LINEAR ALGEBRA (2)

Second Class
Department of Mathematics

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4- If A is 4×6 matrix, proof that columns A form linearly dependent set

5- If A is 5×3 matrix, Proof that columns of A form linearly dependent set.

Linear transformation

Definition :
Let V and W be vector spaces. A linear transformation L of V in to W is a function $L:V\longrightarrow W$
assigning a unique vector L(x) in W to each x in V such that .

a - $L(x+y)=L(x)+L(y)$. for every x and y in V
b- $L(cx)=cL(x)$, for every x in V and every scalar c

Not:
If V=W the linear transformation
$L:V\longrightarrow W$ is also called a linear operator on V.

Example : Let $L: R^3\longrightarrow R^2$ be defined by
$L(x,y,z)=(x,y)$.

To verify that L is linear transformation we let
$X=(x_1, y_1, z_1)$ and $y=(x_2, y_2, z_2)$
Than $L(x+y)=L((x_1, y_1, z_1)+(x_2, y_2, z_2))$
$=L(x_1+x_2, y_1+y_2, z_1+z_2)=(x_1+x_2, y_1+y_2)$
$=(x_1, y_1)+(x_2, y_2)=L(x)+L(y)$

Also if c is a real number .
Then
$L(cx)=L(cx_1, cy_1, cz_1)=(cx_1, cy_1)=c(x_1, y_1)$
$=c\,L(x)$

Example : Let $L: R^3\longrightarrow R^3$ defined by

$L(x, y, z) = (x+1, 2y, z)$. To determine whether L is linear transformation or not

we let $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$

Than $L(X + Y) = L((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= L(x_1+x_2, y_1+y_2, z_1+z_2)$

$=((x_1+x_2)+1, 2(y_1+y_2), z_1+z_2)$

On other hand

$L(x) + L(y) ==(x_1+1, 2y_1, z_1) +(x_2+1, 2y_2, z_2)$

$=((x_1+x_2)+2, 2(y_1+y_2), z_1+z_2)$

Thus $L(x + y) \neq L(x) + L(y)$ L is not linear transformation

Example : Let $L: R^3 \longrightarrow R^3$ be defined by

$L(x) = rx$,r is real number To determine whether L is linear transformation or not

we let $X = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$

Than $L(X + Y) = L((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= L(x_1+x_2, y_1+y_2, z_1+z_2)$

$=(r(x_1+x_2), r(y_1+y_2), r(z_1+z_2))$

$=(rx_1, ry_1, rz_1)+(rx_2, ry_2, rz_2)$

$=rL(x) + rL(Y)$ L is linear transformation

Example : Let $L: R^2 \longrightarrow R^3$ be defined by

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ L is linear transformation since

$X = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $Y = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ Than $L(x + y) = L(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$$L(x) + L(y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Example: $V= C[0,1]$ set of all real –valued function that are continuous function is vector space

let $W=R$ and $L:V \longrightarrow W$ is

$L(f) = \int_0^1 f(x)dx$, L is linear transformation Ch?

Theorem :
Let $L:V \longrightarrow W$ be linear trans formation then
$L(c_1X_1 + c_2X_2, \ldots\ldots + c_nX_n) = c_1 L(X_1) + c_2L(X_2), \ldots\ldots + c_n L(X_n)$
For any vectors $X_1, X_2 \ldots\ldots\ldots, X_n$ and scalars $c_1, c_2, \ldots\ldots, c_n$.

Proof:
$L(c_1X_1 + c_2X_2, \ldots\ldots + c_nX_n) = L(c_1 X_1) + L(c_2X_2) \ldots\ldots + L(c_n X_n)$
$= c_1 L(X_1) + c_2L(X_2), \ldots\ldots + c_n L(X_n)$

Theorem :
Let $L:V \longrightarrow W$ be linear trans formation then

(i) $L(0v) = 0w$

(ii) $L(X-Y) = L(X) - L(Y)$ for X,Y in V
Proof :-
(i) We have $0v = 0v + 0v$, so $L(0v + 0v)$

$L(0v) + L(0v) = L(0v)$ .if
We subtract $L(0v)$ from both sides we obtain $L(0v) = 0w$
(ii) $L(X-Y) = L(X+(-Y)) = L(X) + L(-Y)$

$= L(X) - L(Y)$

Theorem :
Let $L:V \longrightarrow W$ be linear transformation of an n-dimensional vector space V into a vector space W. Also let S= = $\{X_1, X_2, \ldots\ldots, X_n\}$ be a basis for V . if X is any vector in V then L(X) is completely determined by $\{L(X_1), L(X_2), \ldots\ldots, L(X_n)\}$
Proof :-
Since X is in V, we can write $X = c_1X_1 + c_2X_2, \ldots\ldots + c_nX_n$
Where $c_1, c_2, \ldots\ldots, c_n$ are real number
Then
$L(c_1X_1 + c_2X_2, \ldots\ldots + c_nX_n) = L(c_1 X_1) + L(c_2X_2) \ldots\ldots + L(c_n X_n)$
$= c_1 L(X_1) + c_2L(X_2), \ldots\ldots + c_n L(X_n)$

Exercises :

Q1) Is L linear transformation where $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix}$?

Q2) Let L: $R^2 \longrightarrow R^2$ linear transformation and $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

And $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ What is $L\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)$? What is $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$?

Q3)Let $P_2 \longrightarrow P_3$ linear transformation and L(1)=1 ,
$L(t)= t^2$, $L(t^2)=t^3+t$
Find $L(2t^2-5t+3)$, $L(at^2+bt+c)$

The Kernel and Range of linear transformation:

Definition :A Linear trans formation $L:V \longrightarrow W$ is said to be one- to -one if for all X_1 , X_2 in V . $X_1 \neq X_2$ implies $L(X_1) \neq L(X_2)$.
An equivalent statement is that L is one –to-one if for all X_1 , X_2 in V , $L(X_1) = L(X_2)$ implies that $X_1 = X_2$.

Example :
Let L :$R^2 \longrightarrow R^2$ be defined by
$$L(x,y) = (x+y, x-y)$$
to determine whether L is one -one , we let
$$X_1 = (x_1, y_1) \text{ and } X_2 = (x_2, y_2)$$
then if $L(x_1) = L(x_2)$
$$x_1 + y_1 = x_2 + y_2$$
$$x_1 - y_1 = x_2 - y_2$$
adding these equation , we obtain $2x_1 = 2x_2$ or $x_1 = x_2$
which implies that $y_1 = y_2$ Hence $x_1 = x_2$ and L is one –to -one.

Example :Let L :$R^3 \longrightarrow R^2$ be the linear transformation defined by
L(x , y , z) = (x , y)
Since (1,3,3) ≠ (1,3,-2) but

L (1,3,3) = L (1,3,-2) = (1,3)
We conclude that L is not one-to-one .

Definition :
Let $L:V \longrightarrow W$ A linear transformation .The kernel of L denoted by ker (L) . is the subset of V consisting of all vectors X such L(X)=0
Ker L= { X∈V / L(X) =0} .

Example :
Let $L:R^3 \longrightarrow R^2$ defined by L(x , y ,z) = (x , y)

The vector (0,0,2) is in ker L , since L(0,0,2) = (0,0)
However the vector (2,-3,9) is not ker L , since
L(2,-3,9)=(2,-3) to find kerL,we must determine all X in R^3
So that L(x) = 0 that ,
However L(x) = (x_1,x_2) thus $(x_1,x_2) = (0,0)$ So $x_1 = 0$, $x_2 = 0$
and x_3 can be any real number . it is clear that
ker L={(0,0,r) ,r is real number}
Consists of the Z-axis in x,y,z three– dimensional space R^3

Example:
Let $L:R^2 \longrightarrow R^2$ be defined by
L(x,y) = (x+y , x-y) .
Then ker L Consists of of all vectors x in R^2 such that
L(x) = 0 thus we must solve the linear system
x+y = 0
x-y = 0
for x and y . the only solution is x = 0 So ker L = { 0 }

Example:
Let $L:R^4 \longrightarrow R^2$ be defined by

$$L(x,y,z,w) = \begin{bmatrix} x+y \\ z+w \end{bmatrix}$$

Then ker L ={ x in R^2:L(x) = 0} ker L Consists of of all vectors in the form $\begin{bmatrix} r \\ -r \\ s \\ -s \end{bmatrix}$ where r,s any real numbers.

Theorem: If $L:V \longrightarrow W$ is linear trans formation . then Ker L is a subspace of V .

Proof :-
First . observe that Ker L is not an empty set since 0v is in ker L .
Also . let x and y be in Ker L .Then since L is linear transformation .

L (x+y) = L (x) + L(y) = 0w + 0w = 0w So x+y is in Ker L .
Also , if c is a scalar . Then since L is linear transformation

L(cx) = c L(x) = c0w = 0w,So cx is in Ker L .
hence Ker L is subspace of V

Example :
Let $L : R^2 \longrightarrow R^2$ be defined by L(X,Y) = (X+Y, X-Y)
Then Ker L ={0},dim(KerL)=0.

Example :
Let $L:R^3 \longrightarrow R^2$ defined by L(x , y ,z) = (x , y)

Ker L= { X∈ R^2 / L(X) =0} ={(0,0,r): r∈ R } . ,dim(KerL)=1

Example:
Let $L:R^4 \longrightarrow R^2$ be defined by

$L(x,y,z,w) = \begin{bmatrix} x+y \\ z+w \end{bmatrix}$,The basis for KerL is $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ thus

dim(KerL)=2.

Theorem: If $L:V \longrightarrow W$ is linear transformation . then L(X) is one –one if and only if Ker L={0_V}.
Proof :-
Let X∈ Ker L then L(X)=0_W also L(0_V)=0_W Thus L(X)= L(0_V)
Since L(X) is one –one , hence X=0_V Then Ker L={0_V}

Conversely, suppose that Ker L={0_V },assume that $L(x_1) = L(x_2)$ and . $x_1 ,x_2 \in V$ then $L(x_1) - L(x_2)= 0_W$ so $L(x_1-x_2) =0_W$ Then $x_1-x_2 \in$ Ker L thus $x_1-x_2 =0, x_1=x_2$ then L(X) is one –one .

Definition :

Let L:V $\longrightarrow$ W A linear transformation .The range of L denoted by rangL . is the set of all vectors in W that an images under L of vectors in V.

Thus a vector Y is in rangeL if we can find some vectors X in V such that L(X) = Y .If rangL=W ,then L is onto.

Theorem: If L:V $\longrightarrow$ W is linear transformation . then rangeL is a subspace of W .

Proof :-

First . observe that rangeL is not an empty set since $\mathbf{0}_W$ is in rangeL .

Also . let Y_1 and Y_2 be in rangeL .then $Y_1= L(X_1)$,$Y_2= L(X_2)$ .and Y_2 Then since L is linear transformation .

$Y_1 + Y_2 = L(X_1) + L(X_2) = L(X_1+X_2)$ So $Y_1 + Y_2$ is in rangeL .

Also , if c is a scalar and $X \in V$. Then since L is linear transformation

$cY = c\,L(X) = L(cX)$, so $cY \in$ rangeL . hence rangeL is subspace of V.

Example : Let L:$R^3 \longrightarrow R^2$ defined by L(x , y ,z) = (x , y) is L onto?

Choose any vector (x,y) in R^2 , since L(x , y ,z) = (x , y)

However (x , y ,z) in R^3

So that L(x) is onto. And dim(rangeL)=2

Example : Let L:$R^3 \longrightarrow R^3$ defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is L onto?}$$

Let Y= $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in R^3 can we find X in R^3 such that L(X) =Y

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ then } \begin{bmatrix} 1 & 0 & 1 & . & y_1 \\ 0 & 1 & 1 & . & y_2 - y_1 \\ 0 & 0 & 0 & . & y_3 - y_2 - y_1 \end{bmatrix}.$$

Thus a solution exists only for $y_3 - y_2 - y_1 = 0$ and so L is not onto .
To find a basis for the range L

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_1 + x_2 + 2x_3 \\ 2x_1 + x_3 + 3x_3 \end{bmatrix}$$

$$= \mathbf{x}_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \ldots\ldots\ldots(*)$$

Since $y_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$, $y_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$, $y_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

$y_1, y_2, y_3 \in$ range L from the last equation (*) it follows that
$\{y_1, y_2, y_3\}$ spans rang L
Now $\{y_1, y_2\}$ is linearly independent while $y_3 = y_1 + y_2$ thus
y_1, y_2 form a basis for range L , dim(rang L)=2
to find dim(KerL) ,we wish to find all x in R^3 such that L(x)=0

we find $x_1 = -x_3$ and $x_2 = -x_3$ thus

KerL $= \left\{\begin{bmatrix} -r \\ -r \\ r \end{bmatrix}, r \in R\right\}$ A basis for KerL is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, L is not one-one and

dim(KerL)=1.

