of  $Y$  vectors is subset of  $T$  is linearly dependent then  $T$  is linearly dependent a contradiction thus the number  $r$  of  $Y$  must be less than  $n$ .

Corollary 4: If  $S = \{X_1, X_2, \dots, X_n\}$  and  $T = \{Y_1, Y_2, Y_3, \dots, Y_r\}$  are bases for a vector space  $V$  then  $r = n$ .

Proof: Since  $T$  is linearly independent set of vectors by above theorem implies that  $r \leq n$

Similarly  $n \leq r$  hence  $n = r$ .

Definition :The dimension of nonzero vector space  $V$  is the number of vectors in a basis for  $V$ .

We write  $\dim V$  for The dimension  $V$

Remarks:

1-Since  $\{0\}$  is linearly dependent then  $\dim(\{0\}) = 0$

2- The dimension of  $\mathbb{R}^2$  is 2 and  $\dim(\mathbb{R}^4) = 4$ .

3-  $\dim(P_n) = n+1$

4-  $V$  is called finite dimensional vector space if the dimension of  $V$  is finite number .

5-  $C(-\infty, \infty)$  set of continuous function is subspace of the vector space of all real -valued function on  $\mathbb{R}$  are not finite dimension .

Theorem 5: If  $S$  is linearly independent set of vectors in finite dimensional vector space  $V$  there is basis  $T$  for  $V$  which contains  $S$

Proof :Ex. (like example)

This th. Says linearly independent set of vectors can be extended to basis for  $V$  .

Example : Suppose that we wish to find a basis for  $\mathbb{R}^4$  that contains vectors  $X_1 = (1, 0, 1, 0)$ ,  $X_2 = (-1, 1, -1, 0)$

We use theorem 5 as follows

Let  $\{E_1, E_2, E_3, E_4\}$  be natural basis for  $\mathbb{R}^4$  where

$E_1 = (1, 0, 0, 0)$ ,  $E_2 = (0, 1, 0, 0)$ ,  $E_3 = (0, 0, 1, 0)$ ,  $E_4 = (0, 0, 0, 1)$

Let  $S_1 = \{X_1, X_2, E_1, E_2, E_3, E_4\}$  since  $S_1$  spans  $\mathbb{R}^4$  by theorem 2 [Let  $S = \{X_1, X_2, \dots, X_n\}$  set of non-Zero vector space  $V$  and let  $W = \text{space } V$  then some subset of  $S$  is basis for  $W$ ] contains a basis for  $\mathbb{R}^4$

The basis is obtained by deleting every vector that is linear combination of the preceding vectors

We check if  $E_1$  is linear combination of  $X_1, X_2$

The answer is no

Then we retain  $E_1$

We check if  $E_2$  is linear combination of  $X_1, X_2, E_1$

The answer is yes, Then we delete  $E_2$

We check if  $E_3$  is linear combination of  $X_1, X_2, E_1$

The answer is yes, Then we delete  $E_3$

We check if  $E_4$  is linear combination of  $X_1, X_2, E_1$

The answer is no

Then we retain  $E_4$  the basis is  $\{X_1, X_2, E_1, E_4\}$

Theorem 6: Let  $V$  be  $n$ - dimensional vector space and let  $S = \{X_1, X_2, \dots, X_n\}$  set of  $n$  vectors in  $V$

a) If  $S$  is linearly independent then it is basis for  $V$ .

b) If  $S$  spans then it is basis for  $V$ .

proof: a) Let  $S = \{X_1, X_2, \dots, X_n\}$  is linearly independent set of  $n$  vectors in  $V$ . if  $X$  any vector in  $V$  the set  $\{X, X_1, X_2, \dots, X_n\}$  is linearly dependent since the maximal number of linearly independent vectors in  $V$  is  $n$ . thus there is linearly dependent relation

$$cX + c_1X_1 + c_2X_2 + \dots + c_nX_n = 0$$

if  $c=0$  a contradiction to linear independence of  $X_1, X_2, \dots, X_n$

then  $c \neq 0$  and then 
$$X = \left(\frac{-c_1}{c}\right)X_1 + \dots + \left(\frac{-c_{n-1}}{c}\right)X_{n-1}$$

then  $S$  is basis for  $V$ .

b) suppose  $S$  spans for  $V$ .

by [theorem 2: Let  $S = \{X_1, X_2, \dots, X_n\}$  set of non-zero vector space  $V$  and let  $W = \text{space } V$  then some subset of  $S$  is basis for  $W$ .] but this subset must contain  $n$  vectors

thus  $S$  is basis for  $V$ .

