MATHEMATICS

Second Class

Department of Geology

First Semester

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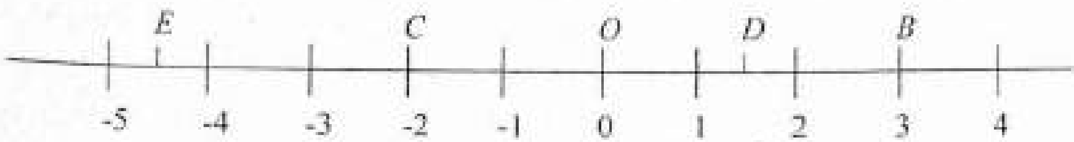
Linear Algebra

Vectors in the plane

Coordinate systems:

We recall that the real numbers system may be visualized as a straight line L, which is usually taken as a horizontal position. A point O called the **origin**, on L; O corresponds to the number 0. A point A is chosen to the right of O and fixing the length of OA as 1 and specifying a positive direction. Thus the positive real numbers lie to right of O; the negative real numbers lie to the left of O(see figure 1).



The absolute value $|x|$ of the real number x is defined by

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

Thus $|3| = 3$, $|-2| = 2$ and $|0| = 0$

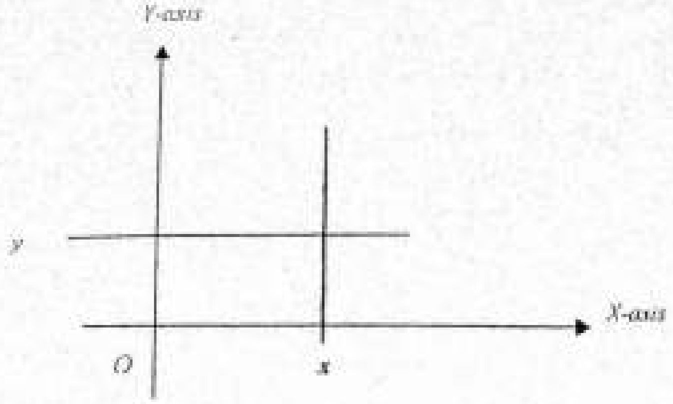
If a, b are two points on the line L, then the distance between the point a and b is |b-a|.

Example: If $a=3$, $b=1.5$, the distance between a and b is
$|b-a|=|1.5-(-3)|=4.5$.

In the plane:

We draw a pair of perpendicular lines intersecting at a point O, called the origin. One of the lines, the x-axis, is usually taken in a horizontal position, the other line, the y-axis, is taken in a vertical position.

We now choose a point on the x-axis to right of O and appoint of the y-axis above O to fix the units of length is used for both axis.



Thus with every point in the plane we associate an order pair (x,y) of real numbers its coordinates. The point p with coordinates x and y is denoted by $p(x,y)$.

Conversely, it is easy to see how we can associate a point in the plane with each order pair (x,y) of real number.

The correspondence given above between points in the plane and ordered pairs of real number is called rectangular coordinate system or the Cartesian coordinate system. The set of all points in the plane is denoted by R^2. It is also called *2*-space.

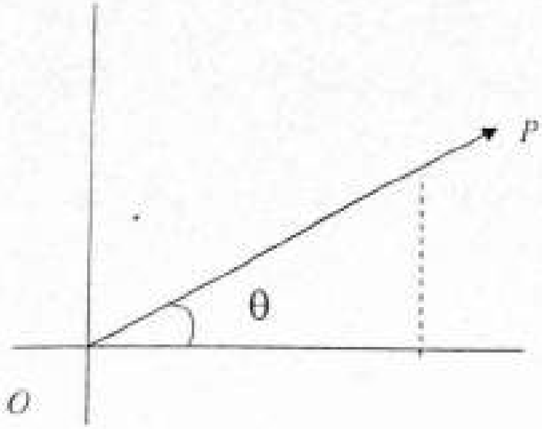
Vectors:

Consider the 2×1 matrix

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

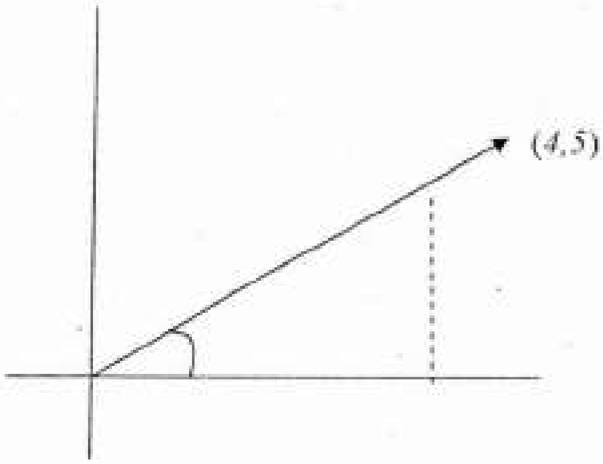
Where x and y are real numbers. With X we associate the directed line segment with initial point at the origin $O\ (0,\ 0)$ and terminal point at $p(x,y)$. It is denoted by $\overline{OP}$; O is called its **tail** and P its **head**.

A directed line segment has a direction, which is the angle made with the positive x-axis, indicated by the arrow at its head



The magnitude of a directed line segment is its length.

Example: Let $X = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ be a vector with the directed line $\overrightarrow{OP}$ with head $p(4,5)$.



Definition: A vector in the plane is a 2×1 matrix $X = \begin{bmatrix} x \\ y \end{bmatrix}$ where x and y are real numbers, called the **component** of X.

Note: Since a vector is a matrix, the vectors

$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

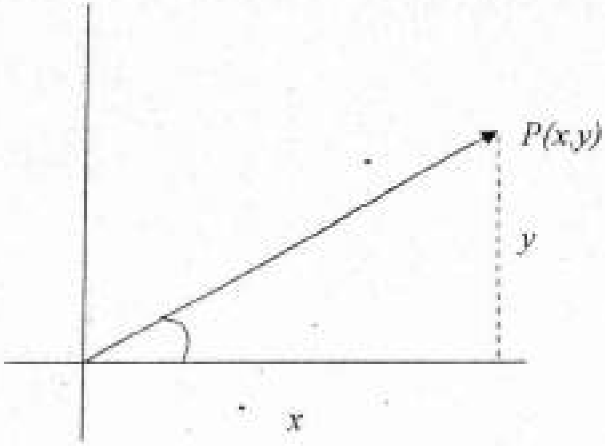
Are said to be **equal** if $x_1 = x_2$ and $y_1 = y_2$. That is two vectors are equal if their respective components are equal.

Example: The vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are not equal.

Length:

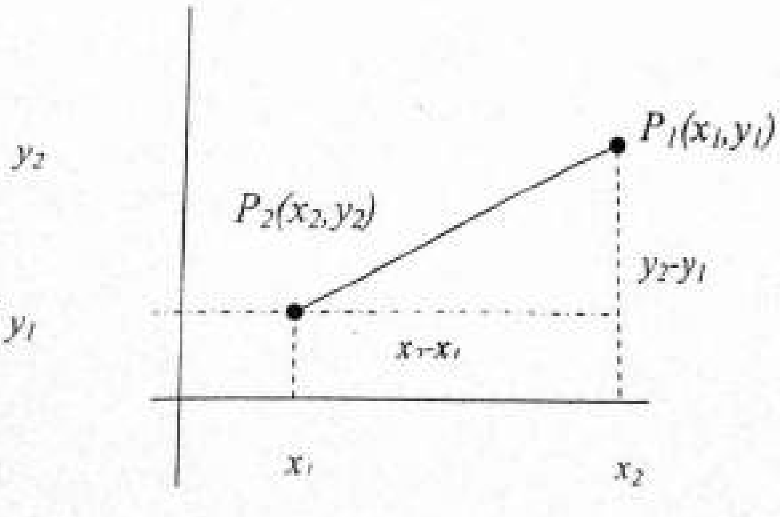
By the Pythagorean theorem the length or magnitude of the vector $X=(x,y)$ is

$$\|X\|=\sqrt{x^2+y^2} \quad \ldots \quad (1)$$



If $P(x_1,y_1)$ and $Q(x_2,y_2)$ are two points on the plan then by the Pythagorean theorem, the length of the directed segment with initial point $P_1(x_1,y_1)$ and terminal point $P_2(x_2,y_2)$ is

$$\|P_1P_2\|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \quad \ldots \quad (2)$$



Example: If $X=(2,-5)$, then

$\|X\| = \sqrt{(2)^2 + (-5)^2} = \sqrt{4+25} = \sqrt{29}$

Example: The distance between $P\ (3, 2)$ and $Q\ (-1, 5)$ is
$\|\vec{PQ}\| = \sqrt{(-1-3)^2 + (5-2)^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

Note: Two vectors $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ is said to be **parallel** if $x_1y_2 = x_2y_1$ or have the same **slopes**.

Vector operations:

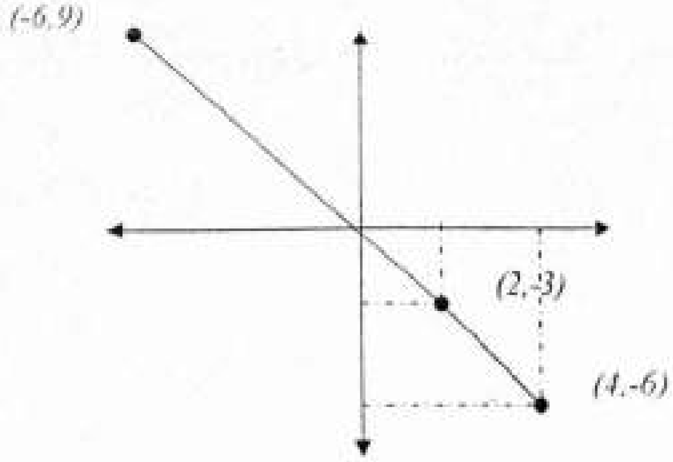
Definition: Let $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ be two vectors in the plane. The **sum** of the vectors X and Y is the vector
(x_1+x_2, y_1+y_2)
And is denoted by $X+Y$.

Example: Let X= (2, 3) and Y= (-5, 6). Then

$Q(x_2,y_2)$ X+Y= (2+ (-5)), 6+3) = (-3, 9)

Definition: If X=(x, y) is a vector and c is a scalar (real number), then the **scalar multiple** cX of X by c is the vector (cx, cy)
If c>0, then cx in the same direction of X
If c<0, then cx in the opposite direction of X

Example: If c=2, d=-3 and X= (2, -3), then
cX=2(2, -3)=(4, -6) and dX= -3(2,-3)=(-6, 9)



Definition: The vector (0,0) is called the **zero vector** and is denoted by O. If X is any vector, then

$$X+0=X.$$

We can also show that

$$X+(-1)X=O.$$

And we can write (-1) X as –X and call it the **negative** of X.
And we write X+ (-1) Y as and call it the difference between X and Y. The vector X-Y is

The angle between two vectors:

The angle between the nonzero vector $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ is the angle θ ,where $0 \le \theta \le \pi$
Appling the law of cosines to the triangle we have

$$\|X-Y\|^2 = \|X\|^2 + \|Y\|^2 - 2\|X\|\|Y\|\cos\theta \quad \dots (2)$$

From (2)

$$\|X-Y\|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= x_1^{\ 2} + x_2^{\ 2} + y_1^{\ 2} + y_2^{\ 2} - 2(x_1x_2 + y_1y_2)$$
$$= \|X\|^2 + \|Y\|^2 - 2(x_1x_2 + y_1y_2)$$

$$\|X\|^2 + \|Y\|^2 - 2(x_1x_2 + y_1y_2) = \|X\|^2 + \|Y\|^2 - 2\|X\|\|Y\|\cos\theta$$

$\|X\| \ne 0$ and $\|Y\| \ne 0$

Then

$$\cos\theta = \frac{x_1x_2 + y_1y_2}{\|X\|\|Y\|} \quad \dots(3)$$

Definition: The **inner product** or **dot product** of the vectors $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ is defined to be :

$$X \bullet Y = x_1x_2 + y_1y_2 \qquad ...(4)$$

Thus we can rewrite (3) as

$$\cos\theta = \frac{X \bullet Y}{\|X\|\|Y\|} \qquad (0 \le \theta \le \pi)...(5)$$

Example: If X=(2,4), Y=(-1,2), then

$X \bullet Y = (2)(-1) + (4)(2) = 6$

Also

$\|X\| = \sqrt{(2)^2 + (4)^2} = \sqrt{20}$ And $\qquad \|Y\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

Hence

$\cos\theta = \frac{6}{\sqrt{20}.\sqrt{5}} = 0.6$ Then $\qquad \theta = 53°8'$ $\qquad$ from table.

Note: If the nonzero vectors X and Y at right angles, then the cosine of the angle θ, $\cos\theta=0$. Hence X.Y=0, conversely if X.Y=0 then $\cos\theta=0$ then $\theta = \frac{\pi}{2} = 90°$

Thus the nonzero vectors X and Y are perpendicular or orthogonal if and only if X.Y=0.

Example:

If X=(2,-4), Y=(4,2)

X.Y=4.2+2(-4)=0 then $X \perp Y$

The properties of the dot product

Theorem: If X,Y and Z are vectors and c is a scalar, then:

(1) $X \bullet X = \|X\|^2 \ge 0$, with equality if and only if X=0,

(2) $X \bullet Y = Y \bullet X$,

(3) $(X+Y) \bullet Z = X \bullet Z + Y \bullet Z$,

(4) $(cX) \bullet Y = X \bullet (cY) = c(X \bullet Y)$.

Proof: **H.W.**

Unit vectors:

A unit vector is a vector whose length is 1 and denoted by U. If X is any nonzero vector, then the vector

$$U = \frac{1}{\|X\|} \bullet X$$

Is a unit vector in the direction of X.

H.W. prove that for any unit vector U, then $\|U\| = 1$.

Example: Let X=(-3,4), then

$\|X\| = \sqrt{(-3)^2 + 4^2} = 5$

Then the vector U is

$U = \frac{1}{5}(-3,4) = (\frac{-3}{5}, \frac{4}{5})$ is the unit vector.

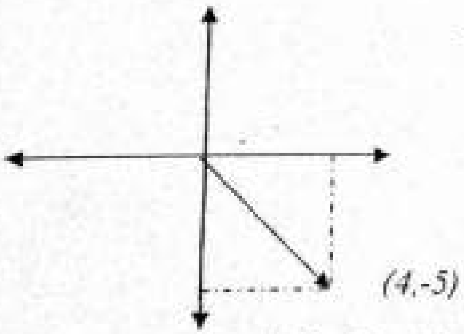
$\|U\| = \sqrt{(\frac{-3}{5})^2 + (\frac{4}{5})^2} = \sqrt{\frac{9+16}{25}} = 1$

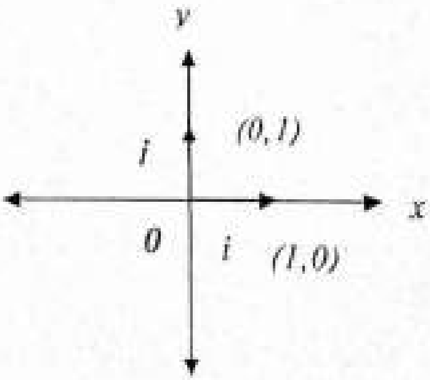
Then U lies in direction of X.

Now, there are two unit vectors in R^2. They are i=(1,0) and j=(0,1).
If X=(x,y) is any vector in R^2, then we write X in the term of I and j as

X=xi+yj

Example: If X=(4,-5) then we can say that X=4i-5j.



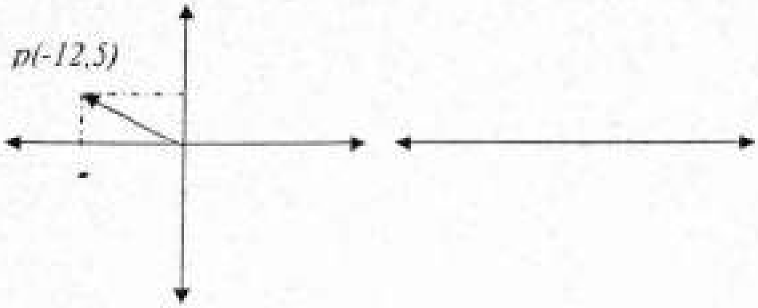


X=4(1,0)+(-5)(0,1)=(4,0)+(0,-5)=(4,-5)

Applications:

Suppose that a force\power of 12 pounds act on a solid in the direction of a negative x-axis and force\power of 5 pounds act on the same solid in the direction of a positive y- axis, find the value and the direction of the magnitude.

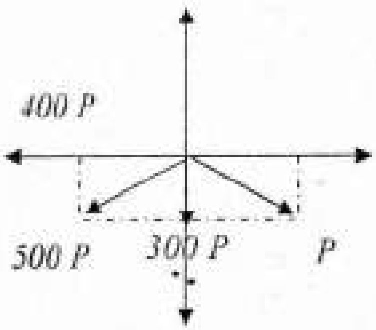
$\|\overrightarrow{OP}\| = \sqrt{12^2+5^2} = \sqrt{144+25} = \sqrt{169} = 13$



Example: A ship is being pushed by a tugboat with a force of 300 pounds along the negative y-axis while another tugboat is pushed along the negative x-axis with a force of 400 pounds. Find the magnitude and sketch the direction of the resultant force.