Theorem: If $L: V \longrightarrow W$ is linear transformation, then

dim (ker L) + dim(rang L)= dim V ……….(1)

proof :-Let n=dim V and k = dim(ker L) if k =n , then kerL=V
which implies that L(x) =0 for every x in V .
hence range L = { 0 }
dim (rang L) =0 , and the conclusion holds .

suppose that $1 \leq k < n$. we shall prove that dim(rang L) = n-k

Let $\{x_1, x_2, \ldots\ldots\ldots, x_n\}$ be a basis for ker L .By theorem [if S is a linear independent set of vectors in finite . dimensional vector space V . then there is a basis T for V , which contains S] .

we can extend this basis to a basis

$S = \{x_1, x_2, \ldots\ldots, x_k, x_{k+1}, \ldots\ldots, x_n\}$ for V.

we now prove that the set $T=\{L(x_{k+1}), L(x_{k+2}), \ldots\ldots, L(x_n)\}$ is a basis for range L , which will prove that (1) holds .

First we show that T spans rang L .

Let y be any vector in rang L .

Then y = L(x) for some x in V . since S is a basis for V , we can write

$X = c_1x_1 + c_2x_2 + \ldots\ldots\ldots + c_nx_n$, where $c_1, c_2, \ldots\ldots, c_n$ are real number then

$y = L(x)$

$= L(c_1x_1 + c_2x_2 + \ldots\ldots + c_kx_k + c_{k+1}x_{k+1} + \ldots\ldots + c_nx_n)$

$= c_1L(x_1) + c_2L(x_2) + \ldots\ldots + c_kL(x_k) + c_{k+1}L(x_{k+1}) + \ldots\ldots + c_nL(x_n)$

$= c_{k+1}L(x_{k+1}) + \ldots\ldots + c_nL(x_n)$

because $x_1, x_2, \ldots\ldots\ldots, x_k$ are in ker L .Hence T spans rangeL .

Now we show that T is linear linearly independent

suppose that

$c_{k+1} L(x_{k+1}) + c_{k+2} L(x_{k+2}) + \ldots\ldots + c_n L(x_n) = 0_w \ldots\ldots$ (2)

$L(c_{k+1}x_{k+1} + c_{k+2}x_{k+2} + \ldots\ldots\ldots + c_nx_n) = 0_w$.

Hence the vector $c_{k+1}x_{k+1} + c_{k+2}x_{k+2} + \ldots\ldots\ldots + c_nx_n$ is in ker L ,

and we can write it as a linear combination of the vectors in the basis for ker L :

$c_{k+1}x_{k+1} + c_{k+2}x_{k+2} + \ldots\ldots + c_nx_n = d_1x_1 + d_2x_2 + \ldots + d_kx_k$

where $d_1, d_2, \ldots\ldots, d_k$ are a real numbers . then

$d_1x_1 + d_2x_2 + \ldots + d_kx_k - c_{k+1}x_{k+1} - c_{k+2}x_{k+2} - \ldots\ldots - c_nx_n = 0_v$

since S is linearly independent m we conclude that

$d_1 = d_2 = \ldots\ldots = d_k = c_{k+1} = \ldots\ldots = c_n = 0$

referring back to Equation (2) , we find that this means that T is linearly independent and is a basis for rang L . if k =0 , the ker L has no basis , we let $\{x_1, x_2, \ldots\ldots\ldots, x_n\}$ be a basis for V.

Corollary;: If L:V $\longrightarrow$ W is linear transformation, and dim V= dimW

(a) if L is one –one then it is onto

(b) if L is onto then L is one –one .

proof: if L is one –one then

KerL=0 ,then dim KerL=0 .From equation (1)

dim(rangeL) =dimV

but dim V= dimW

. dim(rangeL) = dimW

then rangL =W the L is onto .

If L is onto then rangL =W and dim(rangeL) = dimW

but dim V= dimW

hence dim(rangeL) =dimV

From equation (1)

dim KerL=0 thus KerL=0 then L is one –one.

Exercises :

Q1) Let $L: R^2 \longrightarrow R^3$ linear transformation .

Defined by L(x , y)=(x, x + y, y)

a-Find Ker L .

b-Is L one-to –one ? Is L onto ?

Q2) Let $L: R^4 \longrightarrow R^3$ linear transformation

Defined by L(x , y, z,w)=(x + y, z + w , x +z)

a-Find a basis for Ker L .

b- Find a basis for range L .

Q3)Let $P_2 \longrightarrow P_2$ linear transformation Defined by

$L(at^2+bt+c)=(a+c)t^2\ (b+c)t$.

a-Is $t^2 - t - 1$ in Ker L , is $t^2 + t - 1$ in Ker L?

b- Is $2t^2 - t$ in range L , is $t^2 - t + 2$ in range L?

Find a basis for Ker L , a basis for range L .

Q4) Let $L:V \longrightarrow W$ be linear transformation . If

$\{X_1, X_2, \ldots\ldots, X_n\}$ spans V . show that $\{L(X_1), L(X_2), \ldots, L(X_n)\}$

Spans range L .

The matrix of linear transformation

Example : Let L: $R^2 \longrightarrow R^3$ be defined by

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x+y \\ x-y \\ 2x+3y \end{bmatrix}$$ L is linear transformation since

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is any vector in R^3 then $\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so that

$$L(X) = L(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

$$= x\,L(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + y\,L(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) \quad = x\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Coordinate vectors:

Let $L: V \longrightarrow W$ be n-dimensional vector space V with basis $S = \{X_1, X_2, \ldots\ldots, X_n\}$ if $X = a_1 X_1 + a_2 X_2 + \ldots\ldots + a_n X_n$ Is any vector in V then the vector

$$[X]_S = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix}$$ in R^n is called the coordinate vector of X with

respect to the basis S .The components of $[X]_S$ called the coordinates of X with respect to S

Example : Let $S = \{X_1, X_2, X_3\}$ be basis for R^3 where

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

If $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then to find $[X]_S$, we must find c_1, c_2, c_3 such that

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = c_1 X_1 + c_2 X_2 + c_3 X_3 \quad \text{thus} \quad \begin{bmatrix}1 & 0 & 1\\-1 & 1 & -1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

The solution is $c_1 = 2, c_2 = 3, c_3 = -1$

Then $[X]_S = \begin{bmatrix}2\\3\\-1\end{bmatrix}$

Example : Let S= $\{X_1, X_2, \ldots\ldots, X_n\}$ be a basis be n-dimensional vector space V then since

$X_1 = 1X_1 + 0X_2 + \ldots\ldots + 0X_n$ $[X_j]_S = E_J$

Where $\{E_1, E_2, \ldots\ldots\ldots, E_n\}$ a basis for R^n

Example : Let S= $\{t, 1\}$ be a basis for P_1 if P(t)= 5t-2

Then

$[P(t)]_S = \begin{bmatrix}5\\-2\end{bmatrix}$

If T= $\{t+1, t-1\}$ be a basis for P_1

Then $5t - 2 = \frac{3}{2}(t+1) + \frac{7}{2}(t-1)$

Which implies that $[P(t)]_T = \begin{bmatrix}\frac{3}{2}\\\frac{7}{2}\end{bmatrix}$

Theorem : Let $L: V \longrightarrow W$ be n-dimensional vector space V into an m-dimensional vector space w($n \neq 0$, $m \neq 0$) and let S= $\{X_1, X_2, \ldots\ldots, X_n\}$ and = $\{Y_1, Y_2, Y_3, \ldots Y_m\}$ be bases for V and W , respectively . then the mxn matrix A whose j th column is the coordinate vector $[X_j]_T$ of $L(X_j)$ with respect to T is associated with L and has the following property :

if Y= L(X) for some X in V then $[Y]_T = A[X]_S$ where $[X]_S$ and $[Y]_T$ are the coordinate vectors of X and Y with respect to the respective bases S and T .Moreover , A is unique

Example: let L: $R^3 \longrightarrow R^2$ be defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \end{bmatrix}$

Let S = { X_1, X_2, X_3 } and T = { Y_1, Y_2 } be a bases for R^3 and R^2 respectively , where

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } Y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } Y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

we now find the matrix A associated with L :
we have

$$L(X_1) = \begin{bmatrix} 1+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1Y_1 + 0Y_2 \text{ so } [L(X_1)]_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$L(X_2) = \begin{bmatrix} 0+1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1Y_1 + 1Y_2 \text{ so } [L(X_2)]_T = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$L(X_3) = \begin{bmatrix} 0+0 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0Y_1 + 0Y_2 \text{ so } [L(X_3)]_T = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

Hence

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Example : let L: $R^3 \longrightarrow R^2$ be defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \end{bmatrix}$

Let S = { X_1, X_2, X_3 } and T = { Y_1, Y_2 } be a bases for R^3 and R^2 respectively , where

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, Y_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } Y_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ Then}$$

$$L(X_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0Y_1 - 1Y_2 \text{ so } [L(X_1)]_T = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$L(X_2) = \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{3}Y_1 - \frac{2}{3}Y_2$ so $[L(X_2)]_T = \begin{bmatrix}\frac{1}{3}\\ \frac{-2}{3}\end{bmatrix}$,

$L(X_3) = \begin{bmatrix}2\\0\end{bmatrix} = \frac{2}{3}Y_1 - \frac{4}{3}Y_2$ so $[L(X_3)]_T = \begin{bmatrix}\frac{2}{3}\\ \frac{-4}{3}\end{bmatrix}$

Hence the matrix A associated with L is $A = \begin{bmatrix}0 & \frac{1}{3} & \frac{2}{3}\\ -1 & \frac{-2}{3} & \frac{-4}{3}\end{bmatrix}$

so $[L(X)]_T = \begin{bmatrix}0 & \frac{1}{3} & \frac{2}{3}\\ -1 & \frac{-2}{3} & \frac{-4}{3}\end{bmatrix}[X]_S$ ....(*)

to illustrate this equation ,let $X = \begin{bmatrix}1\\6\\3\end{bmatrix}$ Then $L(X) = \begin{bmatrix}7\\3\end{bmatrix}$

$\begin{bmatrix}1\\6\\3\end{bmatrix} = c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix}$ then $c_1 = -3$, $c_2 = 2$, $c_3 = 4$

now $[X]_S = \begin{bmatrix}-3\\2\\4\end{bmatrix}$

then from .(*) $[L(X)]_T = \begin{bmatrix}0 & \frac{1}{3} & \frac{2}{3}\\ -1 & \frac{-2}{3} & \frac{-4}{3}\end{bmatrix}[X]_S = \begin{bmatrix}\frac{10}{3}\\ \frac{-11}{3}\end{bmatrix}$

and $L(X) = \frac{10}{3}\begin{bmatrix}1\\2\end{bmatrix} - \frac{11}{3}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}7\\3\end{bmatrix}$

Definition : The matrix A of previous theorem is called the matrix of L with respect to the bases S and T .

