Chapter Two: Equivalence Between Integral Equations and Ordinary Differential Equations

2.1 Converting Volterra Equation to an ODE

In this section, we will present the technique that converts Volterra integral equations of the second kind to equivalent differential equations. This may be easily achieved by applying the important Leibniz Rule for differentiating an integral. It seems reasonable to review the basic outline of the rule.

2.1.1 Differentiating Any Integral: Leibniz Rule

To differentiate the integral $\int_{\alpha(x)}^{\beta(x)} G(x,t)dt$ with respect to x, we usually apply the useful Leibniz rule given by:

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} G(x,t)dt = G\big(x,\beta(x)\big)\frac{d\beta}{dx} - G\big(x,\alpha(x)\big)\frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial G}{\partial x}dt \qquad (2.1)$$

where $G(x, t)$ and $\frac{\partial G}{\partial x}$ are continuous functions in the domain D in the xt-plane that contains the rectangular region R, $a \le x \le b$, $t_0 \le t \le t_1$, and the limits of integration $\alpha(x)$ and $\beta(x)$ are defined functions having continuous derivatives for $a < x < b$. We note that the Leibniz rule is usually presented in most calculus books, and our concern will be on using the rule rather than its theoretical proof. The following examples are illustrative and will be mostly used in the coming approach that will be used to convert Volterra integral equations to differential equations.

Particular case: If α(x) and β(x) are absolute constants, then (2.1) reduces to:

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} G(x,t)dt = \int_{\alpha(x)}^{\beta(x)} \frac{\partial G}{\partial x}dt$$

Example 2.1. Find $\boldsymbol{\frac{d}{dx}\int_0^x (x-t)^2 u(t)dt}$

In this example α(x)=0 , β(x)=x, hence $\frac{d\alpha}{dx}$=0, $\frac{d\beta}{dx}$=1and $\frac{\partial G}{\partial x} = 2(x-t)u(t)$. Using Leibniz rule (2.1), we find:

$$\boldsymbol{\frac{d}{dx}\int_0^x (x-t)^2 u(t)dt = \int_0^x 2(x-t)u(t)dt}$$

Example 2.2. Find $\frac{d}{dx}\int_0^x (x-t)u(t)dt$

In this example α(x)=0 , β(x)=x, hence $\frac{d\alpha}{dx}$=0, $\frac{d\beta}{dx}$=1and $\frac{\partial G}{\partial x} = u(t)$. Using Leibniz rule (2.1), we find:

$$\frac{d}{dx}\int_0^x (x-t)u(t)dt = \int_0^x u(t)dt$$

Example 2.3. Find $\frac{d}{dx}\int_0^x u(t)dt$

In this example α(x)=0 , β(x)=x, hence $\frac{d\alpha}{dx}$=0, $\frac{d\beta}{dx}$=1and $\frac{\partial G}{\partial x} = 0$. Using the Leibniz rule (2.1), we find:

$$\frac{d}{dx}\int_0^x u(t)dt = u(x)$$

We now turn to our main goal to convert a Volterra integral equation to an equivalent differential equation. This can be easily achieved by differentiating both sides of the integral equation, noting that the Leibniz rule should be used in differentiating the integral as stated above. The differentiating process should be continued as many times as needed until we obtain a pure differential equation with the integral sign removed. Moreover, the initial conditions needed can be obtained by substituting *x* =0 in the integral equation, and the resulting integro-differential equations will be shown. We are now ready to give the following illustrative examples.