Solution:

$\|\overrightarrow{OP}\| = \sqrt{16000+90000} = \sqrt{250000} = 500$ pounds.



Exercises :

1- Find X+Y ,X-Y ,2X and 3X-2Y if X =(2,3) ,Y=(-2,5).

2- Let X=(1,2) ,Y=(-3,4),Z=(x,4) and U= (-2,y) find x and y so that

(a) Z=2X (b) $\frac{3}{2}$U=Y (c) Z+U=X

3-Find the length of X=(-4,-5).

4- Find the distance between (0,3),(2,0).

5-Find X • Y,X=(-2,-3), (2,-1) ,find the cosine of the angle between X,Y.
6- which of the vectors X=(1,2) ,Y =(0,1) ,Z =(-2,-4) ,W=(-2,1), U=(-6,3) are orthogonal ,in same direction ,in opposite direction

7- Show that if Z orthogonal to X and Y then Z orthogonal to rX+ sY ,where r, s are scalars .

n-vectors:

Definition: An n-vectors is an $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Where $x_1, x_2, \cdots, x_n$ and real numbers, which are called **the component** of X.
Since an n-vector is a matrix, the n-vectors

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Are said to be **equal** if $x_i = y_i$ where $1 \le i \le n$.

Example: The 4-vectors $\begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$ are not equal.

Note: The set of all n-vectors is denoted by R^n and called **n-space**.

Vector operations:

Definition: Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Be two vectors in R^n. The **sum** of the vectors X and Y is the vector

$$\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

And it is denoted by X+Y.

Example: If

$$X = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$$

Are vectors in R^3, then

$$X + Y = \begin{bmatrix} 1+2 \\ -2+3 \\ 3+(-3) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Definition: If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector in R^n and c a scalar, then the scalar multiple cX of X and **c** is the vector $\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$.

Example: if $X = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is a vector in R^3 and c=-2, then $cX = (-2)\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \end{bmatrix}$.

Theorem: Let X,Y and Z be any vectors in R^n; let **c and d** be any scalars. Then:

(a) X+Y is a vector in R^n(that is, R^n is closed under the operation of addition)

(1)X+Y=Y+X,

(2)X+(Y+Z)=(X+Y)+Z,

(3)There is a unique vector 0 in R^n, where $0=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$ such that X+0=0+X=X, 0 is called **zero vector.**

(4)There is a unique vector −X, where $-X=\begin{bmatrix}-x_1\\-x_2\\\vdots\\-x_n\end{bmatrix}$ such that X+(-X)=0.

(b) cX is a vector in R^n

(1)c(X+Y)=cX+cY,

(2)(c+d)X=cX+dX,

(3)c(dX)=(cd)X,

(4)1X=X.

Proof: (a) and (b) are immediately from the definitions for vector sum and scalar multiple.

We prove that (c+d)X=cX+dX

$$(c+d)X=(c+d)\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}(c+d)x_1\\(c+d)x_2\\\vdots\\(c+d)x_n\end{bmatrix}=\begin{bmatrix}cx_1+dx_1\\cx_2+dx_2\\\vdots\\cx_n+dx_n\end{bmatrix}.$$

(4) $$X=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}1.x_1\\1.x_2\\\vdots\\1.x_n\end{bmatrix}=\begin{bmatrix}x_1.1\\x_2.1\\\vdots\\x_n.1\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=X.$$

Example: If X and Y are vectors such that $X=\begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $Y=\begin{bmatrix}2\\3\\-3\end{bmatrix}$ then

$$X-Y=\begin{bmatrix}1-2\\-2-3\\3-(-3)\end{bmatrix}=\begin{bmatrix}-1\\-5\\6\end{bmatrix}.$$

Application: The vector in R^n can be used to handle a large amounts of data. Inded a number of computer programming languages.

Example: Suppose that a store handles 100 different items. The inventory on hand at the beginning of the week can be described by the inventory vector A in R^{100}. The number of item sold at the end of the week can be described by the vector S and the vector A-S represents the inventory at the end of the week.

$$A=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{100} \end{bmatrix} \in R^{100}$$

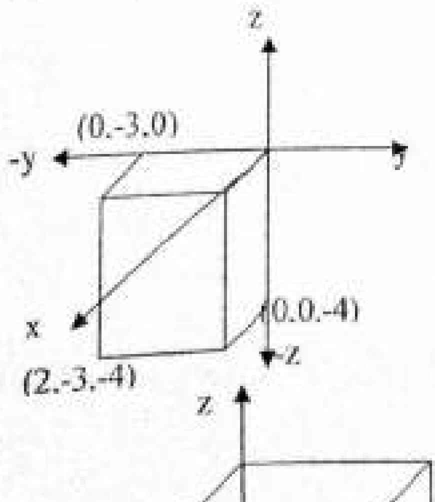
If the store receives a new shipment of goods represented by the vector B. Then its new inventory would be

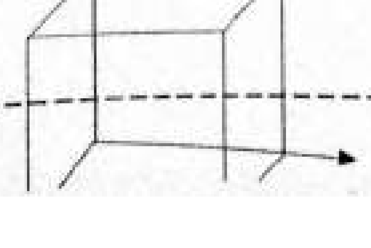
A-S+B

The space R^3:

We draw the three dimension system by fixing the point called **origin** point then we draw the three coordinate axis

Example: Find (2,-3,-4) and (3,5,7) on the coordinate system.

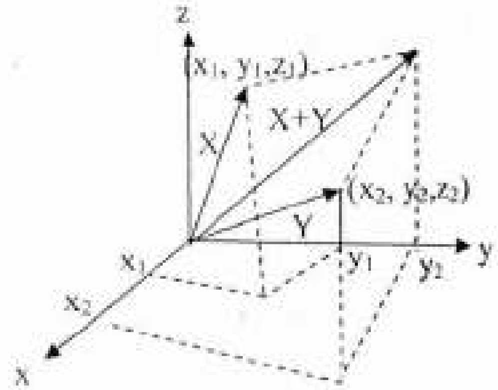




(0,0,7)

(3,5,7)

Note: The sum X+Y of the vectors in R^3 is the diagonal of the parallelogram determined by X and Y.
To illustrate the above note, let $X=(x_1, y_1, z_1)$and $Y=(x_2, y_2, z_2)$ then:



Definition: The length on norm of the vector $X=(x_1, x_2,\dots, x_n)$ in R^n is
$\|X\| = \sqrt{x_1^{\,2} + x_2^{\,2} + \dots + x_n^{\,2}}$
We also define the distance between the points $(x_1, x_2,\dots,x_n)$ and $(y_1, y_2,\dots,y_n)$ by
$\|X-Y\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2}$
Example[*]**:** Let X=(2,3,2,-1) and Y=(4,2,1,3). Then
$\|X\| = \sqrt{2^2+3^2+2^2+(-1)^2} = \sqrt{18}$
$\|Y\| = \sqrt{4^2+2^2+1^2+3^2} = \sqrt{30}$

$\|X-Y\|=\sqrt{(2-4)^2+(3-2)^2+(2-1)^2+(-1-3)^2}=\sqrt{22}$

Definition: If $X=(x_1,x_2,\ldots,x_n)$ and $Y=(y_1,y_2,\ldots,y_n)$ are vectors in R^n, then their **inner product** is defined by:

$X.Y=x_1y_1+x_2y_2+\ldots+x_ny_n$

Also called **dot product**.

Example: If $X=(2,3,2,-1)$ and $Y=(4,2,1,3)$, then

$$X.Y=(2)(4)+(3)(2)+(-1)(3)$$
$$=13$$

Example:(Revenue Monitoring): Consider the store in the above example with (*), if the vector P denoted the price of each of the 100 items, then the dot product $S.P$ given the total revenue received at the end of the week

$$S.P=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{100} \end{bmatrix}\begin{bmatrix} price \\ price \\ \vdots \\ price \end{bmatrix}$$

Theorem(Properties of the Inner product):

If X, Y and Z are vectors in R^n and c is a scalar then:

1. $X.X=\|X\|^2\geq 0$, with equality if and only if $X=0$,

2. $X.Y=Y.X$

3. $(X+Y).Z=X.Z+Y.Z$

4. $(cX).Y=X.(cY)=c(X.Y)$

Theorem(Cauchy-Schwrz Inequality):

If X and Y are vectors in R^n, then

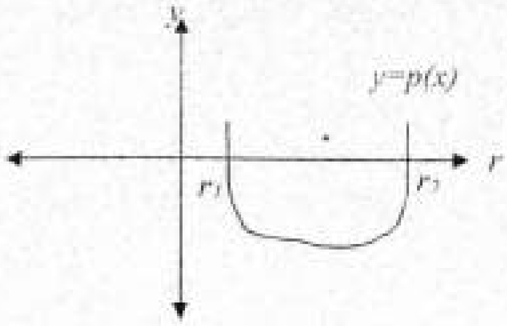
$$|X.Y|\leq\|X\|\,\|Y\|\ldots(3)$$

Proof: If $X=0$, then $\|X\|=0$ and $X.Y=0$, so hold.

Now, suppose that X and Y are nonzero. Let r be a scalar and consider the vector $rX+Y$. Then

$$0\leq(rX+Y).(rX+Y)=r^2X.X+2rX.Y+Y.Y$$
$$=r^2a+2rb+c$$

Where a=X.X, b=X.Y and c=Y.Y



Now , $p(r)=ar^2+2br+c$

$P(r)$ is a quadratic polynomial in **r** (whose grph is a parabola opening upward, since $a>0$)

That is nonzero for all values of r? Why?

This means that either this polynomial has no real roots, or it has real roots then both roots are equal(why?)

(The answer) : if $p(r)$ had two distinct root r_1 and r_2, then it would be negative for some value of r.

Recall that the roots of $p(r)$ are given by quadratic formula as $\frac{-2b+\sqrt{4b^2-4ac}}{2a}$ and $\frac{-2b-\sqrt{4b^2-4ac}}{2a}$ (where $a\neq 0$ since $X\neq 0$). Thus we must have

$$4b^2-4ac\leq 0$$

$$4b^2\leq 4ac$$

Which means that :

$$b^2\leq ac$$

Taking square roots of both side and observing that $b\leq\sqrt{a}\sqrt{c}$ Where

$\sqrt{a}=\sqrt{X.X}=\|X\|$ and $\sqrt{c}=\sqrt{Y.Y}=\|Y\|$

Thus

$$|X.Y|\leq\|X\|\|Y\|.$$

Example: If X=(1,2,-1,2) and Y=(3,1,-1,2) then:

$\|X\|=\sqrt{10}$, $\|Y\|=\sqrt{15}$ and $|X.Y|=10\leq\sqrt{10}\sqrt{15}$.

Definition: The angle between two nonzero vectors X and Y is defined as the unique number θ , $0\leq\theta\leq\pi$ such that :

$$\cos\theta = \frac{X.Y}{\|X\|\|Y\|}$$

It follows from the Cauchy-Schwarz inequality that :

$$\left|\frac{X.Y}{\|X\|\|Y\|}\right| \le 1.$$

Example: Let X=(1,0,0,1) and Y=(0,1,0,1) then we have that

$$\|X\| = \sqrt{2},\ \|Y\| = \sqrt{2} \text{ and } X.Y = 1$$

Thus

$$\cos\theta = \frac{1}{2} \text{ and } \theta = 60^\circ$$

Definition: Two nonzero vector X and **Y** in R^n are said to be **Orthogonal** if $X.Y = 0$. They are said **Parallel** if $|X.Y| = \|X\|\|Y\|$. They are in the **Same direction** if $X.Y = \|X\|\|Y\|$. That is, they are orthogonal if $\cos\theta = 0$, parallel if $\cos\theta = \pm 1$, and in the same direction if $\cos\theta = 1$.

Example: Consider the vectors X=(1,0,0,1) , Y=(0,1,0,1) and Z=(3,0,0,3) then X.Y=0 and Y.Z=0 (check).
Which implies that X and Y are orthogonal and Y and Z are orthogonal too.
Also X.Z=6, $\|X\| = \sqrt{2}$, $\|Z\| = \sqrt{18}$, and $X.Z = \|X\|\|Z\|$
Hence X and Z are in the same direction.

Theorem: (Triangle Inequality) If X and Y are vectors in R^n, then

$$\|X+Y\| \le \|X\| + \|Y\|$$

Proof: By theorem (*)

$$\|X+Y\|^2 = (X+Y).(X+Y)$$
$$= X.X + 2(X.Y) + Y.Y$$
$$= \|X\|^2 + 2(X.Y) + \|Y\|^2$$

By the Cauchy-Schwarz inequality we have:

$$\|X\|^2 + 2(X.Y) + \|Y\|^2 \le \|X\|^2 + 2\|X\|\|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2.$$

Example: Let X=(1,0,0,1) and Y=(0,1,0,1) then

$$\|X+Y\| = \sqrt{4} = 2 < \sqrt{2} + \sqrt{2} = \|X\| + \|Y\|$$

Note: If X and Y are vectors in R^n, then

$$\|X+Y\|^2=\|X\|^2+\|Y\|^2$$

If and only if X and Y are orthogonal.

Definition: A **Unit vector** U in R^n is a vector of unit length. Furthermore, if X is a nonzero vector, then the vector

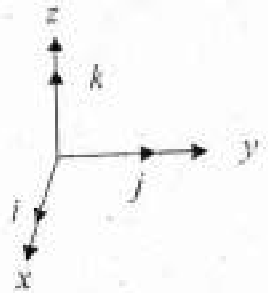
$$U=\left[\frac{1}{\|X\|}\right].X$$

Is a unit vector in the direction of X.

Example: If X= (1,0,0,1), then $\|X\|=\sqrt{2}$ and $U=\frac{1}{\sqrt{2}}(1,0,0,1)$ is a unit vector in the direction of X.

In the case of R^3 the unit vector in the position direction are i=(1,0,0), j=(0,1,0) and k=(0,0,1).

If X=(x, y, z), then X= xi+ yj+ zk.



Example: if X=(2,-1,3), then

X=2i-j+3k

In R^n the unit vector are

$E_1=(1,0,0,\ldots,0)$, $E_2=(0,1,0,\ldots,0)$, …, $E_n=(0,0,\ldots,1)$

And if $X=(x_1, x_2, \ldots, x_n)$ is a vector in R^n, we have

$X= x_1 E_1+x_2 E_2+\ldots+ x_n E_n$.

Exercises :

1- Find X+Y ,X-Y ,2X and 3X-2Y if X =(2,3,5) ,Y=(-2,5,3).

2- Let X=(1,2,2,1) ,Y=(-3,4,-2,-1),Z=(x,4,0,y) and U= (-2,u,v ,4) find x ,u, v and y so that

(a) Z=3X (b) Z-Y=Y (c) Z+U=X

3-Find the length of X=(1,6,-4,-5).

4- Find the distance between (0,3,2),(2,0,4).

5-Find X • Y, X=(-2,-3,-4), (2,-1,2), find the cosine of the angle between X,Y.
6- which of the vectors X=(4,2,6,-8), Y =(-2,3,-1,-1), Z =(-2,-1,-3,4), W=(1,0,0,2) are orthogonal, in same direction, parallel.

7- Prove the parallelogram law. $\|X+Y\|^2+\|X-Y\|^2=2\|X\|^2+2\|Y\|^2$

Cross product in R^3

Definition: if $X=x_1i+x_2j+x_3k$ and $Y=y_1i+y_2j+y_3k$ are two vectors in R^3, then their cross product is the vector $X\times Y$ definedby:

$$X\times Y=(x_2y_3-x_3y_2)i+(x_3y_1-x_1y_3)j+(x_1y_2-x_2y_1)k$$

The cross product $X\times Y$ can also be written as:

$$X\times Y=\begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$X\times Y$ is a vector (from definition)
So

$$X\times Y=\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}i-\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}j+\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}k$$

Example: Let X=2i+j+2k and Y=3i-j-3k. Then

$$X\times Y=\begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix}=-i+12j-5k.$$

Properties of cross product:

Theorem: If X , Y and Z are vectors and c is a scalar, then :

1- $X\times Y=-(Y\times X)$
2- $X\times(Y+Z)=X\times Y+X\times Z$
3- $(X+Y)\times Z=X\times Z+Y\times Z$
4- $c(X\times Y)=(cX)\times Y=X\times(cY)$
5- $X\times X=0$
6- $X\times 0=0\times X=0$

7- $(X \times Y) \times Z = (Z.X)Y - (Z.Y)X$

Also $X \times (Y \times Z) = (X.Z)Y - (X.Y)Z$

8- $(X \times Y).Z = X.(Y \times Z)$

Proof:

$X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$ and $Z = (z_1, z_2, z_3)$

$$X \times Y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \underbrace{(x_2 y_3 - x_3 y_2)}_{u_1} i + \underbrace{(x_3 y_1 - x_1 y_3)}_{u_2} j + \underbrace{(x_1 y_2 - x_2 y_1)}_{u_2} k$$

$$(X \times Y) \times Z = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = (u_2 z_3 - u_3 z_2) i + (u_3 z_1 - u_1 z_3) j + (u_1 z_2 - u_2 z_1) k$$

Substituted instead of u's then we get:

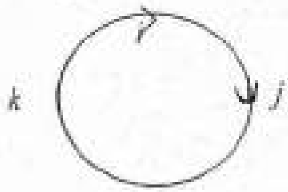
$$= [(x_3 y_1 - x_1 y_3) z_3 - (x_1 y_2 - x_2 y_1) z_2] i + [(x_1 y_2 - x_2 y_1) z_1 - (x_2 y_3 - x_3 y_2) z_3] j + [(x_2 y_3 - x_3 y_2) z_2 - (x_3 y_1 - x_1 y_3) z_1] k$$

Example: From definition we have that:

$i \times i = j \times j = k \times k = 0$ and $i \times j = k$, $j \times k = i$, $k \times i = j$,

Also

$j \times i = -k$, $k \times j = -i$, $i \times k = -j$



Example: Let X=2i+j+2k, Y=3i-j-3k and Z=i+2j+3k

$X \times Y = -i + 12j - 5k$ $(X \times Y).Z = 8$

$Y \times Z = 3i - 12 + 7k$ $X.(Y \times Z) = 8$

Which illustrate equation(8). Also to illustrate (*)

$X \times (Y \times Z) = 31i - 8j - 27k$, $X.Z = 10$, $X.Y = -1$

$(X.Z)Y = 30i - 10j - 30k$, $(X.Y)Z = -i - 2j - 3k$

Hence

$(X.Z)Y - (X.Y)Z = 31i - 8j - 27k$

Also from (8) ,(1) and (5) we get:

$(X \times Y).Y = X.(Y \times Y) = X.0 = 0$

And

$(X \times Y).X = -(Y \times X).X = -Y.(X \times X) = -Y.0 = 0$

Note: If $X \times Y \neq 0$, then $X \times Y$ is orthogonal to both X and Y and to the plane determined by them.