Example: let $L: R^3 \longrightarrow R^2$ be defined by $L\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$

Let $S = \{ X_1, X_2, X_3 \}$ and $T = \{ Y_1, Y_2 \}$ be the natural bases for R^3 , R^2

respectively

$$L(X_1)=\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}=1Y_1+1Y_2 \text{ so } \quad [L(X_1)]_T=\begin{bmatrix}1\\1\end{bmatrix}$$

$$L(X_2)=\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}=1Y_1+2Y_2 \text{ so } \quad [L(X_2)]_T=\begin{bmatrix}1\\2\end{bmatrix}$$

$$L(X_3)=\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=1Y_1+3Y_2 \text{ so } \quad [L(X_3)]_T=\begin{bmatrix}1\\3\end{bmatrix}$$

Then

$$A=\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}$$

Now if $S=\{X_1,X_2,X_3\}$ and $T=\{Y_1,Y_2\}$ be a bases for R^3 and R^2 respectively, where

$$X_1=\begin{bmatrix}1\\0\\1\end{bmatrix},\ X_2=\begin{bmatrix}0\\1\\1\end{bmatrix},\ X_3=\begin{bmatrix}1\\1\\1\end{bmatrix},Y_1=\begin{bmatrix}1\\2\end{bmatrix} \text{ and } Y_2=\begin{bmatrix}-1\\1\end{bmatrix} \text{ Then}$$

$$\text{Then } A=\begin{bmatrix}3&1&0\\-1&1&1\end{bmatrix}$$

Example :Let $L: P_1 \longrightarrow P_2$ be defined by $L(P(t))=tp(t)$.

a) let $S=\{t, 1\}$ and $T=\{t^2, t, 1\}$ be ordered bases for P_1 and P_2 respectively . find the matrix A associated with L .
b) If $P(t)=3t-2$,compute $L(P(t))$ using the matrix obtained in (a)

Solution:

First ,$L(t)=t.t=t^2=1(t^2)+0(t)+0(1)$, so $\quad [L(t)]_T=\begin{bmatrix}1\\0\\0\end{bmatrix}$

$L(1) = t.1=t.$ so $\quad [L(1)]_T = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Hence the matrix of L is $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b)$L(P(t)) = t(P(t))=t(3t-2)=3t^2-2t$.
However We can find L(p(t)using the matrix A as follows since

$$L(P(t))]_T=A[X]_S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

Hence $L(P(t)) = 3t^2-2t+0(1)= 3t^2-2t$

Example :Let $L : P_1 \longrightarrow P_2$ be defined by $L(P(t))= tp(t)$.

c) let $S =\{ t , 1 \}$ and $T=\{ t^2, t-1 ,t+1 \}$ be ordered bases for P_1 and P_2 respectively . find the matrix A associated with L .
d) If $P(t)= 3t-2$,compute $L(P(t))$ using the matrix obtained in (a)

Solution:

$L(t) = t.t= t^2= 1(t^2)+0(t-1)+0(t+1)$, so $\quad [L(t)]_T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$L(1) = t.1=0(t^2)+\frac{1}{2}(t-1)+\frac{1}{2}(t+1)$, so $\quad [L(1)]_T = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

Then the matrix of L is $A = \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$

$L(P(t)) = 3t^2-1(t-1)-1(t+1)=3t^2-2t$

Exercises:

1- Let L: $R^2 \longrightarrow R^3$ be defined by

$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}x-2y\\2x+y\\x+y\end{bmatrix}$ Let S = $\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\}$ and T = $\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix}\}$

a)find the matrix A with respect to S and T .

b)L($\begin{bmatrix}1\\2\end{bmatrix}$),using the matrix obtained in (a)

2-let L: $R^2 \longrightarrow R^2$ be defined by $L(\begin{pmatrix}x\\y\end{pmatrix}) = \begin{bmatrix}x+2y\\x-y\end{bmatrix}$

Let S = $\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix}\}$ and T = $\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\2\end{bmatrix}\}$ be a bases for R^2 find the matrix A with respect to S and T .

3- Let L : $P_2 \longrightarrow P_2$ be defined by

L (at^2+bt+c)= $(a+2c)\,t^2+(b-c)t+(a-c)$.

let S ={ t^2,t , 1 } and T={ t^2-1, t-1 } be ordered bases for P_2 .

(a) find the matrix A L with respect to S and T .

(b)If P(t)=$2t^2-3t+1$ compute L(P(t)),using the matrix obtained in (a)

Math.dep. LINEAR Algebra 2014

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Q1)a-State and Prove Cauchy-Schwrz Inequality .

b-Show that if Z orthogonal to X and Y then Z orthogonal to rX+ sY ,where r, s are scalars .

Q1)a-State and ProveTriangle Inequality

b-Consider of the vectors X=(-3,0,0,-3) , Y=(0,5,0,5) and Z=(-1,0,0,-1) Which of X and Y are orthogonal and in the same direction.

Q1)a-Prove the parallelogram law.$\|X+Y\|^2+\|X-Y\|^2=2\|X\|^2+2\|Y\|^2$

b-which of the vectors X=(4,2,6,-8) ,Y =(-2,3,-1,-1) ,Z =(-2,-1,-3,4) ,W=(1,0,0,2) are orthogonal ,in same direction ,parallel.

Q1)a- Prove the parallelogram law.$\|X+Y\|^2+\|X-Y\|^2=2\|X\|^2+2\|Y\|^2$

Prove $\|X\times Y\|=\|X\|\|Y\|\sin\theta$

Q1)a-Show that $\|X\times Y\|^2+(X.Y)^2=\|X\|^2\|Y\|^2$

b-Prove the Jacobi identity $:(X\times Y)\times Z+(Y\times Z)\times X+(Z\times X)\times Y=0$

Q2) Let W $=\left\{\begin{bmatrix} a & b & c \\ d & e & f\end{bmatrix}, a=2c+1\right\}$ W is subset of vector space V of all 2×3 matrices under usual operations of matrices addition and scalar multiplication is W is subspace of V.

Q2)a-Let W $=\{(a,b,c), b=2a+1\}$ subset of vector space R^3 is W is subspace?

b-Suppose that S=$\{X_1, X_2, X_3\}$ is a linearly independent set of vector in vector space V prove that T = $\{Y_1, Y_2, Y_3\}$ is also linearly independent where $Y_1=X_1+X_2+X_3$, $Y_2=X_2+X_3$, $Y_3=X_3$.

Q1)Let S = $\{X_1, X_2, \ldots\ldots, X_n\}$ is a basis for a vector space V prove every vector in V can be written in one and only one way as a linear combination of the vector in S .

Q2Prove if S = $\{X_1, X_2, \ldots\ldots, X_n\}$ a basis for a vector space V and T = $\{Y_1, Y_2, Y_3, \ldots Y_r\}$ is linearly independent set of vectors in V then $r \le n$.

Q3)Show that S = $\{X_1, X_2, \ldots\ldots, X_n\}$ and T =$\{Y_1, Y_2, Y_3, \ldots Y_r\}$ are bases for a vector space V then $r = n$.

Q3)

1- $\{0\}$ is linearly ------- and dim($\{0\}$)=--------.

2- The dimension of R^2 =----- and dim(P_n)=-------.

3-The homogeneous system AX=0 of linear equation has a nontrivial solution if and only if rank A -------.

4-If the dimension of V is finite number V is called------------vector space.

5-The parallelogram law is-----------------.

6-The $n\times n$ matrix is nonsingular if and only if rank A--------

7- If A is $n\times n$ matrix then rank A $= n$ if and only if---------.

8-The homogeneous system AX=0 of linear equation has a nontrivial solution if and only if rank A ---------.

9-If A is 3×4 matrix the maximum value of rank A is----------.

10-If A is 4×6 matrix, the columns A form linearly ------------ set.

Q4)prove if V be n- dimensional vector space and S = $\{X_1, X_2, \ldots, X_n\}$ set of n vectors in V then

a) If S is linearly independent then it is basis for V .

b) If S is spans then it is basis for V .

Q4)Find a basis for the solution space of homogeneous system

AX=0 where A= $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ What is the dimension ?

Q4)Find a basis for the solution space of homogeneous system

AX=0 where A= $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -3 \\ 1 & 3 & 3 \end{bmatrix}$ What is the dimension.

Q4)Find the rank of A where A= $\begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \end{bmatrix}$

Let $L:R^3 \longrightarrow R^3$ defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ is L onto?

Q2)**Prove**If $L:V \longrightarrow W$ is linear transformation, then
dim (ker L) + dim(rang L)= dim V

Q3)f $L:V \longrightarrow W$ is linear transformation, and dim V= dimW

(c) if L is one –one then it is onto

(d) if L is onto then L is one –one .

Q4)Let $L: R^4 \longrightarrow R^3$ linear transformation
Defined by L(x , y, z)=(x + y, z + w , x +z)

a-Find a basis for Ker L .

b- Find a basis for range L .

Q4)Let $L: R^2 \longrightarrow R^3$ linear transformation
Defined by L(x , y)=(x, x + y, y)

a-Find Ker L .

b-Is L one-to –one ? Is L onto ?

Q1

2-Every group of order≤6 is abelian?

3-Let H_1, H_2 be a subgroup of G then $H_1 \cup H_2$ is a subgroup?

4-If H is a subgroup of G then G\H is a group?

Q3:1-Is the group of order (56) simple?

2-Is the group of order 45 abelian?

Q4:Define the following

Char of ring R, cyclic group, normal subgroup, prime ideal, semiprime ideal, maximal ideal, I.D, P.I.D, nilpotent element, idempotent element, Boolean ring

Q5:

3-show that $Z_2 \cong \{-1,1\}$?

4- Show that for any group G we have $G \backslash Cent G \cong Inn G$?

5-Let $f: G \to G'$ a group homo, prove that if $H \Delta G$ and onto,then $f(H) \Delta G'$?

6-Show that every group of order p^2 is an abelian group (p is prime number)?

7-Let G be a group. Show that if H is normal subgroup of G then $G \backslash H$ is abelian iff $[G:G] \subseteq H$?

12-Show that if $\{H_\alpha\}$ a family of normal subgroup then $\cap H_\alpha$ is a normal subgroup?

13-Given H and K subgroup of a group G prove that if H and K are normal subgroup then HK is normal in G?

14-Show that if H is subgroup of a group G, then $H \Delta G$ iff $(aH)(bH) = abH$ for all $a, b \in G$?

15-Let H, K be a subgroup of a group G then $H \cup K$ is a subgroup iff either $H \subseteq K$ or $K \subseteq H$?

22-Let H and K be a subgroup of What is meant of an internal direct product of H and K, show that if $G = H \oplus K$ then $G \backslash H \cong K$?

24-Show that the only non-trivial homo from Z into Z is the identity?

26-Is $(Z,+) \cong (Q-\{0\}, .)$

30- Let $f: G \rightarrow G'$ a group home with G' is abelianshow that every subgroup of G containing $kerf$ normal?

32- Let $f: G \rightarrow G'$ a group homo which is onto, show that if H and K are subgroups of G with $kerf \subseteq H \cap K$, then $f(H \cap K) = f(H) \cap f(K)$?

34- Let H, K be a subgroup of G then show that if KH is a subgroup of G, then $KH = HK$?

43-Let G be a group: Define $InnG$, show that $\frac{G}{CentG} \simeq InnG$

44-If H_1 and H_2 are subgroups of a group G. Is $H_1 \cup H_2$ a subgroup of G?

45-Let H_1 and H_2 be two subgroups of a group G, show that $H_1 \cup H_2$ is a subgroup if and only if if and only if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

46-Define a proper subgroup. Show that a group G cannot be the union of two of its proper subgroups.

Q6:

1-Exaplain how every field is integral domain and give example to show that converse is not true. What conditions on integral domain induce field(prove your answer)?

2-Show that the only homo from the ring Z into Z are trivial homo or the identity homo?

3-If I,J are ideal of ring R show that I+J is an ideal of R ,what is the relation between I+J and I or J (inclusion)

4-Let M be an ideal of a commutative ring R with one then M is maximal ideal iff R\M is a field?

ORTHONORMEL BASIS IN R^n

Definition : Let $S=\{ x_1 , x_2, \dots\dots\dots, x_n \}$ be set of vectors in R is called orthogonal if any two distinct vector in S are orthogonal that is

If $X_i . X_j = 0$ for $i \neq j$

An orthonormal set of vectors is orthogonal set of unite vectors.