### To find a basis for the solution space of homogeneous system

$AX=0$  we do the following :

1- solve the homogeneous system by gauss-Jordan reduction

2- write the solution as linear combination of vectors, using arbitrary constants as coefficients, the set of vectors  $S$  thus determined spans the solution space  $W$ .

3- find a subset of  $S$  a basis for  $W$ .

Example: find a basis for the solution space of homogeneous system

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 3 & 5 & 0 & 6 & 2 \\ 2 & 3 & 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution : solve the **homogeneous system** by gauss-Jordan reduction

$$\begin{bmatrix} 1 & 0 & 0 & -3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Every solution is of the form

$$X = \begin{bmatrix} 3s+t \\ -3s-t \\ -s-\frac{1}{2}t \\ s \\ t \end{bmatrix} \dots\dots\dots (*) \quad \text{where } s \text{ and } t \text{ are any real number}$$

Every vector in  $W$  is a solution and is then of the form given by (\*)  
Then

$$X = s \begin{bmatrix} 3 \\ -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \dots\dots\dots (**)$$

Let  $s=1, t=0$  And then  $s=0, t=1$

$$X_1 = \begin{bmatrix} 3 \\ -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

That is  $X_1, X_2$  belong to  $W$  from(\*\*) we see  $\{X_1, X_2\}$  spans  $W$   
Since  $S = \{X_1, X_2\}$  linearly independent it is a basis for  $W$  and the  
 $\dim W = 2$ .

**Exercises:**



1-Is the polynomials  $t^2+1$ ,  $t^2-t+1$  is a basis for  $P_2$ ?

2-Find a basis for subspace  $W$  of  $R^3$  spans by

$X_1=(1,2,2)$ ,  $X_2=(3,2,1)$ ,  $X_3=(11,10,7)$ ,  $X_4=(7,6,4)$  what  $\dim W$ ?

3-Find a basis for subspace  $W$  of  $R^3$  such that every vector is  $(a,b,c)$  and  $b=a+c$  what  $\dim W$ ?

4-Find a basis for the solution space of homogeneous system

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

What is the dimension?

5-Proof if  $\dim V = n$  then any set of  $(n-1)$  vectors can not spans  $V$ .

6- proof if  $W$  is subspace of  $n$ - dimensional vector space  $V$  and  $\dim W = \dim V$  then  $W=V$ .

## The Rank of matrix and applications

In the following lecture we obtain a good method for find a basis for vector space  $V$  spanned by given set of vectors

### Definition:

Let  $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$  be an  $m \times n$  matrix

The rows of A  $X_1 = (a_{11}, a_{12}, \dots, a_{1n})$

$$X_2 = (a_{21}, a_{22}, \dots, a_{2n})$$

.....

$$X_m = (a_{m1}, a_{m2}, \dots, a_{mn})$$

considered as vectors in  $R^n$ , span a subspace of  $R^n$  called the row space also columns of A, considered as vectors in  $R^m$  called the column space of A.

**Theorem:** If A and B are two  $m \times n$  row equivalent matrices then the row space of A and B are equivalent.

Proof : Ex.

Example :

Let V be the subspace of  $R^5$  spanned by

$$S = \{(1, -2, 0, 3, -4), (3, 2, 8, 1, 4), (2, 3, 7, 2, 3), (-1, 2, 0, 4, -3)\}$$
 Find a basis for V

Solution :

Let V be the row space of the matrix  $A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$

Now A is row equivalent to the following matrix B in reduced row echelon.

From  $B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$  the row spaces of A and B are identical and a

basis for the row space of B consists of nonzero of B then

$(1, 0, 2, 0, 1), (0, 1, 1, 0, 1), (0, 0, 0, 1, -1)$  a basis for V.