To find the angle between X and Y :

$$\|X \times Y\|^2 = (X \times Y).(X \times Y)$$
$$= X.[Y \times (X \times Y)] \text{ by } (8)$$
$$= X.[(Y.Y)X - (Y.X)Y)] \text{ by } (7)$$
$$= (X.X)(Y.Y) - (Y.X)(Y.X) \text{ by}(2) \text{ and } (4)$$
$$= \|X\|^2\|Y\|^2 - (X \times Y)^2 \text{ by } (1)$$

$$X \times Y = \|X\|\|Y\|\cos\theta$$
$$\|X \times Y\|^2 = \|X\|^2\|Y\|^2 - \|X\|^2\|Y\|^2\cos\theta$$
$$= \|X\|^2\|Y\|^2(1 - \cos^2\theta)$$
$$\therefore \|X \times Y\|^2 = \|X\|^2\|Y\|^2\sin^2\theta$$
$$\therefore \|X \times Y\| = \|X\|\|Y\|\sin\theta \qquad \ldots(9)$$

Note: (1) We do not have $|\sin\theta|$, since $\sin\theta$ is nonnegative for $0 \leq \theta \leq \pi$

(2) X and Y are not parallel if and only if $X \times Y = 0$

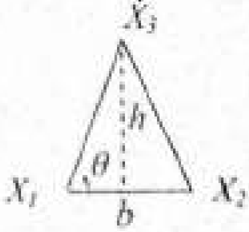
Applications:

Area of triangle:

Area of triangle consider a triangle with vertices X_1, X_2, and X_3. The area of this triangle is

$$A_T = \frac{1}{2}bh$$

Where b is the base and h is the height.



$$b = \|X_2 - X_1\| \text{ and } h = \|X_3 - X_1\|\sin\theta$$

Where

$\sin\theta = \dfrac{h}{\|X_3 - X_1\|}$ so $A_T = \dfrac{1}{2}\|X_2 - X_1\|\|X_3 - X_1\|\sin\theta$

From(9)

$A_T = \dfrac{1}{2}\|(X_2 - X_1)\times(X_3 - X_1)\|$

Example: Find the area of the triangle with vertices $X_1=(2,2,4)$, $X_2=(-1,0,5)$ and $X_3=(3,4,3)$

Solution:

$X_2 - X_1 = -3i - 2j + k$

$X_3 - X_1 = i + 2j - k$

$A_T = \dfrac{1}{2}\|(-3i - 2j + k)\times(i + 2j - k)\|$

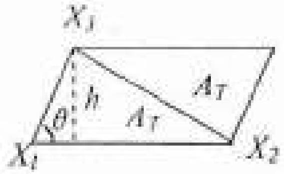
$= \dfrac{1}{2}\|-2j - 4k\|$

$= \|-j - 2k\| = \sqrt{5}$

Area of a parallelogram:

The area A_P of parallelogram with sides X_2-X_1 and X_3-X_1 is $2A_T$.

$A_P = \|(X_2 - X_1)\times(X_3 - X_1)\|$



Example: If $X_1=(2,2,4)$, $X_2=(-1,0,5)$ and $X_3=(3,4,3)$.

Then

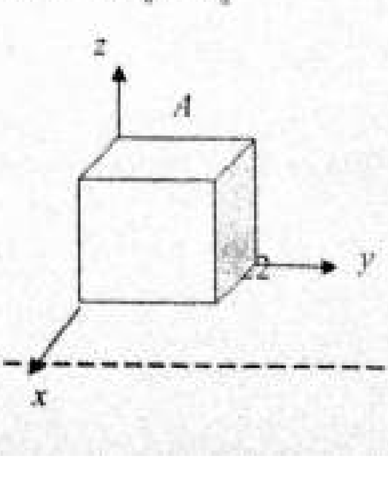
$A_P = 2\sqrt{5}$ (check).

Consider the parallelepiped with vertex at the origin and edges X, Y and Z. The volume of the parallelepiped is the product of the area of the face containing Y, Z and the distance h from the face parallel to it.

$h = \|X\||\cos\theta|$

θ between X and $Y\times Z$, the area of the face determined by Y and Z is $\|Y\times Z\|$

$\therefore\ V = \|Y\times Z\|\|X\||\cos\theta| = |X.(Y\times Z)|$



Example: X=i-2j+3k, Y=i+3j+k and Z=2i+j+2k (H.W.)

Exercises :

1-Show that X and Y are parallel iff $X \times Y = 0$.

2-Show that $\|X \times Y\|^2 + (X.Y)^2 = \|X\|^2 \|Y\|^2$

3-Prove the Jacobi identity :

$$(X \times Y) \times Z + (Y \times Z) \times X + (Z \times X) \times Y = 0$$

Vector space

In the following lectures we study the vector space , subspace ,study the linear Independence , basis and the rank of a matrix .

Definition :

Areal vector space is a set V of elements with two operations$\oplus$ and $\odot$ defined with the following properties .

(a) If X and Y are any elements in V . then $X \oplus Y$ is in V (that is closed under the operation $\oplus$) .

1- $X \oplus Y = Y \oplus X$ for all X ,Y in V .

2- $X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$ for all X ,Y,Z in V

3- There is a uniqueelement 0 in V such that $X \oplus \mathbf{0} = 0 \oplus X = X$for every X in V .

4- For each X in V there exists a unique -X in V such that $X \oplus -X = 0$

(b) If X is any element in V and c is any realnumber then $c \odot X$ is in V .

5-$c \odot (X \oplus Y) = c \odot X \oplus c \odot Y$ for any X, Y in V , and any real number c.

6-$(c+d) \odot X = c \odot X \oplus d \odot Y$ for any X in v and any real numbers c and d .

7-$c \odot (d \odot X) = (cd) \odot X$ for any X in v and real numbers c and d .

8-1 $\oplus$ X = X for any X in v .

$(V, \oplus, \odot)$ is vector space .The operation $\oplus$ is called vector addition .

The operation $\odot$ is called scalar multiplication .

The vector 0 is called Zero vector .

Example 1:

Let R^n be the set of ordered n- tuples $(a_1, a_2, \ldots\ldots, a_n)$ where we define $\oplus$ by $(a_1, a_2, \ldots\ldots, a_n) \oplus (b_1, b_2, \ldots\ldots, b_n)$

$=(a_1+b_1, a_2+b_2, \ldots\ldots, a_n+b_n)$ and $\odot$ by $c \odot (a_1, a_2, \ldots, a_n)$

$=(ca_1, ca_2, \ldots\ldots, ca_n)$

R^n is a vector space .

Example 2:

Let V be the set of ordered triples of real number $(a_1, a_2, 0)$ where we define $\oplus$ by $(a_1, a_2, 0) \oplus (b_1, b_2, 0)$

$=(a_1+b_1, a_2+b_2, 0)$ and $\odot$ by $c \odot (a_1, a_2, 0) = (ca_1, ca_2, 0)$

V is a vector space .

Example 3:

Let V be the set of ordered triples of real number (x, y, z) where we define $\oplus$ by $(x, y, z) \oplus (x', y', z')$

$=(x+x', y+y', z+z'))$ and $\odot$ by $c \odot (x, y, z) = (cx, y, z)$

V is not vector space the property $(c+d) \odot X = c \odot X \oplus d \odot X$ fails to hold

thus $(c+d) \odot (x, y, z) = ((c+d)x, y, z)$,

On other hand $c \odot (x, y, z) \oplus d \odot (x, y, z) = (cx, y, z) \oplus (dx, y, z)$

$=(cx+dx, y+y, z+z) = ((c+d)x, 2y, 2z)$.

Example 3:

Let V be the set of 2×3 matrices under usual operation of matrix addition and scalar multiplication

V is vector space <u>c.h.</u>

Example 4:

Let V be the set of all real –valued function on R .if f and g are in V we define $f \oplus g$ by $(f \oplus g)(t) = f(t) \oplus g(t)$ and if f and c is a scalar define $c \odot f$ by $c \odot f = c f(t)$.

V is vector space c.h.

Example 5:

Let p_n be the set of all real polynomials of degree $\le n$ with zero polynomial .if $p(t) = a_0 t^n + a_1 t^{n-1} + \ldots\ldots + a_{n-1} t + a_n$ and $q(t) = b_0 t^n + b_1 t^{n-1} + \ldots\ldots + b_{n-1} t + b_n$ are in V we define $p(t) \oplus q(t)$ by $p(t) \oplus q(t) = (a_0 + b_0) t^n + (a_1 + b_1) t^{n-1} + \ldots\ldots + (a_{n-1} + b_{n-1}) t + (a_n + b_n)$ and if c is a scalar define $c \odot p(t)$ by

$c \odot p(t) = (ca_0) t^n + (ca_1) t^{n-1} + \ldots\ldots + ca_{n-1} t + ca_n$

the above definition show that the degree of $p(t) \oplus q(t)$ and $c \odot p(t) \le n$

$-p(t) = -a_0 t^n - a_1 t^{n-1} + \ldots\ldots - a_{n-1} t - a_n$ is negative of $p(t)$ and since $a_i + b_i = b_i + a_i$ then $p(t) \oplus q(t) = q(t) + p(t)$

And

$(c+d) \odot p(t) = (c+d) a_0 t^n + (c+d) a_1 t^{n-1} + \ldots\ldots + (c+d) a_{n-1} t + (c+d) a_n$

$= c(a_0 t^n + a_1 t^{n-1} + \ldots\ldots + a_{n-1} t + a_n) + d(a_0 t^n + a_1 t^{n-1} + \ldots\ldots + a_{n-1} t + a_n)$

$= c \odot p(t) \oplus d \odot p(t)$

V is vector space c.h.

Theorem :

If V is a vector space then .

1- $0 \odot X = 0$ for any vector X in V

2- $c \odot 0 = 0$ for any scalar c

3- If $c \odot X = 0$ then either $c = 0$ or $X = 0$

4- $(-1) \odot X = -X$ for any X in V.

Proof:

1) $0X = (0+0)X = 0X + 0X$ by (6) of def. adding $-0X$

$0 = 0X + (-0X) = (0X + 0X) + (-0X)$

$= 0X + [0X + (-0X)]$

$=0X+0=0X.$

2) $c.0 = c.(0+0) = c.0+c.0$

$c.0-c.0=c.0+c.0-c.0$

$0 = c.0$

3) suppose $cX=0$ and $c \neq 0$ then

$0=(\frac{1}{c}).0=(\frac{1}{c})(cx)=[(\frac{1}{c})c]X=1.X$

4) $(-1)X+X=(-1)X+(1)X=(-1+1)X=0X=\mathbf{0}$ so that $(-1)X=-X$

Definition :

Let V be a vector space and W a nonempty subset of V if W is a vector space with respect to the same operations as these in V , then W is called **a subspace** of V .

Example : If $(V,\oplus,\odot)$ is vector space then $\{0\}\subseteq V$, $V\subseteq V$ are two subspaces.

Example :

Let W be the set of ordered triples of real number $(a_1, a_2, 0)$ where we define $\oplus$ by $(a_1, a_2, 0)\oplus(b_1, b_2, 0)$

$=(a_1+b_1, a_2+b_2, 0)$ and $\odot$ by $c\odot(a_1, a_2, 0)=(ca_1, ca_2, 0)$

Then $(W,\oplus,\odot)$ is subspace of $(R^3,\oplus,\odot)$.

Theorem:

Let $(V,\oplus,\odot)$ be a vector space and let **W** be a nonempty subset of V. W is a subspace of V if and only if the following condition hold

1- If **X** , Y are any vectors in W then $X\oplus Y$ is in W

2- If **c** is any real number and X is any vector in W then $c\odot X$ is in **W** .

Proof : H.W.

Example:

Let W be the set of all 2×3 matrices of form

$W=\left\{\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}, a,b,c\in R\right\}$,W is subset of vector space V of all 2×3 matrices under usual operations of matrices addition and scalar multiplication then W is subspace of V.

Solution :

Consider $X=\begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix}$ and $Y=\begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix}$ in W then

$X+Y=\begin{bmatrix} a_1+a_2 & b_1+b_2 & 0 \\ 0 & c_1+c_2 & d_1+d_2 \end{bmatrix}$ is in W also let $r\in R$

$rX=\begin{bmatrix} ra_1 & rb_1 & 0 \\ 0 & rc_1 & rd_1 \end{bmatrix}$ is in W ,W is subspace of V.

Example:

Let W be the sub set of$(R^3,\oplus,\odot)$.

W is ordered triples of real number (a , b , 1),

let $X=(a_1, a_2, 1)$,$Y=(b_1, b_2, 1)$

$X+Y=(a_1+b_1, a_2+b_2, 2)$ Then W is not subspace of $(R^3,\oplus,\odot)$.

Example 5:

Let W be the set of all real polynomials of degree exactly=2

W is subset of p_2 but not subspace of p_2 since

$2t^2+3t+1$ and $-2t^2+t+2$ is polynomial of degree 1 is not in W.

Exercises:

1- Let W $=\left\{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, a=2c+1\right\}$ W is subset of vector space V of all 2×3 matrices under usual operations of matrices addition and scalar multiplication is W is subspace of V.

2- Let W $=\{(a,b,c), b=2a+1\}$ subset of vector space R^3 is W is subspace?

Definition(1 – 7)

Let $X_1, X_2, \ldots\ldots, X_n$ be vectors in a vectors space V. A vector X in V is called linear combination of this vectorsif it can written as X $= c_1X_1+c_2X_2, \ldots\ldots+c_nX_n$ for some real number where $c_1, c_2 \ldots\ldots c_n$ are scalers.

Example: Consider the vector space R^4 . let $X_1=(1,2,1,-1)$, $X_2=(1,0,2,-3)$, $X_3=(1,1,0,-2)$ the vector $X=(2,1,5,-5)$ is linear combination of X_1, X_2, X_3 if we find c_1, c_2, c_3 s.t..

$$X = c_1X_1+c_2X_2+c_3X_3$$

$(2,1,5,-5)=c_1(1,2,1,-1)+c_2(1,0,2,-3)+c_3(1,1,0,-2)$

$(2,1,5,-5)=(c_1,2c_1,c_1,-c_1)+(c_2,0,2c_2,-3c_2)+(c_3,c_3,0,-2c_3)$

$c_1+c_2+c_3=2$

$2c_1+c_3=1$

$c_1+2c_2\quad =5$

$-c_1-3c_2-2c_3=-5$

solving this linear system by Gauss-Jordan we obtain $c_1=1$, $c_2=2$, $c_3=-1$

then X is linear combination of X_1,X_2,X_3

Example: Consider the vector space R^3 . let $X_1=(1,2,-1)$, $X_2=(1,0,-1)$, is the vector $X=(1,0,2)$ is linear combination of X_1,X_2

if we find c_1 ,c_2 s.t..

$$X = c_1X_1+c_2X_2$$

$(1,0,2)=c_1(1,2,-1)+c_2(1,0,-1)$

$c_1+c_2=1$

$2c_1=0$

$-c_1-2c_2=2$

Which has no solution then X is not linear combination of X_1,X_2.