Example : Let $X_1 = (1,0,2)$, $X_2 = (-2,0,1)$ and $X_3 = (0,1,0)$ then $\{ X_1 , X_2 , X_3 \}$ orthogonal set in R^3

Since $X_1 . X_2 = X_2 . X_3 = X_2 . X_3 = 0$

But not orthonormal

The vectors $Y_1 = (\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}})$, $Y_2 = (\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$ are unite vectors in direction of X_1 , X_2 . X_3 is also unite vector then $\{ Y_1 , Y_2 , X_3 \}$ orthonormal set in R^3

Also Span$\{ X_1 , X_2 , X_3 \}$ is the same as Span $\{ Y_1 , Y_2 , X_3 \}$.

Example : Let $S = \{E_1, E_2, \dots\dots\dots, E_n\}$ a natural basis for R^n S is orthonormal set in R^n.

Theorem: Let $S=\{X_1, X_2, \dots, X_n\}$ be orthogonal set in R^n then S is linearly independent.

Proof: Let

$c_1 X_1 + c_2 X_2 + \dots + c_n X_n = 0$ taking the inner product of both sides with X_i

$(c_1 X_1 + c_2 X_2 + \dots + c_n X_n) . X_i = 0 . X_i$

$c_1 (X_1 . X_i) + c_2 (X_2 . X_i) \dots + c_n (X_n . X_i) = 0 . X_i$

since $X_i . X_j = 0$ for $I \neq j$ then $0 = c_i (X_i . X_i) = c_i \|X_i\|^2$

$X_i \neq 0$ then $\|X_i\| \neq 0$ thus $c_i = 0$ $1 \leq i \leq n$

then S is linearly independent.

Corollary: An orthonormal set in R^n then S is linearly independent.

Definition : An orthogonal(orthonormal) basis for a vector space is a basis that is an orthogonal(orthonormal) set.

Theorem:GRAM SCHMIDIT PROCESS

Let W be a non zero subspaces of R^n with basis $S=\{X_1, X_2, \ldots, X_n\}$ then there exists an orthonormal basis $T=\{Z_1, Z_2, \ldots\ldots, Z_n\}$ for W.

Proof:let $Y_1 = X_1$

Let $W=\text{Span}\{X_1, X_2\}$, $w_1=\text{Span}\{Y_1, X_2\}$

$Y_2 = c_1 Y_1 + c_2 X_2$

Now $Y_1 . Y_2 = 0$, $0=(c_1 Y_1 + c_2 X_2). Y_1$

$Y_1 \neq 0$, $Y_1 . Y_1 \neq 0$

$c_1 = -c_2 \dfrac{X_2.Y_1}{Y_1,Y_1}$

let $c_2=1$ we obtain $c_1 = -\dfrac{X_2.Y_1}{Y_1,Y_1}$ thus

$Y_2 = c_1 Y_1 + c_2 X_2 = X_2 - (\dfrac{X_2.Y_1}{Y_1,Y_1}). Y_1$

We have an orthogonal sub set $\{Y_1, Y_2\}$ of W

Next we look at vector Y_3

In the subspace W_2 of W Spanned by $\{X_1, X_2, X_3\}$which is orthogonal to both Y_1, Y_2 of course W_2 is also Spanned by $\{Y_1, Y_2, X_3\}$ then

$Y_3 = d_1 Y_1 + d_2 Y_2 + d_3 X_3$ then

Let $d_3=1$, $Y_3 . Y_1 = 0$, $Y_2 . Y_3 = 0$

$Y_3 . Y_1 = (d_1 Y_1 + d_2 Y_2 + X_3). Y_1 = d_1 (Y_1 . Y_1) + X_3 . Y_1$

$Y_2 . Y_3 = (d_1 Y_1 + d_2 Y_2 + X_3). Y_2 = d_2 (Y_2 . Y_2) + X_3 . Y_2$

$Y_2 \neq 0$

$d_1 = \dfrac{X_3 Y_1}{Y_1.Y_1}$, $d_2 = -\dfrac{X_3.Y_2}{Y_2.Y_2}$

$Y_3 = X_3 - \dfrac{X_3 Y_1}{Y_1.Y_1} Y_1 + - \dfrac{X_3.Y_2}{Y_2.Y_2} Y_2$

At this point we have an orthogonal sub set $\{Y_1, Y_2, Y_3\}$ of W

Next we look at vector Y_4

In the subspace W_3

of W Spanned by $\{X_1, X_2, X_3, X_4\}$ which is orthogonal to

Y_1, Y_2, Y_3 of course W_3 is also Spanned by $\{Y_1, Y_2, Y_3, X_3\}$ then

$$Y_4 = X_4 - \frac{X_4 Y_1}{Y_1.Y_1} Y_1 + - \frac{X_4.Y_2}{Y_2.Y_2} Y_2 + \frac{X_4.Y_3}{Y_3.Y_3} Y_3$$

We continue until we have an orthogonal set $T^* = \{Y_1, Y_2, Y_3, \ldots Y_m\}$

be bases for W if we normalized the Y_i

let $Z_i = \frac{Y_i}{\|Y_i\|}$, $1 \le i \le n$

then $T = \{Z_1, Z_2, \ldots\ldots, Z_n\}$ an orthonormal basis for W.

The Gram Scchmidt Process for orthonormal basis $T = \{Z_1, Z_2, \ldots\ldots, Z_n\}$

For W of R^n with basis $S = \{X_1, X_2, \ldots, X_n\}$

Stepe 1: let $Y_1 = X_1$

Stepe 2:compute $Y_2, Y_3, \ldots Y_m$

By formula $Y_i = X_i - \frac{X_i Y_1}{Y_1.Y_1} Y_1 - \frac{X_i.Y_2}{Y_2.Y_2} Y_2 - \frac{X_i.Y_3}{Y_3.Y_3} Y_3 \ldots\ldots\ldots - \frac{X_i.Y_{i-1}}{Y_{i-1}.Y_{i-1}} Y_{i-1}$

The set $T^* = \{Y_1, Y_2, Y_3, \ldots Y_m\}$ is an orthogonal set .

Stepe 3: let $Z_i = \frac{Y_i}{\|Y_i\|}$ then $T = \{Z_1, Z_2, \ldots\ldots, Z_n\}$ is orthonormal basis for W.

Example: let $S = \{X_1, X_2, X_3\}$ be a basis for R^3 where

$X_1 = (1,1,1)$, $X_2 = (-1,0,-1)$, $X_3 = (-1,2,3)$ use Gram Scchmidt Process to transform to

orthonormal basis for R^3

Sol: Stepe 1: let $Y_1 = X_1$

Stepe 2:compute Y_2, Y_3

$$Y_2 = X_2 - (\frac{X_2.Y_1}{Y_1.Y_1}).Y_1 = (-1,0,-1) + \frac{2}{3}(1,1,1) = (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$$

$$Y_3 = X_3 - \frac{X_3 Y_1}{Y_1.Y_1} Y_1 + - \frac{X_3.Y_2}{Y_2.Y_2} Y_2 = (-2,0,-2)$$

then

$T^* = \{Y_1, Y_2, Y_3\}$ an orthogonal basis in R^3

Let $Z_i = \frac{Y_i}{\|Y_i\|}$

$$Z_1 = \frac{Y_1}{\|Y_1\|} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

$$Z_2 = \frac{Y_2}{\|Y_2\|} = (-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$$

$$Z_3 = \frac{Y_3}{\|Y_3\|} = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$$

then $T = \{Z_1, Z_2, Z_3\}$ an orthonormal basis for R^3

Example: let W be subspace of R^4 with basis $S=\{X_1, X_2\}$ where $X_1=(1,-2,0,1)$, $X_2=(-1,0,0,-1)$ use Gram Scchmidt Process to transform to orthonormal basis for R^4

Sol:

Theorem: Let $S=\{X_1, X_2, \ldots, X_n\}$ be orthonormal basis in R^n and X any vector in R^n then $X = c_1 X_1 + c_2 X_2 + \ldots + c_n X_n$ where $c_i = X.X_i$ $1 \le i \le n$

proof:

$X= \quad c_1 X_1+c_2 X_2+\ldots+c_n X_n$

$X.X_i=(c_1 X_1+c_2 X_2+\ldots+c_n X_n)X_i$

$c_1(X_1.X_i)+ \quad c_2(X_2.X_i) \quad \ldots \quad +c_n(X_n.X_i) \;=X.X_i$

since $X_i.\ X_j=0$ for $I \neq j$ and $(X_i.X_i)=1$ sincf orthonormal

then $X.X_i=c_i$

EX: let X=(4,3,-1) in example 1, write X as linear combination where $T=\{Z_1,Z_2,Z_3\}$

$$Z_1=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}),Z_2=(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}),Z_3=(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}})$$

$X.Z_i=c_i$

Then X= ?

$X=c_1\ Z_1+c_2 Z_2+c_3 Z_3$

Theorem: Let W be an m-dimensional subspace in R^n with orthonormal basis $S=\{X_1, X_2, \ldots, X_m\}$ then every vector X in R^n can be written uniquely as

X=Z+Y where Z is in W and Y is an orthogonal to every vector in W

proof:

Let $Z=(X.X_1)X_1+ \quad (X.X_2)X_2 \quad \ldots \quad +(X.X_m)X_m$

And

Y=X-Z

Since Z is linear combination of $X_1, X_2, \ldots, X_m$ then Z belong to W

We next show that Y is orthogonal to every vector in W .thus let

$Z_1=c_1 X_1+c_2 X_2+\ldots+c_m X_m$ be arbetrary vector in W then

$Y.Z_1=[X-(X.X_1)X_1- \quad (X.X_2)X_2 \quad \ldots \quad -(X.X_m)X_m].[c_1 X_1+c_2 X_2+\ldots+c_m X_m]$

And

since $X_i.\ X_j=0$ for $I \neq j$ and $(X_i.X_i)=1$ sincf orthonormal

then $X.X_i=c_i$

Eigen values And Eigenvectors

Definition: Let A be an n $\times$n matrix. The real number λ is is called an

gen value of A if there exists a nonzero vector X in R^n such that

$$AX=\lambda X \quad ....(1)$$

ery nonzero vector X satisfying (1) is called an **eigenvectors**
A associated with the **Eigen values** λ .

Note:
X=0 always satisfies Equation(1),but we **insist** that an eigenvector **X** be a nonzero vector.

Example 1:if A is identity matrix I_n ,then the only eigenvalue is λ=1;and every nonzero vector in R^n is an eigenvector of A associated with the eigenvalue λ=1:
X=1X.

Example 2: Let $A=\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

hen

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix}= \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So that $X_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of **A** associated with the eigenvalue $\lambda_1=\frac{1}{2}$

Also, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So that $X_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda_2=\frac{-1}{2}$

Figure 5.1 shows that X_1 and AX_1 are parallel, and X_2 and AX_2 are parallel also. this illustrates the fact that if X is an eigenvector of A, then X and AX are parallel.
In figure 5.2 we show X and AX for the cases λ>1,0<λ<1,and λ<0.

mple 3: let $A=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $A\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}=0\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So that $X_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A associated with the **eigenvalue**

$\lambda_1=0$.also,

$X_2=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda_2=1$

Example 3: points out the fact that although the zero vector, by definition, cannot be an eigenvector, the number zero can be eigenvalue.

Example 4: let $A=\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

We wish to find the eigenvalue of A and their associated eigenvectors.

Thus we wish to find all real numbers λ and all nonzero vectors $X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfying(1),that is

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad ....(2)$$

Equation (2) becomes

$$x_1+x_2=\lambda x_1$$
$$-2x_1+4x_2=\lambda x_2$$

or

$$(\lambda-1)x_1- \quad x_2=0$$
$$2x_1+(\lambda-4)x_2=0$$

The homogeneous system of two equations in two unknowns. the homogeneous system in (3) has nontrivial solution if and only if the determinant of its **coefficient** matrix is zero: thus if and only if

$$\begin{vmatrix} \lambda-1 & -1 \\ 2 & \lambda-4 \end{vmatrix}=0$$

This means that $(\lambda-1)(\lambda-4)+2=0$

Or

$\lambda^2-5\lambda+6=0=(\lambda-3)(\lambda-2)$

hence

$\lambda_1=2$ and $\lambda_2=3$are the eigenvalues of A.