**Definition :** The dimension of the row space of A is called the row rank of A and the dimension of the column space of A is called the column rank of A

Example : In example (1) the row rank of A is 3.

Example : Find the column rank of A In example (1) .

Sol: We must find the dimension of the column space of A , that is We must find the dimension of the subspace of  $\mathbb{R}^4$  spanned by column of A. if we write columns of A as row vectors ;we get :

$$A^T = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix} \quad \text{Transform } A^T \text{ to reduced row echelon form then}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \frac{11}{24} \\ 0 & 1 & 0 & \frac{-49}{24} \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{thus the vectors } (1, 0, 0, \frac{11}{24}), (0, 1, 0, \frac{-49}{24}), (0, 0, 1, \frac{7}{3})$$

Form a basis for the row space of  $A^T$  . Then the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{11}{24} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{-49}{24} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{7}{3} \end{bmatrix} \quad \text{form a basis for column space of A}$$

And the column rank of A is 3.

Example : Let  $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} \right\}$  and let  $V$  be the subspace of  $\mathbb{R}^3$  given by  $V = \text{span } S$  for  $V$

Sol: step 1

Let  $A$  be the matrix whose column the given vectors in  $S$

$$A^T = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 3 \\ 3 & -5 & 1 \\ 1 & -4 & 5 \end{bmatrix}$$

Step 2: Transform  $A^T$  to reduced row echelon form then

$$B^T = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3: the nonzero vectors of  $B^T$  written as columns

$$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \text{ form a basis for } V.$$

Observe that the row and the column rank of  $A$  are equal.

**Theorem :** The row rank and the column rank of  $m \times n$  matrix  $A$  are equal .

Proof: Let  $X_1, X_2, \dots, X_n$  be row vectors of  $A$  where



$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{where } X_i = [a_{i1}, a_{i2}, \dots, a_{in}] \quad 1 \leq i \leq m$$

Let row rank  $A = k$  and the set of vectors  $\{Y_1, Y_2, Y_3, \dots, Y_k\}$  form a basis for the row space of  $A$  where

$$Y_i = [b_{i1}, b_{i2}, \dots, b_{in}]$$

Now each of the row vectors is linear combination of  $Y_1, Y_2, Y_3, \dots, Y_k$  (basis)

That means that

$$X_1 = r_{11} Y_1 + r_{12} Y_2 + \dots + r_{1k} Y_k$$

$$X_2 = r_{21} Y_1 + r_{22} Y_2 + \dots + r_{2k} Y_k$$

$$\dots$$

$$\dots$$

$$X_m = r_{m1} Y_1 + r_{m2} Y_2 + \dots + r_{mk} Y_k$$

Where the  $r_{ij}$  are uniquely determined real numbers, then

$$a_{1j} = r_{11} b_{1j} + r_{12} b_{2j} + \dots + r_{1k} b_{kj}$$

$$a_{2j} = r_{21} b_{1j} + r_{22} b_{2j} + \dots + r_{2k} b_{kj}$$

$$\dots$$

$$\dots$$

$$a_{mj} = r_{m1} b_{1j} + r_{m2} b_{2j} + \dots + r_{mk} b_{kj}$$

$$\text{Or } \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = b_{1j} \begin{bmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{m1} \end{bmatrix} + b_{2j} \begin{bmatrix} r_{12} \\ r_{22} \\ \vdots \\ r_{m2} \end{bmatrix} + \dots + b_{kj} \begin{bmatrix} r_{1k} \\ r_{2k} \\ \vdots \\ r_{mk} \end{bmatrix}$$

For  $j=1, 2, \dots, n$

Since every column of  $A$  is linear combination of  $k$  vectors the dimension of the column space of  $A$  is at most  $k$ .

Or column rank  $A \leq \text{row rank } A$

Similarly, we get  $\text{row rank} \leq \text{column rank}$  of  $A$ . Hence row rank and the column rank of  $A$  are equal.

**Definition:** we refer to the rank of  $A$  by  $\text{rank } A$  = The number of non zero rows of  $B$  [where  $B$  is reduced row echelon form  $A$ ].

**Theorem:** The  $n \times n$  matrix is nonsingular if and only if  $\text{rank } A = n$

**Proof:** Suppose that  $A$  is nonsingular then  $A$  is row equivalent to  $I_n$ .

So  $\text{rank } A = n$ .