Example: Consider the vector space R^3. let $X_1=(1,0,1)$, $X_2=(-1,1,0)$, $X_3=(0,0,1)$is the vector $X=(1,1,1)$ is linear combination of X_1,X_2,X_3 if we find c_1 ,c_2,c_3 s.t..

$$X = c_1X_1+c_2X_2+c_3X_3$$

$(1,1,1)=c_1(1,0,1)+c_2(-1,1,0)+c_3(0,0,1)$

$c_1-c_2=1$

$\quad c_2=1$

$c_1 \quad + c_3 = 1$

solving this linear system by Gauss-Jordan we obtain $c_1=2$, $c_2=1$, $c_3=-1$

then X is linear combination of X_1, X_2, X_3

Definition :

Let $S = \{X_1, X_2, \ldots\ldots, X_n\}$ be the set of vectors in a vectors space V. the set spans V, or V is spanned by S, if every vector in V is a linear combination of vector in S.

Example :

Let V be the vector space R^3. let$S=\{X_1, X_2, X_3\}$ set of vectors where $X_1=(1,2,1)$, $X_2=(1,0,2)$, $X_3=(1,1,0)$ is the set S spans V ?

SOL.:

Let $X=(a,b,c)$ be any vector in R^3. and

$$X = c_1X_1+c_2X_2+c_3X_3$$

$(a,b,c)=c_1(1,2,1)+c_2(1,0,2)+c_3(1,1,0)$

$c_1+c_2+c_3=a$

$2c_1+c_3=b$

$c_1+2c_2 \quad = c$

solving this linear system by Gauss-Jordan we obtain

$c_1=\frac{-2a+2b+c}{3}$, $c_2=\frac{a-b+c}{3}$, $c_3=\frac{4a-b-2c}{3}$

since we obtained a solution for every choice of a ,b and c then $S=\{X_1, X_2, X_3\}$ spans V.

Example :

Let V be the vector space R^3. $S=\{i, j, k\}$ spans V

Since for every $X=(a,b,c)$ vector in R^3.

$(a,b,c)=(a,0,0)+(0,b,0)+(0,0,c) = ai + bj+ ck$.

Example :

Let $V=P_2$ be the vector space all polynomials of degree ≤ 2 if and Let $S=\{P_1(t), P_2(t)\}$ where $P_1(t)=t^2+2t+1$ and $P_2(t)=t^2+2$ is S spans V?

Sol ; let $P(t)= at^2+bt+c$ polynomial in P_2 where a,b,c are real number suppose $P(t)=c_1P_1(t)+c_2P_2(t)$ then

$at^2+bt+c = c_1(t^2+2t+1)+c_2(t^2+2)$

$at^2+bt+c = (c_1+c_2)t^2+2c_1t+(c_1+2c_2)$

$c_1+c_2=a$

$2c_1 \quad = b$

$c_1+2c_2=c$

we obtain

$$\begin{bmatrix} 1 & 1 & : & a \\ 2 & 0 & : & b \\ 1 & 2 & : & c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & : & 2a-c \\ 0 & 1 & : & c-a \\ 0 & 0 & : & b-4a+2c \end{bmatrix}$$

If $b-4a+2c \neq 0$ then there is no solution to this system hence S does not span P_2.

Linear independence

Definition :Let $S=\{X_1, X_2, \ldots\ldots, X_n\}$ be the set of vectors in a vectors space V. thenS is said to be linearly dependent if there exist constants $c_1, c_2 \ldots\ldots c_n$ not all zero, such that

$c_1X_1+c_2X_2, \ldots\ldots+c_nX_n=0$,other wise . S is called linearly independent

That is S is linearly independent if the equation

$c_1X_1+c_2X_2, \ldots\ldots+c_nX_n=0$ hold only if $c_1=c_2=\ldots\ldots=c_n=0$

Example: Consider the vector space R^4 . let $X_1=(1,0,1,2)$, $X_2=(0,1,1,2)$, $X_3=(1,1,1,3)$is $S=\{X_1, X_2, X_3\}$ is linearly independent

S0l;

Let $c_1X_1+c_2X_2+c_3X_3=0$ where $c_1, c_2, c_3 \in R$

$c_1(1,0,1,2), +c_2(0,1,1,2)+c_3(1,1,1,3)=(0,0,0,0)$
$(c_1,0,c_1,2c_1), +(0,c_2,c_2,2c_2)+(c_3,c_3,c_3,3c_3)=(0,0,0,0)$

$c_1+c_3=0$
$c_2+c_3=0$
$c_1+c_2+c_3=0$
$2c_1+2c_2+3c_3=0$
we obtain $c_1=0, c_2=0, c_3=0$ then S is linearly independent.

Example :
Let V be the vector space R^3 . let$S=\{X_1,X_2,X_3,X_4\}$ set of vectors where $X_1=(1,2,-1)$, $X_2=(1,-2,1)$, $X_3=(-3,2,-1)$, $X_4=(2,0,0)$ is the set S linearly independent ?
SOL.:
Let $c_1X_1+c_2X_2+c_3X_3+c_4X_4=0$

$c_1(1,2,-1)+c_2(1,-2,1)+c_3(-3,2,-1)+c_4(2,0,0)=0$

$c_1+c_2-3c_3+2c_4=0$
$2c_1-2c_2+2c_3=0$
$-c_1+c_2-c_3=0$
There are infinitely many solution like $c_1=1, c_2=2, c_3=1, c_4=0$
, then S is linearlydependent.

Example :
Let V be the vector space R^3 . $S=\{i,j,k\}$ is linearly independent.

Since

$(0,0,0)=(c_1,0,0)+(0,c_2,0)+(0,0,c_3)$
Then $c_1=0, c_2=0, c_3=0$
In fact $E_1,E_2,\ldots\ldots,E_n$ are linearly independent in R^n.

Example:

Let $V=P_2$ be the vector space all polynomials of degree ≤ 2 if and
Let $S=\{P_1(t), P_2(t), P_3(t)\}$ where $P_1(t)=t^2+t+2$ and $P_2(t)=2t^2+t$
$, P_3(t)=3t^2+2t+2$ is S is linearly independent?

Sol : $c_1P_1(t)+c_2P_2(t)+c_3P_3(t)=0$

$c_1(t^2+t+2)+c_2(2t^2+2)+c_3(3t^2+2t+2)=0$
then
$c_1+2c_2+3c_3=0$
$c_1+c_2+2c_3=0$
$2c_1+2c_3=0$

There are infinitely many solution like $c_1=1, c_2=1, c_3=-1$, then S is linearly dependent.

Remark 1: If $S=\{X_1, X_2, \dots\dots, X_n\}$ be the set of vectors and A matrix whose columns are these vectors .S is linearly independent set Iff $|A|\neq 0$.
Remark 2: If $S=\{X_1, X_2, \dots\dots, X_n\}$ be the set of vectors and X is zero vector then S is linearly dependent set.Why?

Remark 3: Let S_1, S_2 be finite subsets of vector space V and S_1subset of S_2 then
1- If S_1 is linearly dependent set so is S_2.
2-If S_2 is linearly independent so is S_1.

Proof 1) : let $S_1=\{X_1, X_2, \dots\dots, X_k\}$,

$S_2=\{X_1, X_2, \dots X_k, X_{k+1} \dots, X_n\}$ $k\le n$
c.h?

2) $S_1=\{X_1, X_2, \dots\dots, X_k\}$,

$S_2=\{X_1, X_2, \dots X_k, X_{k+1} \dots, X_n\}$ $k\le n$

Let $i\le k$ $, c_1X_1+c_2X_2, \dots\dots+c_iX_i=0$

$c_1X_1+c_2X_2, \dots+c_iX_i+0X_{k+1}\dots+0X_n=0$

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Differential Equations

The equation that involves x, y, and the derivatives is called a differential Equation

$F(x, y, \frac{dy}{dx}) = 0$ is a first order diff. Eq

$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$ is a second order diff. Eq

[1] First order differential Equation

(1) Separable
(2) Homogeneous
(3) Exact
(4) Linear
(5) Bernouli

(1) Variables separable

Any equation which can be put in the form

$$f(x)\,dx = g(y)\,dy$$

is called a variables separable

Example:

Solve: $\frac{dy}{dx} = \frac{x\sqrt{1-y^2}}{5-x^2}$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} = \frac{x\,dx}{5-x^2}$$

$$\sin^{-1} y = -\frac{1}{2}\ln|5-x^2| + c$$

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Exercise 13

solve

(1) $\frac{dy}{dx} = \sqrt{1+x+xy+y}$

(2) $\frac{dy}{dx} = \cos(x+y)$

(3) $\frac{dy}{dx} = \frac{5-x+y}{x-y+2}$

(4) $\frac{dy}{dx} = e^{3x+5y}$

(5) $\frac{dy}{dx} = \sqrt{1+x-y^2-xy^2}$

(2) Homogeneous

Any equation which can be put in the form

$f(\lambda x) = \lambda^n f(x)$

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \quad \frac{y}{x} = v$$

is called homogeneous.

Example solve $\frac{dy}{dx} = \frac{x^3 - 3x^2 y}{x^3 - y^3}$

Solution Divide by x^3.

$$\frac{dy}{dx} = \frac{1 - 3\left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)^3}$$

Let $\frac{y}{x} = v \Rightarrow y = xv \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$\therefore x\frac{dv}{dx} + v = \frac{1-3v}{1-v^3}$$

$$x\frac{dv}{dx} = \frac{1-3v}{1-v^3} - v$$

$$\therefore x\frac{dv}{dx} = \frac{1-3v-v+v^4}{1-v^3}$$

$$x\frac{dv}{dx} = \frac{1-4v+v^4}{1-v^3}$$

$$\int \frac{(1-v^3)\,dv}{1-4v+v^4} = \int \frac{dx}{x}$$

$$-\frac{1}{4}\ln|1-4v+v^4| = \ln x + c$$

$$-\frac{1}{4}\ln\left|1-4\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^4\right| = \ln x + c$$

$v = \frac{y}{x}$ $\quad \frac{dy}{dx} = \frac{x+y}{x-y}$ $\quad (x+y)\,dx - (x-y)\,dy = 0$

$(x+y)\,dv = (x-y)\,dy$

$\frac{dy}{dx} = \frac{(x+y)}{(x-y)}$

$$x\frac{dv}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v \Rightarrow x\frac{dv}{dx} = \frac{1+v-v+v^2}{1-v}$$

$\frac{1+v}{1-v}$

$v = \frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}$$

$x\frac{dv}{dx} + v$

$$\int \frac{dv}{1+v^2} - \int \frac{v\,dv}{1+v^2} = \int \frac{dx}{x}$$

$$\tan^{-1} v - \frac{1}{2}\ln|1+v^2| = \ln|x| + c$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| = \ln|x| + c$$

$$\tan^{-1}\left(\frac{y-2}{x}\right) - \frac{1}{2}\ln\left|1+\left(\frac{y-2}{x}\right)^2\right| = \ln x + c$$

(3) Exact

Any equation which can be put in the form

$p(x,y)\,dx + Q(x,y)\,dy = 0$

provided that $\frac{\partial p}{\partial y} = \frac{\partial Q}{\partial x}$ is called an exact differential Equation.

Example solve $\frac{dy}{dx} = \frac{x^3 - 3x^2y}{x^3 - y^3}$

$$(x^3 - y^3)\,dy - (x^3 - 3x^2y)\,dx = 0$$

$$(x^3 - y^3)\,dy + (3x^2y - x^3)\,dx = 0$$

$$\left.\begin{aligned}\frac{\partial}{\partial x}(x^3 - y^3) &= 3x^2 \\ \frac{\partial}{\partial y}(3x^2y - x^3) &= 3x^2\end{aligned}\right\}\ \text{Equal}$$

$\therefore$ the differential equation is exact

$$x^3 dy - y^3 dy + 3x^2y\,dx - x^3\,dx = 0$$

$$(x^3 dy + 3x^2 y\,dx) - y^3\,dy - x^3\,dx = 0$$

$$d(x^3 y) - y^3\,dy - x^3\,dx = 0$$

$$x^3 y - \frac{y^4}{4} - \frac{x^4}{4} = C$$

Exercise 15

Solve

(1) $\frac{dy}{dx} = \frac{\sin x - \sin y}{x\cos y - \sin y}$

(2) $x^3dy + y^3dx + 3y^2x\,dy + 3x^2y\,dx = x^3dx + y^3dy$

(3) $\tan^{-1}x\,dy + \frac{\sin y}{1+x^2}dx + \frac{y}{1+x^2}dx + \cos y\tan^{-1}x\,dy = 0$

(4) $\frac{dy}{dx} = \frac{x-y}{x+y}$ in two different methods.

Note: Integrating factor is the function that makes the differential Equation exact if the differential Equation is multiplied by it.

Example Solve $x\,dy - y\,dx = x^3y^7\,dx$

Solution It is clear that this diff. Eq. is not exact; divide both sides by x^2

$$\frac{x\,dy - y\,dx}{x^2} = x\,y^7\,dx \qquad * \frac{x^7}{x^7}$$

$$d\left(\frac{y}{x}\right) = x^8\left(\frac{y}{x}\right)^7 dx$$

$$\left(\frac{y}{x}\right)^{-7} d\left(\frac{y}{x}\right) = x^8\,dx$$

$$\frac{\left(\frac{y}{x}\right)^{-6}}{-6} = \frac{x^9}{9} + C$$

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(4) Linear: Any differential Equation of the form

$$\frac{dy}{dx} + p(x)\, y = Q(x)$$

is called linear in y

and it is reduced into an exact if it is multiplied by the integrating factor

$$I.F. = r = e^{\int p(x)\, dx}$$

and the general solution is $\boxed{r \cdot y = \int Q \cdot r\, dx + C}$

Example To solve $\frac{dy}{dx} = \frac{x^5 - 2y}{x}$, we put it

in the form $\frac{dy}{dx} = x^4 - \frac{2}{x} y$

$$\frac{dy}{dx} + \left(\frac{2}{x}\right) y = x^4 \qquad \text{linear in } y$$

$$p(x) = \frac{2}{x}, \quad Q(x) = x^4$$

$$r = e^{\int p(x) dx} = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2$$

thus the general solution is

$$r \cdot y = \int Q\, r\, dx + C$$

$$x^2 \cdot y = \int x^4 \cdot x^2\, dx + C$$

$$x^2 y = \frac{x^7}{7} + C$$

$$\text{i.e. } y = \frac{x^5}{7} + \frac{C}{x^2}$$

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(5) Bernouli : Any equation of the form

$$\frac{dy}{dx} + p(x)\, y^1 = Q(x) \cdot y^n$$

is called Bernouli in y and it is reduced into a linear by the assumption

$$y^{1-n} = Z$$

Example

Solve $\frac{dy}{dx} = \frac{x^4 y^5 - 3y}{x}$

Solution

$$\frac{dy}{dx} = x^3 y^5 - \frac{3}{x} y$$

$$\frac{dy}{dx} + \frac{3}{x} y = x^3 \; y^5 \quad (*) \text{ Bernouli in } y$$

Let $y^{1-(5)} = Z$

$$\Rightarrow y^{-4} = Z$$

$$\therefore -4y^{-5} \frac{dy}{dx} = \frac{dZ}{dx}$$

Multiply equation (*) by $-4y^{-5}$

$$\therefore -4y^{-5} \frac{dy}{dx} - \frac{12}{x} y^{-4} = -4x^3$$

or

$$\frac{dZ}{dx} - \frac{12}{x} Z = -4x^3 \quad \text{linear in } Z$$

$$\therefore r = e^{\int -\frac{12}{x} dx} = e^{-12 \ln x} = e^{\ln(x^{-12})} = x^{-12}$$

thus the general solution is $r \cdot Z = \int Q \cdot r \, dx + C$

$$x^{-12} \cdot y^{-4} = \int (-4x^3) \cdot x^{-12} dx + C$$

$$\frac{1}{x^{12} y^4} = \frac{1}{2} \frac{1}{x^8} + C$$

Exercise 16

(1) $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = ky$, k is constant

(2) $y\ln y\,dx + (1+x^2)\,dy = 0$

(3) $u(\ln v - \ln u)\,dv = v(1+\ln v - \ln u)\,du$

(4) $r\,d\theta + (2\theta - r^2 - 1)\,dr = 0$

(5) $\cosh x\,dy - (y + \cosh x)\sinh x\,dx = 0$

(6) $(e^s + \ln u)\,ds + \left(\frac{s+u}{u}\right)du = 0$

(7) $(r^2 + v^2)\,dv - rv\,dr = 0$

(8) $(r^2 + u^2)\,du - ru\,dr = 0$

(9) $\left(\sin x + \tan^{-1}\left(\frac{y}{x}\right)\right)dx - \left(y - \ln\sqrt{x^2+y^2}\right)dy = 0$

(10) $\frac{dp}{dz} + \frac{2pz}{p^2+z^2} = e^z$

(11) $\frac{dy}{dx} = \frac{y + x\cos^2\left(\frac{y}{x}\right)}{x}$

(12) $\frac{dz}{du} + z\cot u = \sin 2u$

(13) $r\frac{dp}{dr} = r^3 - 2p$

(14) $3x\frac{dz}{dx} - z + x^2 z^4 = 0$

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[2] Second - Order, Linear, diff. eq.s With constant coefficients

(1) Special type $(y' = p)$

(2) Homogeneous type $y'' + ay' + by = 0$

(3) Non-Homogeneous type $y'' + ay' + by = f(x)$

a, b constant numbers.