To find all eigenvectors of A associated with $\lambda_1=2$ we form the linear system

$$AX=2X$$

$$\begin{bmatrix}1 & 1\\ -2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=2\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

This gives

$$x_1+x_2=2x_1$$
$$-2x_1+4x_2=2x_2$$

Or

$$(2-1)\,x_1-\quad x_2=0$$
$$2x_1+(2-4)\,x_2=0$$

Or

$x_1-x_2=0$

$$2x_1-x_2=0$$

Note that we could have obtained this last homogeneous system by merely substituting $\lambda=2$ in (3).

All solution to this last system are given by .

$$x_1=x_2,\ x_2=\text{any real number } r.$$

hence all eigenvectors associated with the eigenvalue $\lambda_1=2$ are given by

$\begin{bmatrix}r\\ r\end{bmatrix}$,r any nonzero real number. In particular, $x_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ is an eigenvector associated with $\lambda_1=2$.

similarly, for $\lambda_2=3$ we obtain, from (3)

$$(3-1)x_1-\quad x_2=0$$
$$2x_1+(3-4)x_2=0$$

Or

$$2x_1-x_2=0$$
$$2x_1-x_2=0$$

All solution to this last homogeneous system are given by

$$x_1=\frac{1}{2}x_2,\quad x_2=\text{any real number } r.$$

hence any eigenvectors associated with the eigenvalue $\lambda_2=3$ are given by

$\begin{bmatrix}\frac{r}{2}\\ r\end{bmatrix}$,r any nonzero real number .in particular $X_2=\begin{bmatrix}1\\ 2\end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2=3$.

Definition: let $A=\left[a_{ij}\right]$ be an $n\times n$ matrix. the determinant

$$f(\lambda)=\left|\lambda I_n - A\right|=\begin{vmatrix} \lambda-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda-a_{nn} \end{vmatrix} \quad (2)$$

Is called the **characteristic polynomial** of A. the equation

$f(\lambda)=\left|\lambda I_n - A\right|=0$

is called the **characteristic equation of A.**

Example 5: let

$$A=\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

The characteristic polynomial of A is (verify)

$$f(\lambda)=\left|\lambda I_3 - A\right|=\begin{vmatrix} \lambda-1 & -2 & -1 \\ -1 & \lambda-0 & 1 \\ -4 & 4 & \lambda-5 \end{vmatrix}=\lambda^3-6\lambda^2+11\lambda-6$$

Theorem: The eigenvalue of **A** are the real roots of the characteristic polynomial of **A**

Proof:

Let λ be an eigenvalue of **A** with associated eigenvector **X.** then

$$AX=\lambda X$$

Which can be rewritten as

$$AX=(\lambda I_n)X$$

Or

$$(\lambda I_n-A)X=0 \qquad (3)$$

A homogeneous system of n equations in n unknowns. This system has a nontrivial solution if and only if the determinant of its coefficient matrix is zero that $\left|\lambda I_n - A\right|=0$

Conversely, if λ is a real root of the characteristic polynomial of **A**, then $\left|\lambda I_n - A\right|=0$, so the homogeneous system (3) has nontrivial solution **X**. Hence λ is the eigenvalue of **A**

Thus to find the Eigen values of a given matrix A, we must find the real roots of its characteristic polynomial $f(\lambda)$.

Example 6: consider the matrix of example 5. the characteristic polynomial is

$f(\lambda)= \lambda^3 - 6\lambda^2 + 11\lambda - 6$

the possible integer roots of $f(\lambda)$ are ± 1, ± 2, ± 3 and ± 6. By substituting these values in $f(\lambda)$, we find that $f(1)=0$ so that $\lambda=1$ is a root of $f(\lambda)$. Hence

$(\lambda-1)$ is a factor of $f(\lambda)$.

Dividing $f(\lambda)$ by $(\lambda-1)$, we obtain

$f(\lambda) = (\lambda-1)(\lambda^2 - 5\lambda + 6)$

Factoring $\lambda^2 - 5\lambda + 6$, we have

$f(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)$

The Eigen value of **A** are then

$\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$.

- To find the eigenvector $\mathbf{X}_1$ associated with $\lambda_1=1$, we form the system

$$(1I_3 - A)X=0,$$

$$\begin{bmatrix} 1-1 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & 1-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Or

$$\begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A solution is $\begin{bmatrix} \frac{-r}{2} \\ \frac{r}{2} \\ r \end{bmatrix}$ for any real number r. thus $X_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector of A associated with $\lambda_1=1$.

To find an eigenvector x_2 associated with $\lambda_2=2$, we form the system

$$(2I_3 - A)X=0$$

That is, $\begin{bmatrix} 2-1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

A solution is $\begin{bmatrix} \frac{-r}{2} \\ \frac{r}{4} \\ r \end{bmatrix}$ for any real number r. thus $X_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ is an eigenvector of A associated with $\lambda_2=2$

To find an eigenvector X_3 associated with $\lambda_3=3$, we form the system

$(3I_3\text{-}A)X=0,$

And find that a solution is $\begin{bmatrix} \frac{-r}{4} \\ \frac{r}{4} \\ r \end{bmatrix}$ for any real number r. thus $X_3=\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ is an eigenvector of A associated with $\lambda_3=3$.

Of course, the characteristic of a given matrix may have imaginary root, and it may even no real roots, however, for the matrices that we are most intersection, symmetric matrices, all the roots of the characteristic polynomial are real.

Example 7: let $A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then the characteristic polynomial of A is

$f(\lambda)=\lambda^2+1,$

which has no real roots, thus A has no eigenvalues.

Problems: Find the eigenvaluesand eigenvectors of

$$A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},B=\begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix},C=\begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Similar Matrices

Definition: A matrix **B** is said to be similar to a matrix **A** if there is a nonsingular matrix **P** such that $\mathbf{B = P^{-1} AP}$.

Example: let $A=\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Let

$$P=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ Then } p^{-1}=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$B = P^{-1}AP=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Thus **B** is similar to **A**.

The properties for similarity:

1. **A** is similar to **A**.
2. if **B** is similar to **A**, then **A** is similar to **B**.
3. **A** is similar to **B** and **B** is similar to **C**, then **A** is similar to **C**.

Proof:?

By property 2 we replace the statements "**A** is similar to **B**" and "**B** is similar to **A**" By " **A** and **B** are similar"

Definition: we shall say that the matrix **A** is **diagonalizable** if it is similar to a diagonal matrix. In this case we also say that **A can be diagonalized.**

Example 9: if **A** and **B** as in above example, then **A** diagonalizable, since it is similar to **B**.

EXC:

1- I f λ is eigenvalue of nonsingular matrix A with associated eigenvectors X .show that $\frac{1}{\lambda}$ is eigenvalue of matrix A^{-1} with eigenvectors X .

2- if A ,B are nonsingular matrix show that AB ,BA are similar .

Theorem: A matrix A is nonsingular iff $|A| \neq 0$.

Corollary :Let $S = \{X_1, X_2, \ldots, X_n\}$ set of non-Zero vectors in R^n and let A be the matrix whose rows (columns) are the vectors in S . linearly independent iff $|A| \neq 0$.

Theorem: An $n \times n$ matrix **A** is diagonalizable if and only if it has n linearly independent eigenvectors. In this case **A** is similar to a diagonal matrix D, with $P^{-1}AP=D$,whose diagonal elements are the eigenvalues of **A**, while **p** is a matrix whose columns are n linearly independent eigenvectors of **A**.

Proof: Suppose that **A** is similar to **D**. then

$$P^{-1}AP = D$$

So that AP=PD Let

$$D=\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix},$$

and let X_j, j=1,2,…,n be the j^{th} column of **p**, Note that the j^{th} column of the matrix AP is AX_j and the j^{th} column of PDis $\lambda_j \mathbf{X}_j$. thus from (6) we have

$$AX_j = \lambda_j \mathbf{X}_j \qquad (7)$$

Since P is nonsingular matrix its columns are linearly independent and so are all zero ,then λ_j is eigenvalues of **A**,

Conversely , suppose that λ_j are eigenvalues corresponding eigenvectors X_j are linearly independent ,let P be matrix its j^{th} column X_j. from cor. And th. Then P is non singular , and $AX_j = \lambda_j \mathbf{X}_j$ then AP=PD and**A** is diagonalizable.

Example:

let**A** let $A=\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.. the eigenvalues are λ_1=2 and λ_2=3 the corresponding eigenvectors

$X_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $X_2=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly independent. Hence **A** is diagonalizable, here

$\mathbf{P}=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{P}^{-1}=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Thus

$$P^{-1}AP=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

Example : let $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

The eigenvalues of **A** are $\lambda_1=1$ and $\lambda_2=1$. eigenvectors associated with λ_1 and λ_2 are vectors of the form $\begin{bmatrix}r\\ 0\end{bmatrix}$.

Where r is any non zero real number.
Since **A** does not have two linearly independent eigenvectors, we conclude that **A** is not diagonalizable.

Corollary: consider the linear transformation $L:R^n \to R^n$ defined by $L(X)=AX$ for X in R^n then **A** is diagonalizable with n linearly independent eigenvectors $X_1, X_2, \ldots, X_n$ if and only if the matrix of **L** with respect to $S=\{X_1, X_2, \ldots, X_n\}$ is diagonal.

Proof:
Suppose that **A** is diagonalizable. Then by theorem it has n linearly independent eigenvectors $X_1, X_2, \ldots, X_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since n linearly independent vectors in R^n form a basis we can conclude that $S=\{X_1, X_2, \ldots, X_n\}$ is a basis for R^n. Now

$$L(X_j)=AX_j$$

$$=\lambda_j X_j = \quad 0X_1+\ldots+0X_{j-1}+.\lambda_j X_j+0X_{j+1}+\ldots+0X_n,$$

so the coordinate vector $[L(X_j)]_s$ of $L(X_j)$ with respect to S is

$$\begin{bmatrix}0\\ \vdots\\ 0\\ \lambda_j\\ 0\\ \vdots\\ 0\end{bmatrix} \leftarrow J^{th}\text{row} \qquad (1)$$

Hence the matrix of **L** with respect to S is

$$\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & & & \vdots\\ 0 & \ldots & 0 & \lambda_n\end{bmatrix}. \qquad (2)$$

Conversely, suppose that there is a basis $S=\{X_1, X_2, \ldots, X_n\}$ for R^n with respect to which the matrix of **L** is diagonal, say of the form (2). Then the coordinate vector of $L(X_j)$ with respect to S is (1), so

$L(X_j) = 0X_1 + \ldots + 0X_{j-1} + \lambda_j X_j + 0X_{j+1} + \ldots + 0X_n = \lambda_j X_j$.

Since $L(X_j) = AX_j$, we have

$AX_j = \lambda_j X_j$,

Which means that $X_1, X_2, \ldots, X_n$ are eigenvectors of A. since they form a basis for R^n. They are linearly independent, and by theorem we conclude that A is diagonalizable.

Theorem: A matrix A is diagonalizable if all the roots of its characteristic polynomial are real and distinct.