Conversely, if  $\text{rank } A = n$ , then  $A$  is row equivalent to  $I_n$ , then  $A$  is nonsingular.

**Corollary:** If  $A$  is  $n \times n$  matrix then  $\text{rank } A = n$  if and only if  $|A| \neq 0$

**Proof:** (H.W).

**Corollary:** Let  $S = \{X_1, X_2, \dots, X_n\}$  set of non-Zero vectors in  $R^n$  and let  $A$  be the matrix whose rows (columns) are the vectors in  $S$ . linearly independent iff  $|A| \neq 0$ .

**Proof:** (H.W).

**Corollary:** The homogeneous system  $AX=0$  of linear equation has a nontrivial solution if and only if  $\text{rank } A < n$ .

**Proof:** The system  $AX=0$  has a nontrivial solution then  $A$  is singular.

By corollary 1  $|A| = 0$  then  $\text{rank } A < n$

Conversely, since  $\text{rank } A < n$  then by corollary 1  $|A| = 0$

Thus  $AX=0$  of linear equation has a nontrivial solution.

**Example:** let  $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$  Transform  $A$  to reduced row echelon form  $B$

We find that  $B = I_3$  thus  $\text{rank } A = 3$  and  $A$  is nonsingular. And The system  $AX=0$  has a nontrivial solution.

Example: let  $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -3 \\ 1 & 3 & 3 \end{bmatrix}$  then A is row equivalent to  $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$  hence

rank  $A < 3$  and A is singular, And The system  $AX=0$  has a nontrivial solution.

**Remark:** The following statements are equivalent for  $n \times n$  matrix A

- 1- A is nonsingular.
- 2-  $AX=0$  has only the trivial solution.
- 3- A is row equivalent to  $I_n$ .
- 4-  $|A| \neq 0$
- 5-  $\text{rank} A = n$
- 6- the rows(columns) of A form linearly independent set of  $n$  vectors in  $R^n$ .

The rank of linear system  $AX=B$ , Where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{Thus } AX=B \text{ has solution iff } B \text{ is a linear}$$

combination of the columns of A, then  $\text{rank } A = \text{rank } [A:B]$

### Exercises:

1- Find the rank of A where  $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -4 & -5 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

2- Find the rank of A where  $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \end{bmatrix}$

3- If A is  $3 \times 4$  matrix what is the maximum value of rank A.

4- If  $A$  is  $4 \times 6$  matrix, proof that columns  $A$  form linearly dependent set

5- If  $A$  is  $5 \times 3$  matrix, Proof that columns of  $A$  form linearly dependent set.

## Linear transformation

### Definition :

Let  $V$  and  $W$  be vector spaces. A linear transformation  $L$  of  $V$  into  $W$  is a function  $L: V \longrightarrow W$

assigning a unique vector  $L(x)$  in  $W$  to each  $x$  in  $V$  such that .

a -  $L(x + y) = L(x) + L(y)$  . for every  $x$  and  $y$  in  $V$

b-  $L(cx) = cL(x)$  , for every  $x$  in  $V$  and every scalar  $c$

### Not:

If  $V=W$  the linear transformation

$L: V \longrightarrow W$  is also called a linear operator on  $V$ .

**Example :** Let  $L: R^3 \longrightarrow R^2$  be defined by

$$L(x, y, z) = (x, y).$$

To verify that  $L$  is linear transformation we let

$$X = (x_1, y_1, z_1) \quad \text{and} \quad y = (x_2, y_2, z_2)$$

$$\begin{aligned} \text{Then } L(x + y) &= L((x_1, y_1, z_1) + (x_2, y_2, z_2)) \\ &= L(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + x_2, y_1 + y_2) \\ &= (x_1, y_1) + (x_2, y_2) = L(x) + L(y) \end{aligned}$$

Also if  $c$  is a real number .

Then

$$\begin{aligned} L(cx) &= L(cx_1, cy_1, cz_1) = (cx_1, cy_1) = c(x_1, y_1) \\ &= cL(x) \end{aligned}$$

**Example :** Let  $L: R^3 \longrightarrow R^3$  defined by