(1) Special type

Ex. To solve $y'' - yy' = 0$

put

$$y' = p \quad \therefore y'' = \frac{dp}{dx}$$

$$\therefore \frac{dp}{dx} - y\,p = 0$$

$$\frac{dp}{dy} \cdot \frac{dy}{dx} - y\,p = 0$$

$$\frac{dp}{dy} \cdot p - y\,p = 0$$

$$p\left(\frac{dp}{dy} - y\right) = 0$$

$p = 0$ or $\frac{dp}{dy} - y = 0$

$\frac{dy}{dx} = 0$; $\frac{dp}{dy} = y$

$dy = 0$; $dp = y\,dy$

$y = C$; $p = \frac{y^2}{2} + C_1$

$$\frac{dy}{dx} = \frac{y^2}{2} + C_1$$

$$\frac{dy}{\frac{y^2}{2} + C_1} = dx$$

$$\sqrt{2}\int \frac{\frac{1}{\sqrt{2}}\,dy}{(\sqrt{C_1})^2 + \left(\frac{y}{\sqrt{2}}\right)^2} = \int dx$$

The solution is

$$y = C \quad \text{or} \quad \frac{\sqrt{2}}{\sqrt{C_1}} \tan^{-1}\left(\frac{\frac{y}{\sqrt{2}}}{\sqrt{C_1}}\right) = x + C_2$$

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(2) Homogeneous, linear, constant coefficients

$$y'' + a y' + b y = 0$$

To solve

$$(D^2 + aD + b)y = 0 \quad , \quad D = \frac{d}{dx}$$

Let $y = e^{mx}$ $\therefore$ $y' = m e^{mx}$

$y'' = m^2 e^{mx}$

$\therefore$ $m^2 e^{mx} + a m e^{mx} + b e^{mx} = 0$

$e^{mx} \neq 0$

$\therefore$ $m^2 + am + b = 0$

characteristic Equation

Theorem: Let m_1, m_2 be the two roots of the characteristic equation $m^2 + am + b = 0$ of the diff. eq. $y'' + ay' + by = 0$ and

(1) If $m_1 \neq m_2$ Then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(2) If $m_1 = m_2 = m$ then $y = c_1 e^{mx} + c_2 x e^{mx}$

(3) If $\left.\begin{array}{l} m_1 = \alpha + i\beta \\ m_2 = \alpha - i\beta \end{array}\right\}$ $i = \sqrt{-1}$ then

$y = e^{\alpha x}[c_1 \sin \beta x + c_2 \cos \beta x]$

Examples

(1) To solve $(D^2 - D - 2)y = 0$

Let $y = e^{mx}$ $\therefore$ $m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m_1 = 2, \quad m_2 = -1$

$\therefore$ $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = c_1 e^{2x} + c_2 e^{-x}$

(2) To solve $y'' - 6y' + 9y = 0$

ch. eq. $m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m_1 = m_2 = 3$

$\therefore y = c_1 e^{3x} + c_2 x e^{3x}$

(3) To solve

$y'' + 2y' + 5y = 0$

$\Rightarrow m^2 + 2m + 5 = 0$

$$m = \frac{-2 \mp \sqrt{(2)^2 - 4(1)(5)}}{2} = \frac{-2 \mp \sqrt{-16}}{2}$$

$\left.\begin{array}{l} m_1 = -1 + 2i \\ m_2 = -1 - 2i \end{array}\right\} \begin{array}{l} \alpha = -1 \\ \beta = 2 \end{array}$

$\therefore y = e^{-x} \left[c_1 \sin 2x + c_2 \cos 2x \right]$

$\begin{array}{r|l} 2 & 84 \\ 2 & 42 \\ & 21 \end{array}$

Higher order

Examples

(1) To solve $(D-2)^5 y = 0$

$\Rightarrow (m-2)^5 = 0$

$\therefore m_1 = m_2 = m_3 = m_4 = m_5 = 2$

$\therefore y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4 x^3 e^{2x} + c_5 x^4 e^{2x}$

(2) To solve $(D^3 + 13D^2 + 12D - 26) y = 0$

$m^3 + 13m^2 + 12m - 26 = 0$

$m = 1$

$$\begin{array}{r|l} & m^2 + 14m + 26 \\ \hline m - 1 & m^3 + 13m^2 + 12m - 26 \\ & \mp m^3 \pm m^2 \\ \hline & 14m^2 + 12m \\ & \mp 14m^2 \pm 14m \\ \hline & 26m - 26 \\ & 26m - 26 \\ \hline & 0 \quad 0 \end{array}$$

$(m-1)(m^2 + 14m + 26) = 0$

$$m_1 = 1, \quad m = \frac{-14 \mp \sqrt{(14)^2 - 4(1)(26)}}{2}$$

$$= \frac{-14 \mp \sqrt{84}}{2} = -7 \mp \sqrt{21}$$

$\therefore y = c_1 e^{x} + c_2 e^{(-7+\sqrt{21})x} + c_3 e^{(-7-\sqrt{21})x}$

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[3] Non-Homogeneous, linear, constant coefficients

To solve $(D^2 + aD + b)y = f(x)$

(1) Find the homogeneous solution (y_h)

(2) Find the particular solution (y_p) by **one** of the following methods

(i) undetermined coefficients method

(ii) D-operator method

(iii) Variation of parameters

(3) Find the general solution as

$$y = y_h + y_p$$

(i) The undetermined coefficients method

$f(x)$	y_p
(1) e^{ax}	$k e^{ax}$
(2) x^n	$A_1 x^n + A_2 x^{n-1} + \dots + A_n$
(3) $\sin ax$, $\cos ax$, $2\sin ax + 3\cos ax$, $4\sin ax - 7\cos ax$	$A \sin ax + B \cos ax$
$x e^{ax}$	$(Ax + B) e^{ax}$
$e^{ax} \cos bx$	$(A \cos bx + B \sin bx) e^{ax}$
$x \sin ax$	$(Ax + B)(C \sin ax + D \cos ax)$ or $(A_1 x \sin ax + A_2 x \cos ax + A_3 \sin ax + A_4 \cos ax)$

Examples

(1) To solve $(D^2-D-2)y = 2e^{4x}$

$$m^2-m-2=0 \Rightarrow (m-2)(m+1)=0$$

$$\Rightarrow m_1=2, \quad m_2=-1$$

$$y_h = c_1 e^{2x} + c_2 e^{-x}$$

Let $y_p = k e^{4x}$

$$\therefore y_p' = 4ke^{4x}, \quad y_p'' = 16ke^{4x}$$

$$\therefore 16ke^{4x} - 4ke^{4x} - 2ke^{4x} = 2e^{4x}$$

$$\Rightarrow 10k = 2 \quad \therefore k = \frac{1}{5}$$

$\therefore y_p = \frac{1}{5}e^{4x}$ thus

$$y = y_h + y_p = (c_1 e^{2x} + c_2 e^{-x}) + \left(\frac{1}{5}e^{4x}\right)$$

(2) Solve $(D^2-D-2)y = 7e^{3x} - 5e^{6x}$

(3) Solve $(D^2-D-2)y = \sinh 3x$

(4) To solve $(D^2-D-2)y = 4x-2x^2$,

find

$$y_h = c_1 e^{2x} + c_2 e^{-x}$$

Let $y_p = Ax^2+Bx+C$

$$\therefore y_p' = 2Ax+B, \quad y_p'' = 2A$$

$$(2A) - (2Ax+B) - 2(Ax^2+Bx+C) = 4x-2x^2$$

$$x^2; \quad -2A = -2 \Rightarrow A=1$$

$$x^1; \quad -2A-2B = 4 \Rightarrow B=-3$$

$$x^0; \quad 2A-B-2C=0 \Rightarrow C=5/2$$

$\therefore$

$$y_p = x^2-3x+5/2$$

and

$$y = (c_1 e^{2x} + c_2 e^{-x}) + (x^2-3x+5/2)$$

(5) To solve $(D^2-3D-4)y = \sin 2x$

$$m^2-3m-4=0 \Rightarrow (m-4)(m+1)=0 \Rightarrow m_1=4,\ m_2=-1$$

$$\therefore y_h = c_1 e^{4x} + c_2 e^{-x}$$

Let $y_p = A\sin 2x + B\cos 2x$

$$y_p' = 2A\cos 2x - 2B\sin 2x$$

$$y_p'' = -4A\sin 2x - 4B\cos 2x$$

$\therefore$

$$(-4A\sin 2x - 4B\cos 2x) - 3(2A\cos 2x - 2B\sin 2x) - 4(A\sin 2x + B\cos 2x) = \sin 2x$$

$$\sin 2x;\ -4A + 6B - 4A = 1 \Rightarrow 6B - 8A = 1$$

$$\cos 2x;\ -4B - 6A - 4B = 0 \Rightarrow 4B + 3A = 0$$

$$\therefore A = -\frac{2}{25},\quad B = \frac{3}{50}$$

$$y_p = -\frac{2}{25}\sin 2x + \frac{3}{50}\cos 2x$$

and

$$y = \left(c_1 e^{4x} + c_2 e^{-x}\right) + \left(\frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x\right)$$

(6) To solve $(D^2-D-2)y = e^{2x}$

$$y_h = c_1 e^{2x} + c_2 e^{-x}$$

Let $y_p = Axe^{2x}$, $y_p' = 2Axe^{2x} + Ae^{2x}$

$$y_p'' = 4Axe^{2x} + 4Ae^{2x}$$

$\therefore$

$$(4Axe^{2x} + 4Ae^{2x}) - (2Axe^{2x} + Ae^{2x}) - 2(Axe^{2x}) = e^{2x}$$

$$\Rightarrow 3Ae^{2x} = e^{2x} \Rightarrow A = \frac{1}{3}$$

$$\therefore y_p = \frac{1}{3}xe^{2x}$$

and

$$y = \left(c_1 e^{2x} + c_2 e^{-x}\right) + \left(\frac{1}{3}xe^{2x}\right)$$

(7) To solve $(D^2 - D)y = 4x - 2x^2$

$$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m_1 = 0,\ m_2 = 1$$

$$y_h = c_1 e^{0\cdot x} + c_2 e^{x} = c_1 + c_2 e^{x}$$

Let

$$y_p = (Ax^2 + Bx + C) * x$$

$$\therefore\ y_p = Ax^3 + Bx^2 + Cx$$

(8) $(D^2 - 6D + 9)y = 7e^{3x}$

$$m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m_1 = m_2 = 3$$

$$\therefore\ y_h = c_1 e^{3x} + c_2\, x e^{3x}$$

Let $y_p = Ax^2 e^{3x}$, $y_p' = 3Ax^2 e^{3x} + 2Axe^{3x}$

$$y_p'' = 9Ax^2 e^{3x} + 12Axe^{3x} + 2Ae^{3x}$$

$$\therefore\ (9Ax^2 e^{3x} + 12Axe^{3x} + 2Ae^{3x}) - 6(3Ax^2 e^{3x} + 2Axe^{3x}) + 9(Ax^2 e^{3x}) = 7e^{3x}$$

$$\Rightarrow 2A = 7 \quad \therefore\ A = \frac{7}{2}$$

$$\therefore\ y_p = \frac{7}{2} x^2 e^{3x}$$

and

$$y = (c_1 e^{3x} + c_2 x e^{3x}) + \left(\frac{7}{2} x^2 e^{3x}\right)$$

(9) To solve $(D^3 - 2D^2 + 4D - 8)y = e^{x}$

$$m^3 - 2m^2 + 4m - 8 = 0$$

$$m = 2$$

$$\begin{array}{r|l} & m^2 + 4 \\ m-2 & m^3 - 2m^2 + 4m - 8 \\ & m^3 - 2m^2 \\ & 0 \quad 0 \quad 4m - 8 \\ & \quad\quad 4m - 8 \\ & \quad\quad 0 \quad 0 \end{array}$$

$$(m-2)(m^2+4) = 0$$

$$m_1 = 2,\quad m_2 = -2i,\ m_3 = +2i \quad \left\{ \alpha = 0,\ \beta = 2 \right.$$

$$y_h = c_1 e^{2x} + e^{0}[c_2 \sin 2x + c_3 \cos 2x]$$

Let $y_p = Ae^{x}$

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(ii) D-operator Method

(1) $\frac{1}{F(D)}\left(e^{ax}\right) = \frac{1}{F(a)} * e^{ax}$, $F(a) \neq 0$

(2) $\frac{1}{F(D)}\left(e^{ax} \cdot f(x)\right) = e^{ax} \times \frac{1}{F(D+a)}\left(f(x)\right)$

if $f(x) = 1$ then $\frac{1}{F(D)}\left(e^{ax}\right) = e^{ax} * \frac{1}{F(D+a)}(1)$

(3) $\frac{1}{F(D^2)}\begin{pmatrix} \sin ax \\ \text{or} \\ \cos ax \end{pmatrix} = \begin{pmatrix} \sin ax \\ \text{or} \\ \cos ax \end{pmatrix} * \frac{1}{F(-a^2)}$

(4) $\frac{1}{F(D)}\left(x^n\right)$ = Long Division or Using

$\frac{1}{1-x} = 1 + x + x^2 + \cdots$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

Examples

(1) $(D^2 - D - 2)y = 2e^{4x}$

$y_p = \frac{1}{D^2 - D - 2}\left(2e^{4x}\right) = 2e^{4x} * \frac{1}{4^2 - 4 - 2}$

$y_p = \frac{1}{5} e^{4x}$

(2) $(D^2 - D - 2)y = 7e^{3x} - 5e^{6x}$

$y_p = \frac{1}{D^2 - D - 2}\left(7e^{3x}\right) - \frac{1}{D^2 - D - 2}\left(5e^{6x}\right)$

$y_p = \frac{1}{3^2 - 3 - 2} * 7e^{3x} - \frac{1}{6^2 - 6 - 2} * 5e^{6x}$

$\therefore y_p = \frac{7}{4} e^{3x} - \frac{5}{28} e^{6x}$

(3) $(D^2 - D - 2)y = 4x - 2x^2$

$y_p = \frac{1}{D^2 - D - 2}\left(4x - x^2\right) = \frac{1}{-2 - D + D^2}\left(4x - 2x^2\right)$

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$$-\tfrac{1}{2} + \tfrac{1}{4}D - \tfrac{3}{8}D^2 + \dots$$

$$-2 - D + D^2 \,\Big|\, 1$$

$$\mp 1 \mp \tfrac{1}{2}D \pm \tfrac{1}{2}D^2$$

$$-\tfrac{1}{2}D + \tfrac{1}{2}D^2$$

$$\pm\tfrac{1}{2}D \pm \tfrac{1}{4}D^2 \mp \tfrac{1}{4}D^3$$

$$\therefore y_p = \frac{1}{-2-D+D^2}(4x-2x^2) \qquad \tfrac{3}{4}D^2 \quad \tfrac{1}{4}D^3$$

$$= \left[-\tfrac{1}{2} + \tfrac{1}{4}D - \tfrac{3}{8}D^2 + \dots\right](4x-2x^2)$$

$$= \tfrac{1}{2}(4x-2x^2) + \tfrac{1}{4}(4-4x) - \tfrac{3}{8}(-4)$$

$$\therefore y_p = x^2 - 3x + \tfrac{5}{2}$$

(4) $(D^2 - 3D - 4)y = \sin 2x$

$$y_p = \frac{1}{D^2 - 3D - 4}(\sin 2x)$$

$$= \frac{1}{-(2)^2 - 3D - 4}(\sin 2x) = \frac{1}{-3D-8}\sin 2x$$

$$= -\frac{1}{3D+8}(\sin 2x) \quad * \frac{3D-8}{3D-8}$$

$$= -\frac{3D-8}{9D^2-64}(\sin 2x) = -\frac{3D-8}{9(-(2)^2)-64}(\sin 2x)$$

$$= \frac{1}{100}(3D-8)\sin 2x = \frac{6}{100}\cos 2x - \frac{8}{100}\sin 2x$$

$$\therefore y_p = \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$$

(5) $(D^2 - D - 2)y = e^{2x}$

$$y_p = \frac{1}{D^2 - D - 2}(e^{2x})$$

$$y_p = e^{2x} * \frac{1}{(D+2)^2-(D+2)-2}(1)$$

$$= e^{2x} * \frac{1}{D^2+3D}(1) = e^{2x} * \frac{1}{D+3}\left(\frac{1}{D}(1)\right)$$

$$= e^{2x}\frac{1}{D+3}(x)$$

$$= \frac{1}{3}e^{2x}\frac{1}{1+\frac{D}{3}}(x) \quad \text{Using } \frac{1}{1+x} = 1-x+x^2-x^3\ldots$$

$$= \frac{1}{3}e^{2x} * \left[1-\frac{D}{3}+\frac{D^2}{9}-\cdots\right](x)$$

$$= \frac{1}{3}e^{2x}\left[x-\frac{1}{3}\right] = \frac{1}{3}xe^{2x} - \frac{1}{9}e^{2x}$$

$$\therefore y = y_h + y_p = \left(c_1e^{2x}+c_2e^{-x}\right)+\frac{1}{3}xe^{2x}-\frac{1}{9}e^{2x}$$

$$= \left[\left(c_1-\frac{1}{9}\right)e^{2x}+c_2e^{-x}\right]+\frac{1}{3}xe^{2x}$$