Proof:

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalue of A and let $S=\{X_1, X_2, \ldots, X_n\}$ be a set ofassociated eigenvectors, we wish to show that S is linearly independent. Suppose that S is a linearly dependent set of vectors, then theorem (*) implies that vector X_j is linear combination of the preceding vectors in S ,we can assume that $S_1=\{X_1, X_2, \ldots, X_{j-1}\}$ is linearly independent, for otherwise one of the vectors in S_1 is a linear combination of the preceding ones, and we can choose a new set S_2 and so on, we thus have that S_1 is linearly independent and that

$$X_j = c_1 X_1 + c_2 X_2 + \ldots + c_{j-1} X_{j-1}, \qquad (1)$$

Where $c_1, c_2, \ldots, c_{j-1}$ are real numbers, (multiplying on the left) both sides of equation (1) by A we obtain

$$\begin{aligned} A X_j &= A(c_1 X_1 + c_2 X_2 + \ldots + c_{j-1} X_{j-1}) \\ &= c_1 AX_1 + c_2 AX_2 + \ldots + c_{j-1} AX_{j-1} \qquad (2) \end{aligned}$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen value of A and $X_1, X_2, \ldots, X_j$ its associated eigenvectors, we know that $A X_i = \lambda_i X_i$ for I =1,2,...,j , substituting in (2), we have

$$\lambda_j X_j = c_1 \lambda_1 X_1 + c_2 \lambda_2 X_2 + \ldots + c_{j-1} \lambda_{j-1} X_{j-1} \qquad (3)$$

multiplying (1) by λ_j, we obtain

$$\lambda_j X_j = \lambda_j c_1 X_1 + \lambda_j c_2 X_2 + \ldots + \lambda_j c_{j-1} X_{j-1} \qquad (4)$$

subtracting (4) from (3), we have

$$\begin{aligned} 0 &= \lambda_j X_j - \lambda_j X_j \\ &= c_1(\lambda_1 - \lambda_j)X_1 + c_2(\lambda_2 - \lambda_j)X_2 + \ldots + c_{j-1}(\lambda_{j-1} - \lambda_j)X_{j-1} \end{aligned}$$

Since S_1 is linearly independent, we must have

$c_1(\lambda_1 - \lambda_j) = 0, \qquad c_2(\lambda_2 - \lambda_j) = 0, \ldots, \qquad c_{j-1}(\lambda_{j-1} - \lambda_j) = 0.$

Now

$\lambda_1 - \lambda_j \neq 0, \qquad \lambda_2 - \lambda_j \neq 0, \ldots, \lambda_{j-1} - \lambda_j \neq 0$

(because the λ_i are distinct), which implies that

$c_1 = c_2 = \ldots = c_{j-1} = 0$.

From (1) we conclude that $X_j = 0$, which is impossible if X_j is an eigenvector. Hence S is linearly independent, and from theorem (*) it follows that **A** is diagonalizable.

If all roots of characteristic polynomial of **A** are real and not all distinct, then **A** may or may not be diagonalizable. characteristic polynomial of **A** can be written as the product of **n** factors, each of the form $\lambda - \lambda_j$, thus the characteristic polynomial can be written as

$$(\lambda-\lambda_1)^{k1}\,(\lambda-\lambda_2)^{k2}\ldots(\lambda-\lambda_r)^{k_r},$$

Where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct Eigen value of **A** ,and $k_1, k_2, \ldots, k_r$ are integers whose sum is n.the integer k_i is called the **multiplicity** of λ_i.

Example: let $A=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic polynomial of **A** is

$f(\lambda) = \lambda(\lambda-1)^2$, so the eigenvalue of **A** are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 1$, thus $\lambda_2 = 1$ is an eigenvalue of multiplicity 2.

we now consider the eigenvectors associated with the eigenvalues $\lambda_2 = \lambda_3 = 1$, they are obtained by solving the linear system $(1I_3 - A)X = 0$:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A solution is any vector of the form $\begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$, where r is any real number, so the dimension of the solution space of $(1I_3 - A)X = 0$ is 1.ther do not exist two linearly independent eigenvector. thus **A** cannot be diagonalized.

EXC:

1- I f λ is eigenvalue of nonsingular matrix A with associated eigenvectors X .show that $\frac{1}{\lambda}$ is eigenvalue of matrix A^{-1} with eigenvectors X .

2- if A ,B are nonsingular matrix show that AB ,BA are similar .

Example: let

$$A=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}.$$

The characteristic polynomial of **A** is $f(\lambda)=\lambda(\lambda-1)^2$ so the eigenvalues of **A** are $\lambda_1=0$. $\lambda_2=1$, and $\lambda_3=1$ is eigenvalue of multiplicity 2.

now we again consider the solution space$(1I_3-A)X=0$, that is of

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

A solution is any vector of the form $\begin{bmatrix}0\\ r\\ s\end{bmatrix}$ for any real numbers **r** ands .thus we can take as

eigenvectors X_2 and $X_3$the vectors $X_2=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$ and $X_3=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

Now we look for an eigenvector associated with $\lambda_1=0$.

we have to solve $(0I_3-A)X=0$

$$\text{or}\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ -1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

A solution is any vector of the form $\begin{bmatrix}t\\ 0\\ -t\end{bmatrix}$ for any real number t. thus $X_1=\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$ is an eigenvector

associated with $\lambda_1=0$. sinceX_1, X_2 **and** $X_3$are linearly independent, **A** can be diagonalzed.

Thus an $n\times n$ matrix may fail to be diagonalizable either because not all the roots of its characteristic polynomial are real numbers, or it does not have n linearly independent eigenvectors.

Eigenvalues and eigenvectors satisfy many important properties:

1-If A is an upper(lower) triangular matrix, then the Eigenvalue of **A** are the elements on the main diagonal of **A** .

2- If λ a fixed eigenvalue of **A** ,thenthe set S consisting of all eigenvectors of **A** associated with λ as well as the zero vector is a subspace of $\mathbf{R}^n$ called the **Eigenspace associated with** λ .

Problems:

1- Which of the following matrices are diagonalizable :

$A=\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, C=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

2- let $\mathbf{A}=\begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$ compute A^9

3- If λ is eigenvalue of **A** ,with eigenvector X .Prove λ^k is eigenvalue of $\mathbf{A}^k$ with same eigenvector X.

4- A is called nilpotent if $\mathbf{A}^k=0$,prove if A is nilpotent then the only eigenvalue of A is 0..

Symmetric matrix

Definition: a matrix A is calledsymmetric matrix if $\mathbf{A}=\mathbf{A}^T$.

Ex: $A=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is symmetric matrix

Theorem: All the roots of the characteristic polynomial of a symmetric matrix are real numbers.

Corollary: if A is a symmetric matrix all of whose eigenvalues are distinct, then **A** is diagonalizable.

Proof:
Since **A** is symmetric, all its eigenvalues are real. From theorem **(*)** it follows that **A** can be diagonalzed.

Theorem : if **A** is a symmetric matrix, then eigenvectors that belong to distinct eigenvalues of **A** are orthogonal.
Proof:
First, we shall let the reader verify the property that if **X** and **Y** are vectors in $\mathbf{R}^n$,then

$$(\mathbf{AX}).\mathbf{Y}=\mathbf{X}.(\mathbf{A}^T\ \mathbf{Y}) \quad \text{........(H.W.)}$$

Now let X_1 and X_2 be eigenvectors of A associated with the distinct eigenvalues λ_1 and λ_2 of A, we then have

$A X_1 = \lambda_1 X_1$ and $A X_2 = \lambda_2 X_2$.

Now

$$\lambda_1 (X_1 . X_2) = (\lambda_1 . X_1) . X_2 = (A X_1) . X_2$$
$$= X_1 .(A^T X_2) = X_1 . (A X_2)$$
$$= X_1 .(\lambda_2 X_2) = \lambda_2 (X_1 . X_2)$$

Where we have used the fact that $A = A^T$.

Thus

$\lambda_1 (X_1 . X_2) = \lambda_2 (X_1 . X_2)$

and subtracting, we obtain

$$0 = \lambda_1 (X_1 . X_2) - \lambda_2 (X_1 . X_2)$$

$= (\lambda_1 - \lambda_2)(X_1 . X_2)$

Since $\lambda_1 \neq \lambda_2$, we conclude that $X_1 . X_2 = 0$.

Example :let $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

We find that the characteristic polynomial of **A** is (verify)

$f(\lambda) = (\lambda + 2)(\lambda - 4)(\lambda + 1)$

so the eigenvalues of **A** are

$\lambda_1 = -2, \quad \lambda_2 = 4, \quad \lambda_3 = -1$

then we find the associated eigenvectors by solving the linear system

$(\lambda I_3 - A) = 0$;

and obtain the respective eigenvectors

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Its clear that $\{ X_1, X_2, X_3 \}$ is an orthogonal set of vectors in $\mathbf{R}^3$ (and this thus linearly independent by theorem *),

thus **A** is diagonalizable and is similar to

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We recall that if **A**can be diagonal zed, then there exists a nonsingular matrix **P** such that $P^{-1}AP$ is diagonal, moreover the columns of **P** are eigenvectors of **A**

Now if the eigenvectors of **A** form an orthogonal set S, as happens when **A** is symmetric and the eigenvalue of **A** are distinct, then since any scalar multiple of an eigenvector of **A** is also an eigenvector of **A** ,we can normalize S to obtain an orthonormal set

$T=\{X_1\ X_2\ \dots\ X_n\}$

Of eigenvectors of **A** ,the J^{th} column of P is eigenvector X_j associated with λ_j, and we now examine what type of matrix **P** must be, we can write **P** as

$P=\{X_1\ X_2\ \dots\ X_n\}$.

Then

$$P^T=\begin{bmatrix} x_{1^T} \\ x_{2^T} \\ \vdots \\ x_{n^T} \end{bmatrix}$$

Where X_i^T, $1\le i\le n$,is the transpose of the $n\times 1$ matrix (or vector) X_i, we find that the I,j^{th} entry in $P^T P$ is $X_i.X_j$ (verify). Since

$$X_i.X_j=\begin{cases}1 & ifi=j\\ 0 & ifi\neq j\end{cases}$$

Then $P^T P=I_n$. thus$P^T=P^{-1}$, such matrices are important enough to have a special name.

Definition:A nonsingular matrix **A** is called **orthogonal** if

$A^{-1}=A^T$.

Of course, we can also say that a nonsingular matrix **A** is orthogonal if

$A^T A=I_n$.

Example:let $A=\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$.

It is easy to check that $A^T A=I_n$,henceA is an orthogonal matrix.

Example :let **A** be the matrix of example1, we already know that the set of eigenvectors $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\2\end{bmatrix},\begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$

Is orthogonal. If we normalize these vectors, we find that

$$T=\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-\frac{1}{\sqrt{5}}\\0\\\frac{2}{\sqrt{5}}\end{bmatrix},\begin{bmatrix}\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{bmatrix}\right\}$$

Is an orthogonal set of vectors, the matrix **P** such that that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal is the matrix whose columns are the vectors in T, thus

$$\mathbf{P}=\begin{bmatrix} 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

we leave it to the reader to verify that **P** is an orthogonal matrix and that

$$\mathbf{P}^{-1}\mathbf{AP}=\mathbf{P}^{T}\mathbf{AP}=\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Theorem :the $n\times n$ matrix **A** is orthogonal if and only if the columns of **A** form an orthonormal set of vectors in R^n.

Remark :If the $n\times n$ matrix **A** is orthogonal then $|A|\neq 0$

Theorem :if **A** is a symmetric $n\times n$ matrix, then there exists an orthogonal matrix **P** such that $P^{-1}AP=D$, a diagonal matrix. The eigenvalues of **A** lie on the main diagonal of **D** .

Example : let

$$A=\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

The characteristic polynomial of **A** is

$f(\lambda)=(\lambda+2)^2(\lambda-4)$,

so the eigenvalues are

$\lambda_1=-2, \quad \lambda_2=-2, \quad \lambda_3=4$

that is, -2 is an eigenvalue whose multiplicity is 2.

to find the eigenvectors associated with λ_1 and λ_2, we solve the homogeneous linear system $(-2I_3-A)X=0$;

$$\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (4)$$

A basis for the solution space of (4) consists of the eigenvectors

$$X_1=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } X_2=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Now X_1 and X_2 are not orthogonal, since $X_1 . X_2 \neq 0$, we can use the Gram -Schmidt process to obtain an orthonormal basis for the solution space of (4) (the eigenspace of $\lambda_1=-2$) as follows, let

$$Y_1 = X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

And

$$Y_2 = X_2 - \left(\frac{x_2 . y_1}{y_1 . y_2}\right) Y_1 = \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix}.$$

Let

$$Y^*_2 = 2\, Y_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

The set { Y_1, Y^*_2 } is an orthogonal set of vectors, Normalizing these eigenvectors, we obtain

$$Z_1 = \frac{y_1}{|y_1|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Z_2 = \frac{y_2}{|y_2|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

The set { Z_1, Z_2 } is an orthonormal basis of eigenvectors of **A** for the solution space of (4).