$$\therefore y = \left(K_1e^{2x}+c_2e^{-x}\right)+\frac{1}{3}xe^{2x}$$

(6) $(D^2-6D+9)y = xe^{3x}$

$$y_p = \frac{1}{D^2-6D+9}\left(xe^{3x}\right)$$

$$= \frac{1}{(D-3)^2}\left(xe^{3x}\right) = e^{3x} * \frac{1}{[(D+3)-3]^2}(x)$$

$$= e^{3x} * \frac{1}{D^2}(x) = \frac{1}{6}x^3e^{3x}$$

Exercise 17 Solve

(1) $(D^3-2D^2+4D-8)y = e^x+e^{2x}$

(2) $(D^3-D^2)y = x-x^2$

(3) $(D^3-2D^2+4D-8)y = \sin x$

(4) $(D^2+1)y = \sin^2 x$

(5) $(D^2+4)^2 y = \cos 3x$

(6) $(D^2+9)y = \cos 3x$

(7) $(D^2-D-2)y = \sinh 2x$

(8) $(D^2-D-2)y = x\cosh x$

(9) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 = 0$

(10) $(D^2-D)y = 2x-3e^x$

(iii) Method of variation of parameters

To find y_p for $y'' + ay' + by = f(x)$

[1] Find $y_h = c_1 y_1 + c_2 y_2$

[2] Assume $y_p = u_1(x) y_1 + u_2(x) y_2$

[3] Solve

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = 0 \end{cases}$$

Example (1) To solve $y'' + y = \sec x$

$m^2 + 1 = 0 \Rightarrow m = \pm i$ $\therefore \alpha = 0, \beta = 1$

$$y_h = e^{0}[c_1 \sin x + c_2 \cos x]$$

$$y_h = c_1 \sin x + c_2 \cos x$$

Let $y_p = u_1 \sin x + u_2 \cos x$

$\therefore$

$$u_1' \sin x + u_2' \cos x = 0$$

$$u_1'(\cos x) + u_2'(-\sin x) = \sec x$$

$$\Rightarrow u_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\sec x \cos x}{-\sin^2 x - \cos^2 x} = \frac{-1}{-1} = 1$$

$$\therefore u_1(x) = x$$

$$\Rightarrow u_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec x \end{vmatrix}}{-1} = \frac{\sin x \sec x}{-1} = \frac{-\sin x}{\cos x}$$

$$u_2(x) = \ln|\cos x|$$

$$\therefore y_p = x \sin x + \cos x \cdot \ln|\cos x|$$

and $y = (c_1 \sin x + c_2 \cos x) + (x \sin x + \cos x \ln|\cos x|)$.

Example (2) Using the method of variation of parameters to solve

$$(D^3 - D^2)y = e^x$$

$m^3 - m^2 = 0 \Rightarrow m^2(m-1) = 0 \Rightarrow m_1 = m_2 = 0,\ m_3 = 1$

$$y_h = c_1 + c_2 x + c_3 e^x$$

let $y_p = u_1(x)\cdot 1 + u_2(x)\, x + u_3(x)\, e^x$

and solve

$$\begin{cases} u_1' * 1 + u_2' * x + u_3' e^x = 0 \\ u_1' * 0 + u_2' * 1 + u_3' e^x = 0 \\ u_2' * (0) + u_3' (e^x) = e^x \end{cases}$$

$\Rightarrow$

$u_3' = 1 \Rightarrow \boxed{u_3(x) = x}$

$u_2' = -u_3' e^x \Rightarrow u_2' = -e^x$

$\Rightarrow \boxed{u_2(x) = -e^x}$

and $u_1' = -x u_2' - e^x u_3'$

$= -x(-e^x) - e^x(1) = xe^x - e^x$

$\therefore \boxed{u_1(x) = xe^x - 2e^x}$

$\therefore y_p = (xe^x - 2e^x) + (-e^x)x + (x)e^x = xe^x - 2e^x$

and $y = (c_1 + c_2 x + c_3 e^x) + (xe^x - 2e^x)$ Or

$y = (c_1 + c_2 x + k e^x) + xe^x$

Exercise 18 solve

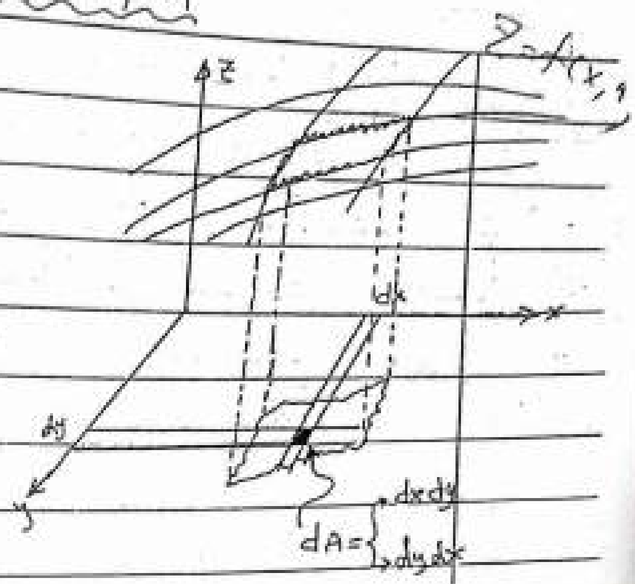
(1) $(D^2+4)y = \tan 2x$

(2) $(D^2+4)y = \sin^2 x$

الموضوع ....
التاريخ ....

Double Integral

$$\iint_R f(x,y)\, dA = \text{Volume}$$



$$dV = z \cdot dA$$

$$V = \int_R dv = \iint_R z\, dA = \begin{cases} \iint_R f(x,y)\, dy\, dx \\ \iint_R f(x,y)\, dx\, dy \end{cases}$$

Special case

when $z = f(x,y) = 1$, it is reduced into

$$\left(\begin{matrix}\text{Area} \\ \text{inside } R\end{matrix}\right) = \iint_R dA = \iint_R dy\, dx = \iint_R dx\, dy$$

Examples (1) Find $\iint_R xy\, dA$, R: the region bounded by

$y = x^2$, $y = 3x$

Answer

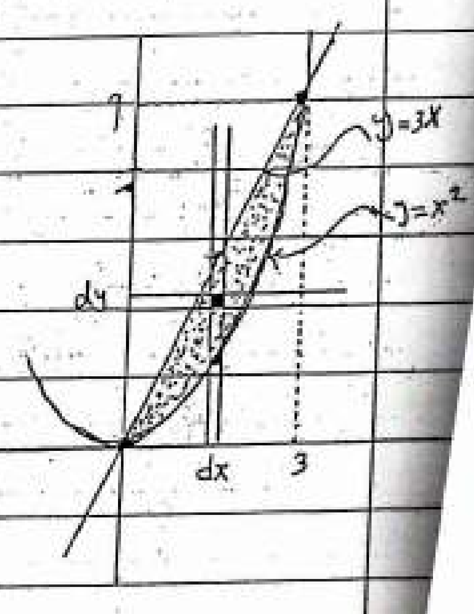
put $y_1 = y_2 \Rightarrow x^2 = 3x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0 \qquad x = 3$

$y = 0 \qquad y = 9$



Method (1)

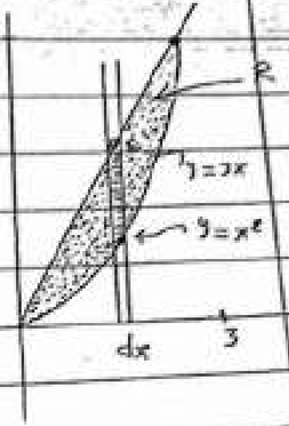
$$\iint_R xy\,dA = \int_0^3 \int_{x^2}^{3x} xy\;dy\;dx$$

dx نثبت
dy ونغير
ونرسم dx خط

$$= \int_0^3 x\,\frac{y^2}{2}\Big]_{x^2}^{3x}\,dx$$

$$= \int_0^3 \frac{x}{2}\left\{(3x)^2 - (x^2)^2\right\}dx$$

$$= \frac{243}{8}$$



Method (2)

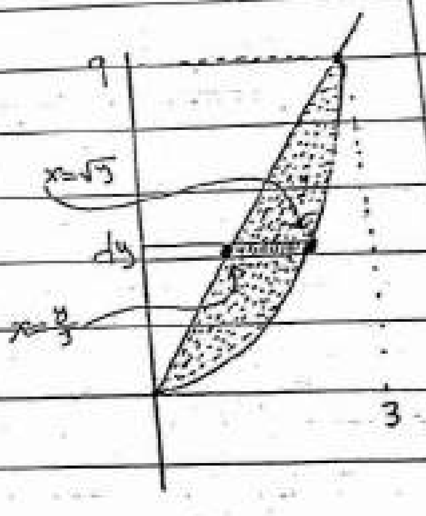
$$\iint_R xy\,dA = \int_0^9 \int_{\frac{y}{3}}^{\sqrt{y}} xy\;dx\;dy$$

$$= \int_0^9 y\,\frac{x^2}{2}\Big]_{\frac{y}{3}}^{\sqrt{y}}\,dy$$

$$= \int_0^9 \frac{y}{2}\left\{(\sqrt{y})^2 - \left(\frac{y}{3}\right)^2\right\}dy$$

$$= \int_0^9 \frac{y}{2}\left\{y - \frac{y^2}{9}\right\}dy$$

$$= \frac{243}{8}$$

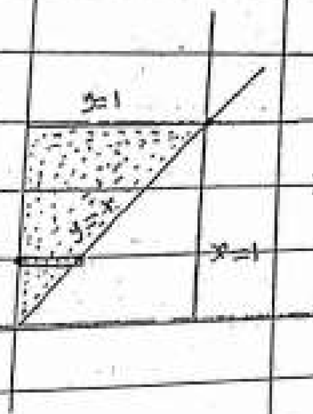


(2) Find $\displaystyle\int_0^1\int_x^1 e^{y^2}\,dy\,dx$

To find this integral we reverse the order of the integral

R: $x=0$ to $x=1$

$y=x$ to $y=1$ ..



$$\int_0^1\int_x^1 e^{y^2}\,dy\,dx = \int_0^1\left(\int_0^y e^{y^2}\,dx\right)dy$$

$$= \int_0^1 e^{y^2}x\Big]_0^y\,dy$$

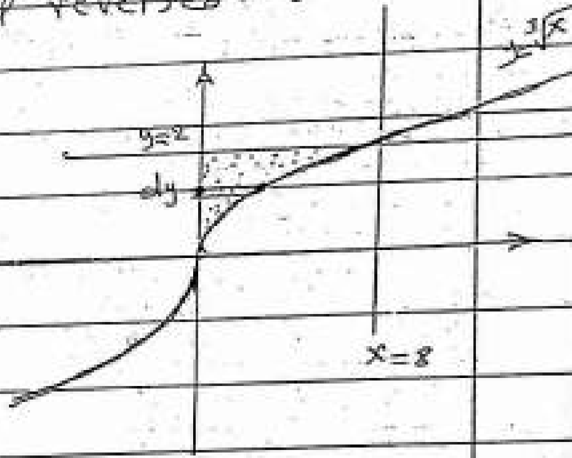
$$= \int_0^1 e^{y^2}(y-0)\,dy$$

$$= \frac{1}{2}e^{y^2}\Big]_0^1 = \frac{1}{2}\left[e^1-1\right]$$

(3) Find an equivalent integral to the integral

$$\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$$

by the order of the integral reversed, and then evaluate.



Answer

R: $x=0$ to $x=8$

$y=\sqrt[3]{x}$ to $y=2$

$$\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$$

$$= \int_0^2\left(\int_0^{y^3}\frac{1}{y^4+1}\,dx\right)dy =$$

$$= \int_0^2 \frac{1}{y^4+1}\,x\Big]_0^{y^3}\,dy$$

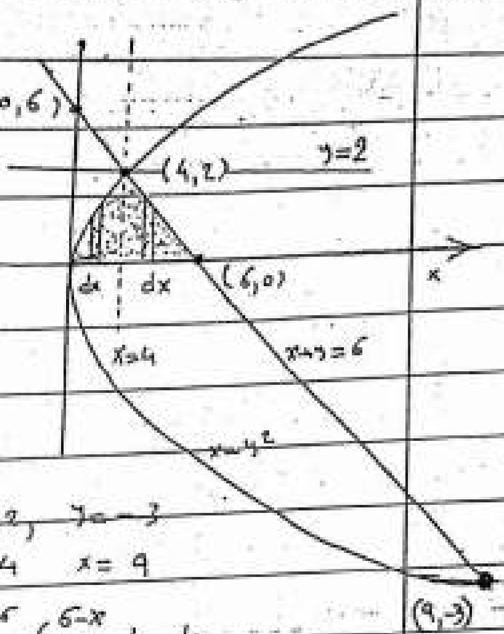
$$= \int_0^2 \frac{y^3-0}{y^4+1}\,dy = \frac{1}{4}\ln\left|y^4+1\right|\Big]_0^2$$

$$= \frac{1}{4}\ln 17$$

(4) Find an equivalent integral to the integral

$$\int_0^2\int_{y^2}^{6-y} dx\,dy$$

by the order of the integral reversed and then evaluate.



Answer

$y=0$ to $y=2$

$x=y^2$ to $x=6-y$

$\Rightarrow y^2=6-y$

$y^2+y-6=0$

$(y+3)(y-2)=0 \rightarrow y=2,\ y=-3$

$x=4 \qquad x=9$

$$\therefore \int_0^2\int_{y^2}^{6-y} dx\,dy = \int_0^4\int_0^{\sqrt{x}} dy\,dx + \int_4^6\int_0^{6-x} dy\,dx$$

Exercise 19

Find an equivalent integral by the order of the integral reversed, for each of the following integrals, and, then, evaluate:

(1) $\displaystyle\int_0^4\int_{\sqrt{x}}^{6-x} dy\,dx$

(2) $\displaystyle\int_0^2\int_{\sqrt{x}}^{6-x} dy\,dx$

(3) $\displaystyle\int_0^1\int_{y^2}^{6-y} dx\,dy$

(4) $\displaystyle\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\,dx\,dy$

(5) $\displaystyle\int_0^1\int_y^{2-y} dx\,dy$

(6) $\displaystyle\int_0^2\int_0^{4-x^2} \frac{x e^{2y}}{4-y}\,dy\,dx$

(7) $\displaystyle\int_0^4\int_{\sqrt{y}}^2 \cos(4x^3+5)\,dx\,dy$

(8) $\displaystyle\int_0^8\int_0^{e^x} dy\,dx$

(9) $\displaystyle\int_0^1\int_0^x dy\,dx + \int_1^2\int_0^{\frac{1}{x}} dy\,dx$

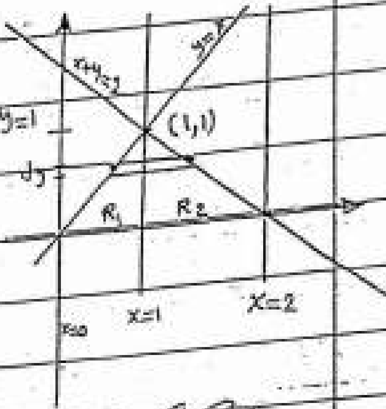
Example (5)

Reverse the order of the integral

$$\int_0^1\int_0^x dy\,dx + \int_1^2\int_0^{2-x} dy\,dx$$

R_1 : $x=0$ to $x=1$ $\qquad$ R_2 : $x=1$ to $x=2$

$y=0$ to $y=x$ $\qquad$ $y=0$ to $y=2-x$



$y_1=x$, $y_2=2-x$

put $y_1=y_2$ then

$x=2-x \Rightarrow x=1$
$y=1$

Let $R_1 \cup R_2 = R$

$$\therefore \iint_{R_1} f(x,y)\,dy\,dx + \iint_{R_2} f(x,y)\,dy\,dx = \iint_R f(x,y)\,dx\,dy$$

thus $\int_0^1\int_0^x dy\,dx + \int_1^2\int_0^{2-x} dy\,dx = \int_0^1\int_y^{2-y} dx\,dy$

Transformation into polar coordinates

$$\iint_R f(x,y)\,dy\,dx = \iint_R f(x,y)\,dx\,dy = \iint_R F(r,\theta)\cdot r\,dr\,d\theta$$

where $x=r\cos\theta$, $y=r\sin\theta$, $r^2=x^2+y^2$, $\theta=\tan^{-1}\frac{y}{x}$ and

$$dA = dx\,dy = dy\,dx \longrightarrow r\,dr\,d\theta$$

$$\begin{pmatrix}\text{Area inside}\\ \text{the region } R\end{pmatrix} = \iint_R dx\,dy \longrightarrow \iint_R r\,dr\,d\theta$$

Examples

(1) To find $\int_0^a \int_0^{\sqrt{a^2-y^2}} \frac{dx\,dy}{\sqrt{5+2x^2+2y^2}}$

change into polar coordinates

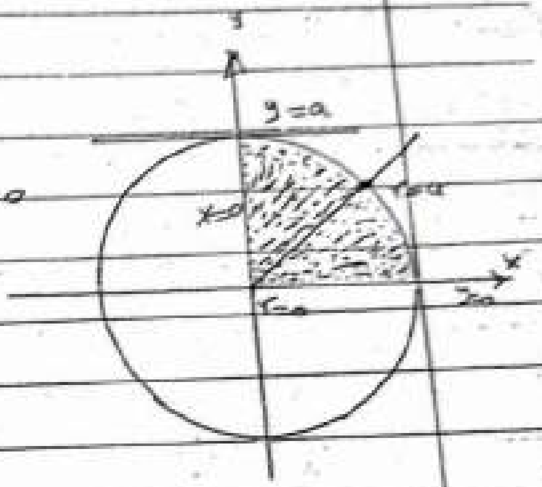
R: $y=0$ to $y=a$

$x=0$ to $x=\sqrt{a^2-y^2}$

$\Rightarrow x^2=a^2-y^2$

$\Rightarrow x^2+y^2=a^2,\ x>0$

$r=a$



$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} \frac{dx\,dy}{\sqrt{5+2x^2+2y^2}}$$

$$= \int_0^{\pi/2} \int_0^a \frac{r\,dr\,d\theta}{\sqrt{5+2r^2}}$$

$$= \int_0^{\pi/2} d\theta \cdot \frac{1}{4}\int_0^a (5+2r^2)^{-\frac{1}{2}} (4r\,dr)$$

$$= \left(\frac{\pi}{2}\right)\cdot\left(\frac{1}{4}\right)\cdot \frac{(5+2r^2)^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_0^a$$

$$= \frac{\pi}{4}\left\{\sqrt{5+2a^2}-\sqrt{5}\right\}$$

(2) To find $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

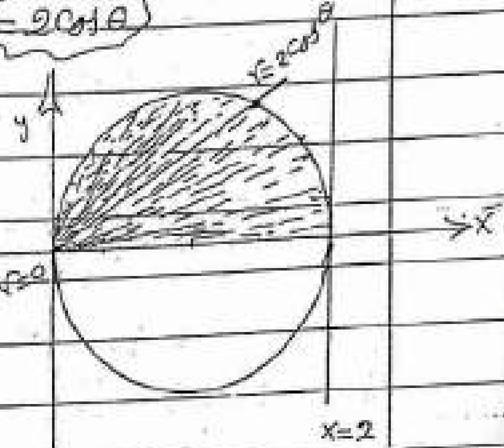
~~change into polar coordinates~~

R: ~~$x=0$ to $x=2$~~

~~$y=0$ to $y=\sqrt{2x-x^2}$~~

~~$y^2=2x-x^2$~~

~~$x^2+y^2=2x \longrightarrow \boxed{r=2\cos\theta}$~~



$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\int_0^{2\cos\theta} r^2\,dr \right] d\theta$$

$$= \int_0^{\pi/2} \left\{ \frac{r^3}{3} \right\}\Big|_0^{2\cos\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \left[(2\cos\theta)^3 - 0 \right] d\theta = \frac{8}{3} \int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \frac{8}{3} \int_0^{\pi/2} \cos^2\theta \cdot \cos\theta\, d\theta = \frac{8}{3} \int_0^{\pi/2} (1 - \sin^2\theta) \cos\theta\, d\theta$$

$$= \frac{8}{3} \int_0^{\pi/2} (\cos\theta - \sin^2\theta \cos\theta)\, d\theta$$

$$= \frac{8}{3} \left[\sin\theta - \frac{\sin^3\theta}{3} \right]_0^{\pi/2} = \frac{8}{3} \left[1 - \frac{1}{3} \right]$$

$$= \frac{16}{9}$$

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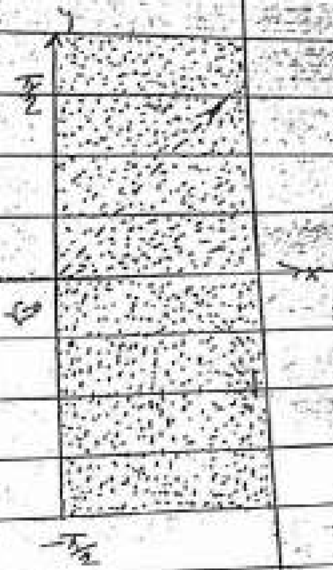
(3) To find $\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-x^2-y^2}\,dx\,dy = I$

change into polar coordinates



R: y runs from ∞ to ∞

x runs from 0 to ∞

$$\therefore I = \int_{-\pi/2}^{\pi/2}\int_{0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \cdot \int_{0}^{\infty} e^{-r^2}\, r\,dr$$

$$= \pi \cdot \lim_{b\to\infty}\int_{0}^{b} e^{-r^2}\, r\,dr$$

$$= -\frac{\pi}{2}\lim_{b\to\infty}\int_{0}^{b} e^{-r^2}\,(-2r\,dr)$$

$$= \frac{\pi}{2}\lim_{b\to\infty}\left[e^{-r^2}\right]_{0}^{b}$$

$$= -\frac{\pi}{2}\lim_{b\to\infty}\left\{e^{-b^2} - e^{-0}\right\}$$

$$= -\frac{\pi}{2}\left[0 - 1\right]$$

$$= \frac{\pi}{2}$$

Exercise 20

[1] Find each of the following integrals

(1) $\displaystyle\int_0^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{\sqrt{(7+3x^2+3y^2)^3}}$

(2) $\displaystyle\int_0^{a}\int_{-\sqrt{a^2-y^2}}^{0} \frac{dx\,dy}{\sqrt[3]{8-x^2-y^2}}$

(3) $\displaystyle\int_0^{1}\int_{1-\sqrt{1-y^2}}^{y} x\,dx\,dy$

(4) $\displaystyle\int_0^{1}\int_{y}^{\sqrt{2y-y^2}} dx\,dy$

[2] The integral $\displaystyle\int_{-1}^{1}\int_{x^2}^{1} dy\,dx$ represents the area of a region of the xy-plane. Sketch the region and express the same area as a double integral with the order of the integral reversed.

[3] Find $\displaystyle\int_0^{\infty} e^{-x^2}\,dx$

[4] Show that
$$\int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy = a^2\beta\left(\ln a-\frac{1}{2}\right)$$
where $a>0$, and $0<\beta<\frac{\pi}{2}$.

[5] Evaluate $\displaystyle\iint \frac{dx\,dy}{[1+x^2+y^2]^2}$ over the region enclosed by $(x^2+y^2)^2-(x^2-y^2)=0$.

[6] Find the volume of the solid bounded by $x=y^2+z^2$ and $x=1-y^2$

[7] Find the volume of the region bounded by the plane $z=a$, the cylinder $x^2+y^2=a^2$ and the cylinder $az=a^2-x^2$.

[8] Find the volume of the region enclosed by $z=x^2+y^2$ and $z=\frac{1}{2}(x^2+y^2)+1$.

[9] Find the volume of the region bounded above by $x^2+y^2+z^2=2a^2$ and below by $az=x^2+y^2$.

[10] Find the volume of the region enclosed by the cylinder $y = \cos x$ and the planes $z = y$, $x = 0$, $x = \frac{\pi}{2}$, and $z = 0$.

[11] Find the volume of the region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = x^2 + y^2$.

[12] Find the volume of the region bounded by the planes $x + 2y + 3z = 6$, $x = 0$, $y = 0$, $z = 0$ in six methods

[13] Find the volume of the region in the first octant that lies between the two cylinders $r = 1$, and $r = 2$ and that is bounded below by the xy-plane and above by the surface $z = xy$.

[14] Find the volume of the region common to $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$

[15] Show that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$.

[15] Using double integral to find the area common to $r = 1 + \sin\theta$ and $r = 4\sin\theta$

Infinite Series

Sequence

A sequence $\{a_n\}$ is a function whose domain is the set of natural numbers.

Example $\{a_n\}_{n=1}^{\infty} = \left\{\frac{n+1}{n}\right\}_{n=1}^{\infty} = 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \dots \to 1$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

~~Definition~~ The sequence $\left\{\frac{n+1}{n}\right\}$ converges to 1

Convergence of a Sequence

A Sequence $\{a_n\}$ converges to L means that $\boxed{\lim_{n\to\infty} a_n = L}$

Where L is a finite single real number, otherwise it diverges.

Exercise 21 Determine the convergence of

[1] $\left\{\frac{2n^2}{3n^2+n+1}\right\}$ [2] $\left\{\frac{5n+2}{3n^2+n+1}\right\}$ [3] $\left\{\frac{5n^6+n^2}{3n^3+n^2+5}\right\}$

[4] $\left\{2^n\right\}$ [5] $\left\{\frac{1}{3^n}\right\}$ [6] $\left\{\frac{\ln n}{n}\right\}$

[7] $\left\{n^{\frac{1}{n}}\right\}$ [8] $\left\{n-\sqrt{n^2+3}\right\}$ [9] $\left\{\frac{(-1)^n}{n}\right\}$

[10] $\left\{n-\sqrt{n^2+3n}\right\}$ [11] $\left\{\sqrt{n^2+2n}-\sqrt{n^2+7n}\right\}$ [12] $\left\{\left(1+\frac{1}{n}\right)^n\right\}$

[13] $\left\{\ln n - \sinh^{-1} n\right\}$ [14] $\left\{n-\ln \cosh n\right\}$ [15] $\left\{\left(1-\frac{7}{n}\right)^{3n}\right\}$

Series

$$\sum a_n = a_1 + a_2 + a_3 + \cdots$$

Example (1): $\sum\limits_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.999999$

$= 2$

The series $\sum\limits_{n=0}^{\infty} \frac{1}{2^n}$ converges to 2

Example (2): $\sum\limits_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$

The series does not converges to any definite finite number; it diverges

Definition (convergence of a Series)

The series $\sum\limits_{n=1}^{\infty} a_n$ converges to S means that $\lim\limits_{n \to \infty} S_n = S$

S_n = nth partial Sum $= a_1 + a_2 + a_3 + \cdots + a_n$, and S is the total sum of the series, provided that S is a real single number, otherwise it diverges

Examples: Use the definition (Finding S_n) to determine the convergence or divergence of each of the following Series:

(1) $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ (2) $\sum\limits_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$ (3) $\sum\limits_{n=1}^{\infty} (-1)^n$

(1) $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$,

$$a_n = \ln\left(\frac{n}{n+1}\right)$$

$$a_n = \ln n - \ln(n+1)$$

$a_1 = \ln 1 - \ln 2$

$a_2 = \ln 2 - \ln 3$

$a_3 = \ln 3 - \ln 4$

$\vdots$

$a_{n-1} = \ln(n-1) - \ln n$

$a_n = \ln n - \ln(n+1)$

$$\therefore S_n = a_1 + a_2 + \cdots + a_n = \ln 1 - \ln(n+1)$$
$$= -\ln(n+1) \to -\infty \quad \text{as } n \to \infty$$

$\therefore$ The series diverges. And $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\ln\left(\frac{n}{n+1}\right)\right]$
$$= \lim_{n\to\infty} \ln\left(\frac{1}{1+\frac{1}{n}}\right)$$
$$= \ln 1 = 0$$

(3) $\sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+1 \cdots$

$S_1 = -1$

$S_2 = -1+1 = 0$

$S_3 = -1+1-1 = -1$

$S_4 = -1+1-1+1 = 0$

$\vdots$

$$\therefore S_n = \begin{cases} \to -1 \\ \to 0 \end{cases}$$

Thus $\lim_{n\to\infty} S_n$ does not exist and then the series diverges. And $\lim_{n\to\infty} a_n$ does not exist

(2) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$

$$a_n = \frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$
$$= \frac{A(n+3) + B(n+1)}{(n+1)(n+3)}$$

$\Rightarrow 1 = A(n+3) + B(n+1)$

$\Rightarrow \boxed{A = \frac{1}{2}}$ and $\boxed{B = -\frac{1}{2}}$

$$a_n = \frac{1/2}{n+1} - \frac{1/2}{n+3}$$

$$a_n = \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+3}\right]$$

$$a_1 = \frac{1}{2}\left[\frac{1}{2} - \cancel{\frac{1}{4}}\right]$$

$$a_2 = \frac{1}{2}\left[\frac{1}{3} - \cancel{\frac{1}{5}}\right]$$

$$a_3 = \frac{1}{2}\left[\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right]$$

$$a_4 = \frac{1}{2}\left[\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right]$$

$$a_5 = \frac{1}{2}\left[\cancel{\frac{1}{6}} - \frac{1}{8}\right]$$

$$a_6 = \frac{1}{2}\left[\cancel{\frac{1}{7}} - \frac{1}{9}\right]$$

$$\vdots$$

$$a_{n-2} = \frac{1}{2}\left[\frac{1}{n-1} - \cancel{\frac{1}{n+1}}\right]$$

$$a_{n-1} = \frac{1}{2}\left[\cancel{\frac{1}{n}} - \frac{1}{n+2}\right]$$

$$a_n = \frac{1}{2}\left[\cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right]$$

$$\therefore S_n = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$\lim S_n = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3}\right] = \frac{5}{12}$$

$\therefore$ the Series $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$ converges to $\frac{5}{12}$

And

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{(n+1)(n+3)}\right) = 0$$

Geometric Series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ it converges to $\frac{1}{1-x}$ when $-1<x<1$ otherwise it diverges

Or $\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad , -1<x<1$.

Exercise 21

[1] Use the definition to determine whether each of the series converges or diverges.

(1) $\sum_{n=1}^{\infty} \ln\left(\frac{n+3}{n+5}\right)$ (2) $\sum_{n=1}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$ (3) $\sum_{n=2}^{\infty} \frac{2^{n+1}}{5^{3n+2}}$

(4) $\sum_{n=1}^{\infty} \frac{3^n - 5^{2n}}{12^n}$ (5) $\sum_{n=1}^{\infty} \frac{1}{n(n+9)}$ (6) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

(7) $\sum \frac{1}{n(n+1)(n+2)}$ (8) $\sum_{k=1}^{\infty} \frac{1}{k^2+12k+27}$ (9) $\sum_{n=2}^{\infty} \frac{(-1)^{n+7}}{5^{n+3}}$

[2] Find two positive integers p and r such that $x = \frac{p}{r}$ where

[a] $x = 9.3512312312313\ldots$

[b] $x = 7.2423252525252525\ldots$

[c] $x = 2.44444\ldots$

[3] Show that $0.99999\ldots = 1$

Tests for Convergence

[1] The nth general term test

If $\lim a_n \neq 0$ then $\sum a_n$ diverges.

[2] The comparison test

(a) If $a_n \le b_n$, $\sum b_n$ converges then $\sum a_n$ converges

(b) If $a_n \ge b_n$, $\sum b_n$ diverges then $\sum a_n$ diverges.

[3] The integral test

If $f(x)$ is a decreasing, continuous, positive function defined for $x \ge 1$ then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ both converges or diverges

[4] The limit comparison test

If $\lim_{n\to\infty} \frac{a_n}{b_n} \neq 0$ then $\sum a_n$ and $\sum b_n$ converge or diverge together

[5] The limit ratio test

(a) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$ then $\sum a_n$ converges.

(b) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$ then $\sum a_n$ diverges.

Fails if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

[6] The nth root test

(a) If $\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1$ then $\sum a_n$ converges.

(b) If $\lim \sqrt[n]{a_n} = L > 1$ then $\sum a_n$ diverges

No information if $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$

Examples

[1] $\sum_{n=1}^{\infty} \left(\frac{n+2}{n+3} \right)^{5n}$

Answer Using the important limit

$$\boxed{\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x}$$

Thus $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+2}{n+3} \right)^{5n}$

$$= \lim_{n\to\infty} \left[\frac{(1+\frac{2}{n})^n}{(1+\frac{3}{n})^n} \right]^5 = \left(\frac{e^2}{e^3}\right)^5 = e^{-5} \neq 0$$

thus the Series diverges.

[2] $\sum_{n=1}^{\infty} \frac{1}{n^2}$,

Let $f(x) = \frac{1}{x^2}$, $x \geq 1$

It is clear that f is a positive, decreasing, continuous for $x \geq 1$

$$\therefore \int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{1}{x} \Big]_1^b \right)$$

$$= -\lim_{b\to\infty} \left\{ \frac{1}{b} - \frac{1}{1} \right\} = 1 \quad \therefore \int_1^{\infty} f(x)\,dx \text{ Converges}$$

and therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

[3] $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$,

Let $f(x) = \frac{1}{\sqrt{x}}$, $x \geq 1$

it is easy to show that f satisfies all the three condition of the integral test.

$$\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{\sqrt{x}}\,dx = 2 \lim_{b\to\infty} \left(\sqrt{x} \Big]_2^b \right)$$

$$= \lim_{b\to\infty} 2(\sqrt{b} - \sqrt{2}) \longrightarrow \infty \text{ as } b\to\infty$$

thus the integral diverges and therefore the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges

<u>Theorem</u> (P series)

$\sum \frac{1}{n^p}$ converges for $p > 1$
diverges for $p \leq 1$

[4] $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$f(x) = \frac{1}{n \ln n}$, $n \geq 2$

$$\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_2^b \frac{dx}{x \ln x}$$

$$= \lim_{b\to\infty} \left(\ln(\ln x) \Big]_2^b \right) = \lim_{b\to\infty} \left(\ln\ln b - \ln\ln 2 \right) = \infty$$

The series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

[5] $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^5}$ Since $\sin^2 n \leq 1$ $\therefore \frac{\sin^2 n}{n^5} \leq \frac{1}{n^5}$

and since $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges, p-Series, $p = 5 > 1$

$\therefore \sum \frac{\sin^2 n}{n^5}$ converges.