Now we find a basis for the solution space of system$(4I_3 - A)X = 0$,

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (5)$$

To consist of

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Normalizing this vector, we have the eigenvector

$$Z_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

As an orthonormal basis for the solution space of (5),

since eigenvectors associated with distinct eigenvalue are orthogonal, we observe that Z_3 is orthogonal to both $\mathbf{Z}_1$ and $\mathbf{Z}_2$, Thus the set { $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3$ } is an orthonormal basis of R^3 consisting of eigenvectors of **A**

The matrix **P** is the matrix whose j^{th} column is $\mathbf{Z}_j$:

$$P=\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

We leave it to the reader to verify that

$$p^{-1}AP=P^{T}AP=\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Example : let

$$A=\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

The characteristic polynomial of **A** is

$f(\lambda)=(\lambda+1)^2(\lambda-3)^2$,

so the Eigen values of **A** are

$\lambda_1=-1$, $\lambda_2=-1$, $\lambda_3=3$, $\lambda_4=3$.

We find (verify) that a basis for the solution space of

$$(-1I_3-A)X=0 \qquad (6)$$

Consists of the eigenvectors

$$X_1=\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad X_2=\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Which are orthogonal, Normalizing these eigenvectors, we obtain

$$Z_1=\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Z_2=\begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

As an orthonormal basis of eigenvectors for the solution space of (6). We also find(verify) that a basis for the solution space of

$$(3I_3-A)X=0 \qquad (7)$$

Consists of the eigenvectors

$$X_3=\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad \textbf{and} \quad X_4=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Which are orthogonal, Normalizing these eigenvectors, we obtain

$$Z_3=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\\0\end{bmatrix} \quad \text{and} \quad Z_4=\begin{bmatrix}0\\0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

As an orthonormal basis of eigenvectors for the solution space of (7). Since eigenvectors associated with distinct Eigen value are orthogonal, we conclude that

$$\{\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4\}$$

Is an orthonormal basis of R^4 consisting of eigenvectors of **A** ,the matrix **P** is the matrix whose J^{th} column is $\mathbf{Z}_j$:

$$\mathbf{P}=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

Suppose now that **A** is an $n\times n$ matrix for which we can find an orthogonal matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix **D**

,thus$\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$ or $\mathbf{A}=\mathbf{P}\ \mathbf{AP}^{-1}$.

Since $\mathbf{P}^{-1}=\mathbf{P}^{T}$, we can write $\mathbf{A}=\mathbf{P}\ \mathbf{DP}^{T}$.

Then $\mathbf{A}^{T}=(\mathbf{P}\ \mathbf{DP}^{T})^{T}=(\mathbf{P}^{T})^{T}\mathbf{D}^{T}\mathbf{P}^{T}=\mathbf{P}\ \mathbf{DP}^{T}=\mathbf{A}$

($\mathbf{D}=\mathbf{D}^{T}$, since **D** is a diagonal matrix). Thus **A** is symmetric.

Then

If P an orthogonal matrix such that $\mathbf{P}^{-1}\mathbf{AP}$ Thus **A** is symmetric

Exercises:

1-diagonalize $A=\begin{bmatrix}2&1&0\\1&2&0\\0&0&4\end{bmatrix}, B=\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$

2- Let $A = \begin{bmatrix} 9 & -1 & -2 \\ -1 & 9 & -2 \\ -2 & -2 & 6 \end{bmatrix}$ findorthogonal matrix such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonalizable .

3- Show that if A is an orthogonal matrix then A^T is also orthogonal

4- Show that if A is an orthogonal matrix then det(A)= ∓ 1

5- Show that if A is an orthogonal matrix then A^{-1} is also orthogonal

6- Show that if A ,B orthogonal matrices then AB is an orthogonalmatrix

LINES AND PLANES

Lines in $\mathbf{R}^2$:

Any two distinct points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ in R^2 (Figure 7.1)
Determine a straight line whose equation is

$$ax + by + c = 0, \quad \textbf{(1)}$$

where a, b , and c are real numbers, and a and c are not both zero, since P_1 and P_2 lie on the line, their coordinates satisfy equation (1):

$ax_1 + by_1 + c = 0$ **(2)**

$ax_2 + by_2 + c = 0.$ **(3)**

we now write (1), (2), and (3) as a linear system in the unknowns a, b, and c, obtaining

$$x\,a + y\,b + c = 0$$

(4)

$x_1 a + y_1 b + c = 0$

$x_2 a + y_2 b + c = 0$.

we seek a condition on the values x and y that allows (4) to have a nontrivial solution a, b and c, since (4) is a homogeneous system, it has a nontrivial solution if and only if the determinate of the coefficient matrix is zero that is. if and only if

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \qquad (5)$$

Thus every point P(x, y) on the line satisfies (5) and conversely, a point satisfying (5) lies on the line .

Example 1: find the equation of the line determined by the points $P_1(-1, 3)$ and $P_2(4, 6)$.

Solution:

Substituting in (5), we obtain

$$\begin{vmatrix} x & y & 1 \\ -1 & 3 & 1 \\ 4 & 6 & 1 \end{vmatrix} = 0$$

Expanding this determinant in cofactors about the first row, we have (verify)

$$-3x + 5y - 18 = 0$$

Lines in R^3

We may recall that in R^2 a line is determined by specifying its slop and one of its points. In R^3 a line is determined by specifying its direction and one of its points. Let U=(u, v, w) be a nonzero vector in R^3, and let $P_0=(x_0, y_0, z_0)$ be a point in R^3, let X_0 be a position vector in of P_0, then the line L through P_0 and parallel to U consists of the points P(x, y, z) whose position vector **X** satisfy (figure 7.2)

$$\mathbf{X} = \mathbf{X}_0 + t\,\mathbf{U} \qquad (-\infty<t<\infty), \qquad (6)$$

Equation (6) is called **a parametric equation** of L ,since it contains the parameter t, which can be assigned any real number, Equation (6) can also be written in terms of the components as

$$X = x_0 + tu$$

$$Y = y_0 + tv$$

$$Z = z_0 + tw,$$

Which are called **parametric equation** of L.

Example 2: parametric equations of the line through the point $P_0(-3,2,1)$, which is parallel to the vector U=(2,-3,4), are

$$X = -3 + 2t$$

$$Y = 2 - 3t \qquad (-\infty<t<\infty)$$

$$Z = 1 + 4t.$$

Example 3: find parametric equations of the line L through the point $P_0(2,3,-4)$ and $P_1(3,-2,5)$

Solution:

The desired line is parallel to the vector $U=\overrightarrow{P_0P_1}$. Now

$$U=(3-2,\ -2-3,\ 5-(-4))=(1,-5,9).$$

Since P_0 lies on the line, we can write the parametric equations of L as

$$X=2+t$$
$$Y=3-5t$$
$$Z=-4+9t.$$

In Example 3 we could have used the point P_2 instead of P_1, in fact we could use any point on the line in the parametric equations of L. Thus a line can be represented in infinitely many ways in parametric form, if u, v, and w are nonzero in (7), we can solve each equation for t and equate the results to obtain the equations in **symmetric form** of the line through P_0 and parallel to U:

$$\frac{x-x_0}{u}=\frac{y-y_0}{v}=\frac{z-z_0}{w}.$$

The equations in symmetric form of the line are usual in some analytic geometry applications.

Example 4: The equations in symmetric form of the line in Example 3 are

$$\frac{x-2}{1}=\frac{y-3}{-5}=\frac{z+4}{9}.$$

Planes in R^3:

A plane in R^3 can be determined by specifying a point in the plane and vector perpendicular to it, A vector perpendicular to a plane is called **normal** to the plain.

To obtain an equation of the plane passing through the point $P_0=(x_0, y_0, z_0)$ and having The nonzero vector N= (a, b, c) as a normal we proceed as follows, A point P(x, y, z) Lies in the plane if and only if the vector $\overrightarrow{p_1p_2}$ is perpendicular to N (figure 7.3)

Thus P(x, y, z) lies in the plane if and only if

$$N.\overrightarrow{p_1p_2}=0 \qquad (8)$$

Since

$\overrightarrow{p_1p_2}=(x-x_0, y-y_0, z-z_0)$,

We can write (8) as

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0, \qquad (9)$$

Example 5: Find an equation of the plane passing through the point (3, 4, -3) and perpendicular to the vector N=(5, -2, 4).

Solution:

Substituting in (9), we obtain the equation of the plane as

$$5(x-3)-2(y-4)+4(z+3)=0 \qquad \textbf{(10)}$$

A plane is also determined by three nonlinear points, as we show in the following example.

Example 6: Find an equation of the plane passing through the points $p_1(2, -2, 1)$, $p_2(-1, 0, 3)$, and $p_3(5, -3, 4)$.

Solution:

The nonparallel vectors $\overrightarrow{p_1p_2} = (-3, 2, 2)$ and $\overrightarrow{p_1p_3} = (3, -1, 3)$ lie in the plane, since the points p_1, p_2 and p_3 lie in the plane, the vector

$$N = \overrightarrow{p_1p_2} \times \overrightarrow{p_1p_3} = (8, 15, -3)$$

Is then perpendicular to both $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_1p_3}$ is thus a normal to a plane, using the vector N and the point $p_1(2, -2, 1)$ in (9), we obtain an equation of the plain as

$$8(x-2) + 15(y+2) - 3(z-1) = 0$$

If we multiply out and simplify, (9) can be written as

$$ax + by + cz + d = 0 \qquad (11)$$

Example 7: The equation for the plane in Example 6 can be rewritten in the form given by equation (11) as

$$8x + 15y - 3z + 17 = 0 \qquad (12)$$

Example 8: A second solution to Example 6 is as follows, let the equation of the desired plane be

$$ax + by + cz + d = 0 \qquad (13)$$

where a, b, c and d are to be determined. Since p_1, p_2 and p_3 lie in the plane, their coordinates satisfy (13), thus we obtain the linear system

$$\begin{aligned} 2a - 2b + c + d &= 0 \\ -a \qquad + 3c + d &= 0 \\ 5a - 3b + 4c + d &= 0. \end{aligned}$$

Solving this system, we have

$$a = \tfrac{8}{17}r, \quad b = \tfrac{15}{17}r, \quad c = -\tfrac{3}{17}r, \quad d = r,$$

where r is any real number, letting r=17, we obtain

$$a = 8, \quad b = 15, \quad c = -3, \quad d = 17,$$

which yields (12) as in the first solution.

Example 9: A third solution to Example 6 is as follows, proceeding as in the case of a line in R^2 determined by two distinct points P_1 and P_2, it is not difficult to show that an equation of the plane through the non collinear points $P_1(x_1, y_1, z_2)$, $P_2(x_2, y_2, z_2)$, And $P_3(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

In our example, the equation of the desired plane is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & -2 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 5 & -3 & 4 & 1 \end{vmatrix} = 0,$$

Expanding this determinant in cofactors about the first row, we obtain equation (12)
The equation of a line in symmetric form can be used to determine two plans whose Intersection is the given line.

Example 10: find two plans whose intersection is the line

$$X = -2 + 3t \qquad Y = 3 - 2t \qquad Z = 5 + 4t. \qquad (-\infty < t < \infty)$$

Solution:

First ,find equations of the line in symmetric form as

$$\frac{x+2}{3} = \frac{y-3}{-2} = \frac{z-5}{4}$$

The given line is then the intersection of the plans

$$\frac{x+2}{3} = \frac{y-3}{-2} \quad \text{and} \quad \frac{x+2}{3} = \frac{z-5}{4}$$

Thus the given line is the intersection of the planes

$$2x + 3y - 5 = 0 \quad \text{and} \quad 4x - 3z + 23 = 0$$

Two planes are either parallel or they intersect in a straight line, they are parallel if their normal are parallel, in the following example we determine the line of intersection of two planes.

Example 11: Find parametric equations of the line of the intersection of the planes $\pi_1 : 2x + 3y - 2z + 4 = 0$ and $\pi_2 : x - y + 2z + 3 = 0$.

Solution:

Solving the linear system consisting of the equations of π_1 and π_2, we obtain

$$X = \frac{-13}{5} - \frac{4}{5}t \qquad Y = \frac{2}{5} + \frac{6}{5}t \qquad Z = 0 + t \qquad (-\infty < t < \infty)$$

As parametric equation of the line L of intersection of the planes

Three planes in R^3 may intersection in a plane, in a line in a unique point, or may not intersection at all, these possibilities can be detected by solving the linear system consisting of their equations.

ıadratic forms

efinition: if A is asymmetric matrix ($A = A'$), then the function $Q: R^n \to R^n$ real- valued function defined on R^n) defined by

$$Q(x) = X^T AX,$$

Where

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}.$$

Is called **a quadratic form in n variables** $x_1, x_2, \ldots, x_n$ The matrix A is called **the matrix of the quadratic form Q.** we shall also denote the quadratic form by **Q(x).**

xample 1: the left side of equation (1) is the quadratic form in the variable X and Y: where

$$Q(x) = X^T AX$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Example 2: the left side of equation (1) is the quadratic form

$$Q(x) = X^T AX,$$

Where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}.$$

Example 3: The following expressions are quadratic form:

(a) $3x^2 - 5xy - 7y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & \frac{-5}{2} \\ \frac{-5}{2} & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$

1- $3x^2 - 7xy + 5xz + 4y^2 - 4yz - 3z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & \frac{-7}{2} & \frac{5}{2} \\ \frac{-7}{2} & 4 & -2 \\ \frac{5}{2} & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$

Suppose now that $Q(x) = X^T AX$ is a quadratic form. To simplify the quadratic form, we change form the variable $x_1, x_2, \ldots, x_n$ to the variables $y_1, y_2, \ldots, y_n$, where we assume that the old variables are related to the new variables by $X = PY$ for some orthogonal matrix **P** ,then

$Q(x) = X^T AX = (PY)^T A(PY) = Y^T (P^T AP)Y = Y^T BY$,

Where $B = P^T AP$, and if **A** is a symmetric matrix then $P^T AP$ is also symmetric thus

$Q'(Y) = Y^T BY$

Is another quadratic form and $Q(x) = Q'(Y)$.

efinition : Two $n \times n$ matrices A and B are said to be **congruent** if $B = P^T AP$ for a nonsingular matrix

efinition : Two quadratic forms **Q** and Q' with matrices A and **B**, respectively are said to be **equivalent if A and B are congruent .**

xample 4 : consider the quadratic form in the variables x and y defined by

$$Q(x) = 2x^2 + 2xy + 2y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \ldots(3)$$

We now change form the variables x and y to the variables x' and y', suppose the old variables are related to the new variables by the equations

$$X = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y' \quad \text{and} \quad Y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \quad \ldots(4)$$

Which can be written in matrix form as

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = PY.$$

Where the orthogonal (hence nonsingular) matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Substituting in (3) we obtain

$Q(x) = X^T AX = (PY)^T A(PY) = Y^T P^T APY$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{Q}'(\mathbf{Y}).$$

Thus the matrices

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{and} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Are congruent and the quadratic forms Q and Q' are equivalent.
The equation

$$Q(x) = 2x^2 + 2xy + 2y^2 = 9 \quad .....(5)$$

Represents a conic section ,since Q is a quadratic form defined in example 4, it is equivalent to the quadratic form

$$Q'(Y) = 3x'^2 + y'^2.$$

Now the equation

$$Q'(Y) = 3x'^2 + y'^2 = 9 \quad ......(6)$$

Is the equation of an ellipse.

Theorem 8: Any quadratic form in n variables $\mathbf{Q(x) = X^T AX}$ is equivalent by means of an orthogonal (principal nal matrix P to a quadratic form, $Q'(Y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

And $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the Eigen values of the matrix A of Q.

Proof :

If A is the matrix of **Q**, then since **A** is symmetric we know by theorem * that A can be diagonalized by an orthogonal matrix, this mean that there exists an orthogonal matrix P such that

$$\mathbf{B = P^{-1} AP}$$ is diagonal matrix, since P is orthogonal ,

$\mathbf{P^{-1} = P^T}$, so $\mathbf{B = P^T AP}$

moreover ,the elements on themain diagonal of **B** are the Eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, the quadratic form Q' with matrix **B** is given by

$$Q'(Y) = \lambda_1 y_1^{\,2} + \lambda_2 y_2^{\,2} + \ldots + \lambda_n y_n^{\,2}$$

Q and Q' are equivalent .

xample 5: consider the quadratic form Q in the variables x, y and z, defined by

$$Q(x) = 2x^2 + 4y^2 + 6yz - 4z^2 .$$

The matrix of Q is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & -4 \end{bmatrix}$$

And the Eigen values of A are

$\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_3 = -5$

Let Q' be the quadratic form in variables x', y', and z' defined by

$$Q'(Y) = 2x'^2 + 5y'^2 - 5z'^2 .$$

Then **Q** and $\mathbf{Q}'$ are equivalent by means of some orthogonal matrix, since

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Is the matrix of $\mathbf{Q}'$, **A** and **D** are congruent matrices.

xample 6: consider the conic section whose equation is (5),

$$Q(x) = 2x^2 + 2xy + 2y^2 = 9,$$

This conic section can also be described by equation (6)

$$Q'(Y) = 3x'^2 + y'^2$$

Which can be written as

$$\frac{x'^2}{3} + \frac{y'^2}{9} = 1$$

This is the equation of an ellipse .

whose major axis is along the y'-axis, the semi major axis is of length 3, the semi major axis of length $\sqrt{3}$, we now note that there is q very close connection between the eigenvectors of the matrix **Q** in (5) and the location of the x'-and y'-axes

sinceX = PY, we have $\mathbf{Y} = \mathbf{P}^{-1}\mathbf{X} = \mathbf{P}^{T}\mathbf{X}$ (**P** is orthogonal), thus

$$x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \text{ and } y' = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

This means that in terms of the x- and y-axes, the $\mathbf{x}'$-axis lies along the vector

$$X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

and the y'-axis lies along the vector

$$X_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now $\mathbf{X}_1$ and $\mathbf{X}_2$ are the columns of the matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

Which in turn are eigenvectors of the matrix **A**, thus the x'-and y'-axes lie along the Eigenvectors of the matrix **A**.

The situation described in example 6 is turn in general, thus the principal axes of a conic Or surface lie along the eigenvectors of the matrix of the quadratic form.

Let **Q(X)** = $\mathbf{X}^T \mathbf{AX}$ be a quadratic form in n variable, then we know that **Q** is equivalent To the quadratic form $Q'(Y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix **A** of **Q** ,we can label the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ so that all the Positive eigenvalues of **A** ,if any, are listed first , followed by all the negative eigenvalues Of **A** ,if any, followed by the zero eigenvalues, if any, thus let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be positive, $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_r$ be negative, and $\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n$ be zero, we now define the diagonal matrix **H** whose entries on the main diagonal are

$\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \ldots, \frac{1}{\sqrt{\lambda_p}}, \frac{1}{\sqrt{-\lambda_{p+1}}}, \frac{1}{\sqrt{-\lambda_{p+2}}}, \ldots, \frac{1}{\sqrt{-\lambda_r}}, 1, 1, \ldots, 1$, with n – r ones, let **D** be the diagonal Matrix whose entries on the main diagonal are $\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_r, \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n$;

A and **D** are congruent let $\mathbf{D}_1 = \mathbf{H}^T \mathbf{DH}$ be the matrix whose diagonal elements are $1, 1, \ldots, 1, -1, 0, 0, \ldots, 0$ (p ones, n – r zeros);

D and $\mathbf{D}_1$ are then congruent, it follows that **A** and $\mathbf{D}_1$ are congruent, in terms of quadratic forms,

Theorem 9: A quadratic form **Q(x)** = $\mathbf{X}^T \mathbf{AX}$ in n variables is equivalent to a quadratic form $Q'(Y) = y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \ldots - y_r^2$ for some $0 \le p, r \le n$.

It is clear that the rank of matrix D_1 is r, the number of nonzero entries On its diagonal, now it can be shown that the convergent matrices have Equal ranks, since the rank of D_1 is r and the rank of A is r, we also refer to r as the **rank** of the quadratic form Q whose matrix is A, the Difference between the number of positive Eigen values and the number Of negative Eigen values is s = p –(r – p) = 2p – r and called the **Signature** of the quadratic form, thus if Q and Q′ are equivalent Quadratic form then they have equal ranks and signatures.

ample 7: consider the quadratic form:

$$Q(X) = 3x_2^2 + 8x_2x_3 - 3x_3^2 = \mathbf{X}^T\mathbf{AX}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The Eigen values of A are

$\lambda_1 = 5$, $\lambda_2 = -5$, and $\lambda_3 = 0$,

In this case A is congruent to

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we let

$$H = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Then

$$D_1 = H^T DH = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And A are congruent, and the given quadratic form is equivalent to the Canonical form

$\mathbf{Q'} = y_1^2 - y_2^2$

The rank of Q is 2, and since p = 1, the signature s = 2p – r = 0,

efinition : A symmetric n × n matrix A is called positive definite if $\mathbf{X}^T\mathbf{AX} > \mathbf{0}$ For every onzerovector X in R^n.

Theorem 10: A symmetric matrix A is positive definite if and only if all the eigenvalues of A are ositive.

A quadratic form is then called **positive definite** if its matrix is positive definite .

eorem 4.82 (cayley_hamilton): if $f(\lambda)$ is the characteristic polynomial of the mapping $\in L(V)$, then $f(T) = 0$.

oof: As mentioned above, we argue using matrices, let $A \in M_n(F)$ be the matrix of T relative o some fixed basis for V, by the corollary to theorem 4 – 33, the adjoint of the matrix $A - \lambda I$ is known to satisfy the identity

$$(A - \lambda I)\, adj\,(A - \lambda I) = \det(A - \lambda I)I = f(\lambda)I$$

Since $A - \lambda I$ is of order n, the elements of adj $(A - \lambda I)$ are signed determinants of order n-1 When expanded, these yield polynomials in λ of degree at most n-1, combining the terms Involving powers of λ^i into a coefficient matrix B, we may write adj$(A - \lambda I)$ as the polynomial matrix

$$adj\,(A - \lambda I) = B_{n-1}\lambda^{n-1} + \ldots + B_1\lambda + B_0,$$

where each $B_i \in M_n(F)$, if the characteristic polynomial of T is given by

$$f(\lambda) = (-1)^n \lambda^n + b^{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0 \qquad b_i \in F$$

then our adjoint identity can be written in the more detailed form

$$(A - \lambda I)(B_{n-1}\lambda^{n-1} + \ldots + B_1\lambda + B_0) = ((-1)^n \lambda^n + b^{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0)I.$$

Both sides of this equation are polynomial matrices in λ of degree n, Since two polynomial Matrices are equal if and only if their corresponding coefficients are equal, we obtain the Following relations:

$$-B_{n-1} = (-1)^n I$$
$$AB_{n-1} - B_{n-2} = b_{n-1}I$$
$$AB_{n-2} - B_{n-3} = b_{n-2}I.$$
$$\vdots$$
$$AB_1 - B_0 = b_1 I$$
$$AB_0 = b_0 I$$

Now multiply these matrix equations on the left by $A^n, A^{n-1}, \ldots, A, I$, respectively, and add the Results, the terms on the left-hand side will cancel out in pairs, leaving only the zero matrix, This gives

$$0 = (-1)^n A^n + b^{n-1}A^{n-1} + \ldots + b_1 A + b_0 I = f(A).$$

Through the usual association of matrices in $M_n(F)$ with elements of L(V), the foregoing Equation can be written as $f(T) = 0$, which is the result we require there is a fairly short, but Erroneous, "proof" of this last result which runs as follows formally substitute the matrix A For the indeterminate λ in the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$, the net effect Is that $f(\lambda) = \det(A - AI) = \det(A - A) = \det 0 = 0$.

The fallacy in this attractive argument is that the right-most zero is a scalar whereas theorem

4 – 82 asserts that $f(\lambda)$ should equal the zero matrix of order n,
The cayley – Hamilton theorem has a number of interesting applications, two of which we Shall examine below, first it provides an easy method of expressing any polynomial in a Matrix $A \in M_n(F)$ as a polynomial of degree at most n – 1, let $f(\lambda) = \det(A - \lambda I)$ be the Characteristic polynomial of A and $g(\lambda)$ be an arbitrary polynomial, we may divide $g(\lambda)$

V