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[6] $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$, $a_n = \frac{n^3}{2^n}$, $a_{n+1} = \frac{(n+1)^3}{2^{n+1}}$

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2}\cdot\left(\frac{n+1}{n}\right)^3 = \lim \frac{1}{2}\left(1+\frac{1}{n}\right)^3 = \frac{1}{2} < 1$$

∴ The series $\sum \frac{n^3}{2^n}$ converges.

$n! = n(n-1)(n-2)\ldots 3.2.1$

$(n+1)! = (n+1)\underbrace{(n)(n-1)(n-2)\ldots 3.2.1}_{n!}$

$0! = 1$, $1! = 1$ $2! = 2$

$\Rightarrow (n+1)! = (n+1)\,n!$ $3! = 6$, $4! = 24$

[7] $\sum \frac{n!}{n^n}$, $a_n = \frac{n!}{n^n}$, $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{(n+1)}\,\cancel{n!}}{(n+1)^n \cdot \cancel{(n+1)}^1} \cdot \frac{n^n}{\cancel{n!}}\right) = \lim_{n\to\infty}\left(\frac{n^n}{(n+1)^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e^1} = \frac{1}{2.7} < 1$$

∴ The series $\sum \frac{n^n}{n!}$ converges.

[8] $\sum_{n=1}^{\infty} \frac{5n+3}{7n^3+2n+1}$ let $\sum_{n=1}^{\infty} \frac{5n+3}{7n^3+2n+1} = \sum_{n=1}^{\infty} a_n$

and choose

$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, p-series, $p = 2 > 1$ converges

and find

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left[\frac{\left(\frac{5n+3}{7n^3+2n+1}\right)}{\left(\frac{1}{n^2}\right)}\right] = \lim_{n\to\infty} \frac{5n^3+3n^2}{7n^3+2n+1}$$

$$= \frac{5}{7} \neq 0$$

Thus the series $\sum_{n=1}^{\infty} \frac{9n+3}{7n^3+2n+1}$ also converges.

[9] $\sum_{n=3}^{\infty} \left(1-\frac{3}{n}\right)^{7n^2}$

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(1-\frac{3}{n}\right)^{7n^2}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(1-\frac{3}{n}\right)^{7n} = \lim \left[\left(1+\frac{(-3)}{n}\right)^{n}\right]^{7}$$

$$= \left(e^{-3}\right)^{7} = e^{-21} = \frac{1}{e^{21}} < 1$$

The series $\sum_{n=3}^{\infty} \left(1-\frac{3}{n}\right)^{7n^2}$ converges

[10] $\sum_{n=2} \frac{1}{n^3 \ln n}$

Since $\ln n > 1$, $n > 2$

$\therefore n^3 \ln n > n^3$

$$\frac{1}{n^3 \ln n} < \frac{1}{n^3}$$

but $\sum \frac{1}{n^3}$ is a p-series, $p=3>1$ converges

$\therefore \sum \frac{1}{n^3 \cdot \ln n}$ also converges

Exercise 22

Discuss the behaviour of each of the following series :-

[1] $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$ [2] $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,|\cos n|}$ [3] $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

[4] $\sum_{n=2}^{\infty} \frac{1}{(n^2+3n)\ln n}$ [5] $\sum_{n=2}^{\infty} \frac{1}{(2n+3)\ln n}$ [6] $\sum_{n=1}^{\infty} \frac{(2n+1)!}{(3n+2)!}$

[7] $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2+3n+11}$ [8] $\sum_{n=2}^{\infty} \frac{1}{1+\ln n}$ [9] $\sum_{n=1}^{\infty} \frac{5^{2n+1}}{(3n+2)!}$

[10] $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ [11] $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ [12] $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

[13] $\sum_{n=1}^{\infty} \frac{1}{n^{(1+\frac{1}{n})}}$ [14] $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ [15] $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$

Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \ , \quad a_n > 0$$

Note

(1) $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |(-1)^{n-1} a_n|$ converges

(2) $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges conditionally if $\sum_{n=1}^{\infty} |(-1)^{n-1} a_n|$ diverges and $\lim a_n = 0$

(3) $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ diverges if $\sum_{n=1}^{\infty} |(-1)^{n-1} a_n|$ diverges and $\lim_{n\to\infty} a_n \neq 0$

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Power Series (Series of functions)

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Interval of convergence

The values of x for which the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely and to find it one may use a certain test

Examples Find the radius of convergence of

(1) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n \, n^4}$ (2) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n x^n$

Answers

(1) $a_n = \frac{(x-2)^n}{3^n \, n^4}$, $a_{n+1} = \frac{(x-2)^{n+1}}{3^{n+1} (n+1)^4}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3^{n+1} (n+1)^4} \cdot \frac{3^n \, n^4}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{3} \cdot (x-2) \cdot \underbrace{\frac{n^4}{(n+1)^4}}_{\to 1} \right|$$

$= \left|\frac{x-2}{3}\right| < 1$ the series converges

$$\left|\frac{x-2}{3}\right| < 1 \Rightarrow 1 < \frac{x-2}{3} < 1 \Rightarrow -3 < x-2 < 3$$

$$\Rightarrow -1 < x < 5$$

$\therefore$ The series converges for $-1 < x < 5$

and The series diverges for $x > 5$, $x < -1$

At $x = 5 \Rightarrow \sum \frac{(5-2)^n}{3^n \, n^4} = \sum \frac{1}{n^4}$ is a p-series , $p = 4 > 1$

at $x = -1 \Rightarrow \sum \frac{(-1-2)^n}{3^n \cdot n^4} = \sum \frac{(-1)^n}{n^4}$ is absolutely conv

Interval of convergence is $-1 \le x \le 5$

Radius of convergence is $R = 3$

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(2) $a_n = \left(\frac{n+1}{n+2}\right)^n x^n$,

$$\lim_{n\to\infty} \left|\sqrt[n]{a_n}\right| = \lim \left|\sqrt[n]{\left(\frac{n+1}{n+2}\right)^n x^n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) \cdot |x|$$

$= |x| < 1$ converges

$\Rightarrow |x| < 1 \Rightarrow -1 < x < 1$

at $x = 1 \Rightarrow \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n = ?$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$$

$$= \lim \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{e^1}{e^2} = \frac{1}{e} \neq 0$$

$\therefore$ the Series $\sum \left(\frac{n+1}{n+2}\right)^n$ diverges because $\lim_{n\to\infty} a_n \neq 0$

at $x = -1 \Rightarrow \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n (-1)^n$ this is an alternating series Which is a divergent series since $\lim a_n \neq 0$

Interval of convergence is

$-1 < x < 1$

Radius of convergence is $R = 1$

Exercise 23

Find the radius of convergence of

[1] $\sum_{n=1}^{\infty} \frac{(x+1)^n}{5^n \sqrt{n}}$

[2] $\sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n (\sqrt[4]{n}+\sqrt[3]{n}+2)}$

[3] $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{4^n (n^5+2n^2+4)}$

[4] $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

[5] $\sum_{n=1}^{\infty} \frac{(x+2)^{3n+4}}{(7n+2)!}$

[6] $\sum_{n=1}^{\infty} (x-\frac{1}{x})^n$

[7] $\sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{x+1}{3x+2}\right)^n$

[8] $\sum_{n=1}^{\infty} x^n$

Taylor's and Maclaurin's Series

Theorem (Taylor)

If f is defined at $x=a$ and is differentiable at $x=a$ of order n then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Taylor's expansion of f about $x=a$.

Special case

If $a=0$ then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Maclaurin's Series for $f(x)$

Examples

(1) Using Taylor series to find the approximate value for $\cos(61^\circ)$.

Answer

Let $f(x) = \cos x$, $a = \frac{\pi}{3} = 60^\circ$

$\therefore f(x) = \cos x \Rightarrow f(\frac{\pi}{3}) = \frac{1}{2}$

$f'(x) = -\sin x \Rightarrow f'(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x \Rightarrow f''(\frac{\pi}{3}) = -\frac{1}{2}$

$\vdots$

$$f(x) = f(\tfrac{\pi}{3}) + f'(\tfrac{\pi}{3})(x - \tfrac{\pi}{3}) + \frac{f''(\frac{\pi}{3})}{2!}(x - \tfrac{\pi}{3})^2 + \frac{f'''(\frac{\pi}{3})}{3!}(x - \tfrac{\pi}{3})^3 + \cdots$$

$$\therefore \cos x = \frac{1}{2} + (-\tfrac{\sqrt{3}}{2})(x - \tfrac{\pi}{3}) + \frac{(-\frac{1}{2})}{2!}(x - \tfrac{\pi}{3})^2 + \cdots$$

for all x near $\frac{\pi}{3}$

If $x = 61^\circ$ then

$$x - \tfrac{\pi}{3} = 61^\circ - 60 = 1^\circ = \tfrac{\pi}{180} \text{ rad}$$

thus

$$\cos 61^\circ \cong \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{4}\left(\frac{\pi}{180}\right)^2.$$

(2) Find Maclaurin Series for $\frac{1}{1-x}$ and then find M.S. for $\ln(1-x)$

Answer

Let $f(x) = \frac{1}{1-x}$, $a = 0$

$\therefore f(x) = \frac{1}{1-x} \Rightarrow f(0) = 1$

$f'(x) = (-1)(1-x)^{-2} \cdot (-1) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1 = 1!$

$f''(x) = (-2)(1-x)^{-3} \cdot (-1)$

$= 2(1-x)^{-3} \Rightarrow f''(0) = 2 = 2!$

$f'''(x) = (2)(-3)(1-x)^{-4} \cdot (-1)$

$= 2(3)(1-x)^{-4} \Rightarrow f'''(0) = 2 \cdot 3 = 3!$

but $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$$\frac{1}{1-x} = 1 + (1)x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \cdots$$

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots}$$

put $x = t$

$$\therefore \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$$

$$\int_0^x \frac{1}{1-t}\,dt = \int_0^x (1 + t + t^2 + \cdots)\,dt$$

$$-\ln(1-t)\Big]_0^x = t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots\Big|_0^x$$

$$-[\ln(1-x) - \ln 1] = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

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Examples

(1) Using Series to find $\lim_{x\to 0}\left[\frac{(e^{3x}-1)}{\sin x}\right]$

Answer since

$$e^{3x} = 1+(3x)+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\cdots$$

and $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

$$\therefore \lim_{x\to 0}\left(\frac{e^{3x}-1}{\sin x}\right)$$

$$=\lim_{x\to 0}\frac{\left(1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\cdots\right)-1}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots}$$

$$=\lim_{x\to 0}\frac{x\left(3+\frac{9}{2}x+\frac{9}{2}x^2\cdots\right)}{x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}\cdots\right)} = 3$$

(2) Using Series to prove that $\sin 2x = 2\sin x\cos x$

Since $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

$$\sin 2x = 2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots$$

$$\sin 2x = 2x-\frac{4}{3}x^3+\frac{4}{15}x^5-\cdots$$

thus, The Right Hand side of the identity is

$$2\sin x\cos x = 2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\cdot\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\cdots\right)$$

$$=2\left\{(0)x^0+(1)x^1+(0)x^2+\left(-\frac{1}{3!}-\frac{1}{2!}\right)x^3+\cdots\right\}$$

$$=2x-\frac{4}{3}x^3+\cdots = \sin 2x$$

Q.E.D.

(3) Using Series to find the approximate value for: $\int_{0}^{0.1} \sin x^2\, dx$

Answer

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\therefore \sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots$

$$\int_0^{0.1} \sin x^2\, dx = \int_0^{0.1} \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right) dx$$

$$= \frac{x^3}{3} - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)} \Big]_0^{0.1}$$

$$= \frac{(0.1)^3}{3} - \frac{(0.1)^7}{42} + \frac{(0.1)^{11}}{1320}$$

$$\cong 0.000333331$$

Exercise 24

[1] Using M.S. to show that

(a) $\sin^2 x + \cos^2 x = 1$ (b) $\cosh^2 x - \sinh^2 x = 1$

(c) $e^{ix} = \cos x + i \sin x$ (d) $\sinh(ix) = i \sin x$

(e) $\cosh(ix) = \cos x$ (f) $e^{-ix} = \cos x - i \sin x$

(g) $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ (h) $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

[2] Using the identity $\frac{x^3 - 1}{x - 1} = x^2 + x + 1$ to find the M.S. for $\ln(1 + x + x^2)$ and then find $\lim_{x \to 0} \frac{\ln(1 + x + x^2)}{x}$

[3] Find the approximate value for (a) $\sin(29°)$ (b) $\int_0^{0.1} e^{x^3} dx$

[4] Using the series $\frac{1}{1-x}$ to find M.S. for (a) $\ln(1+x)$ (b) $\tan^{-1} x$

[5] Using the series $(1-x)^n$ to find M.S. for (a) $\sin^{-1} x$ (b) $\sinh^{-1} x$

[6] Find M.S. for $\frac{e^{3x}}{1-x}$, $e^{2x} \sin x$, $\tan^{-1} x \cdot \cos x$, $\sec x$

[7] Find M.S. for $\tan x$, $\tanh x$, $\frac{e^x}{\cos x}$, $\frac{x e^{3x}}{\sin x}$.

[8] Using Series to find

(a) $\lim_{x \to 0} \left(\frac{1 - e^{x^9}}{x^3 - \sin x^3} \right)$

(b) $\lim_{x \to 0} \left(\frac{1 - \cos(x^5)}{1 - \cosh(x^4)} \right)$

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Matrices

$A_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (2 \text{ by } 2 \text{ matrix})$

Properties

(1) Addition and Subtraction

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \mp \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1 \mp x_1 & b_1 \mp y_1 \\ a_2 \mp x_2 & b_2 \mp y_2 \end{bmatrix}$$

(2) $k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

(3) Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ r & s \end{bmatrix} = \begin{bmatrix} ax+br & ay+bs \\ cx+dr & cy+ds \end{bmatrix}$$

(4) Identity Matrix (I) with the property that $A*I = I*A = I$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) $A*B \neq B*A$ in general.

(6) Transpose Matrix (A^T)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

(7) Singular Matrix

A is a singular matrix, if $\det A = 0$.

(8) Inverse Matrix (A^{-1}) with the property that

$A*A^{-1} = A^{-1}*A = I$

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Note : A^{-1} exists if and only if A is a non-Singular matrix.

Finding A^{-1}

To find A^{-1} for

$$A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$$

[1] Find det A

$$\det A = 4(3) - (-4)(1) + 5(-3) = 1 \neq 0$$

$\therefore$ A is a non-Singular matrix

thus A^{-1} exists.

[2] Find the cofactor matrix

$$\text{Cof}A = \begin{bmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ -3 & 2 & 4 \end{bmatrix}$$

[3] Find the adjoint Matrix

$$Adj(A) = (\text{cof}A)^T = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}$$

[4] Find the inverse Matrix A^{-1}

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{1}{1} \begin{bmatrix} 3 & 1 & -3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}$$

It is obvious that

$$A * A^{-1} = A^{-1} * A = I$$

Solution of system of n linear equations

To solve

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The system can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Or

$$A * X = B \quad (\text{matrix Equation})$$

To solve this matrix equation, multiply both sides by A^{-1}

$$A^{-1} * A * X = A^{-1} * B$$
$$I * X = A^{-1} * B$$
$$\therefore \boxed{X = A^{-1} * B} \text{ — the solution}$$

Ex. Using inverse matrix to solve

$$4x_1 - 4x_2 + 5x_3 = 1$$
$$-2x_1 + 3x_2 - 3x_3 = 0$$
$$3x_1 - 3x_2 + 4x_3 = 2$$

Answer

Let $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & +3 & -3 \\ 3 & -3 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

then the system will be

$$A * X = B$$

The solution is

$$X = A^{-1} * B$$

thus we must find A^{-1}

$$\therefore A^{-1} = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}$$ [see p.124]

$$\therefore X = A^{-1} * B$$

$$= \begin{bmatrix} 3 & 1 & -3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+0-6 \\ 1+0+4 \\ -3+0+8 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow$$

$x_1 = -3$, $x_2 = 3$, $x_3 = 5$

Eigen Values and Eigen Vectors

Eigen values

The roots of the equation $\det(A - \lambda I) = 0$ are called the eigen values for the square matrix A

* Note: Each value of λ (root of equation $\det(A - \lambda I) = 0$) gives a vector called eigen vector.

Example: Find the eigen values and the eigen vectors for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$

$6 - 5\lambda + \lambda^2 - 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda - 1) = 0 \Rightarrow \lambda = 1, \ \lambda = 4.$

If $\lambda = 1$ then $A - \lambda I = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 2x_2 = 0 \\ x_1 + 2x_2 = 0 \end{matrix} \Rightarrow x_1 = -2x_2$

let $x_2 = t \ \therefore x_1 = -2t$

First eigen Ratio $\frac{x_1}{x_2} = \frac{-2}{1}$

First Eigen vector $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$