Example 2.4. Find the initial value problem equivalent to the Volterra integral equation**:** $u(x) = 1 + \int_0^x u(t)dt$

Differentiating both sides of the integral equation and using the Leibniz rule we find:

$$u'(x) = u(x)$$

The initial condition can be obtained by substituting *x* = 0 into both sides of the integral equation; hence we find *u*(0) = 1. Consequently, the corresponding initial value problem of the first order is given by**:**

$$u'(x) - u(x) = 0, u(0) = 1$$

Example 2.5. Convert the following Volterra integral equation to an initial value problem**:** $u(x) = x + \int_0^x (t-x)u(t)dt$

Differentiating both sides of the integral equation, we obtain:

$$u'(x) = 1 - \int_0^x u(t)dt$$

We differentiate both sides of the resulting integro-differential equation to remove the integral sign, therefore, we obtain:

$$u''(x) = -u(x)$$

or equivalently

$$u''(x) + u(x) = 0$$

The related initial conditions are obtained by substituting $x = 0$ in $u(x)$ and in $u'(x)$ in the equations above, and as a result we find $u(0) = 0$ and $u'(0) = 1$. Combining the above results yields the equivalent initial value problem of the second order given by:

$$u''(x) + u(x) = 0\,, u(0) = 0, u'(0) = 1$$

Example 2. 6. Find the initial value problem equivalent to the Volterra integral equation: $u(x) = x^3 + \int_0^x (x-t)^2 u(t)dt$

Differentiating both sides of the above equation three times, we find:

$$u'(x) = 3x^2 + 2\int_0^x (x-t)u(t)dt$$

$$u''(x) = 6x + 2\int_0^x u(t)dt$$

$$u'''(x) = 6 + 2u(x)$$

The proper initial conditions can be easily obtained by substituting $x = 0$ in $u(x)$, $u'(x)$ and $u''(x)$ in the obtained equations above. Consequently, we obtain the nonhomogeneous initial value problem of third order given by:

$$u'''(x) - 2u(x) = 6\,, u(0) = 0\,, u'(0) = 0\,, u''(0) = 0$$

Exercises 2.1.

In exercises 1-4, find $\frac{d}{dx}$ for the given integrals by using the Leibniz rule:

1. $\int_0^x (x-t)^3 u(t)dt$
2. $\int_x^{x^2} e^{xt}dt$
3. $\int_0^x (x-t)^4 u(t)dt$
4. $\int_x^{4x} sin(x+t)dt$

In exercises 5-8, convert each of the Volterra integral equations to an equivalent initial value problem:

5. $u(x) = e^x + \int_0^x (x-t)u(t)dt$
6. $u(x) = 2 + 3x + 5x^2 + \int_0^x [1 + 2(x-t)]u(t)dt$
7. $u(x) = x - cosx + \int_0^x (x-t)u(t)dt$
8. $u(x) = -5 + 6x + \int_0^x (5 - 6x + 6t)u(t)dt$

2.2 Converting IVP to Volterra Equation

In this section, we will study the method that converts an initial value problem to an equivalent Volterra integral equation. Before outlining the method needed, we wish to recall the useful transformation formula**:**

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_{n-1}} f(x_n)dx_n \dots dx_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt \qquad \textbf{(2.2)}$$

that converts any multiple integral to a single integral. This is an essential and useful formula that will be employed in the method that will be used in the conversion technique. We point out that this formula appears in most calculus texts. For practical considerations, the formulas:

$$\int_0^x \int_0^x f(t)dt\,dt = \int_0^x (x-t)f(t)dt \qquad \textbf{(2.3)}$$

$$\int_0^x \int_0^x \int_0^x f(t)dt\,dt\,dt = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt \qquad \textbf{(2.4)}$$

are two special cases of the formula given above, and the most used formulas that will transform double and triple integrals respectively to a single integral for each. For simplicity reasons, we prove the first formula (2.3) that converts a double integral to a single integral. Noting that the right-hand side of (2.3) is a function of *x* allows us to set the equation:

$$I(x) = \int_0^x (x-t)f(t)dt \qquad \textbf{(2.5)}$$

Differentiating both sides of (2.5), and using the Leibniz rule, we obtain:

$$I'(x) = \int_0^x f(t)dt \qquad \textbf{(2.6)}$$

Integrating both sides of (2.6) from 0 to *x*, noting that *I*(0) = 0 from (2.5), we find:

$$I(x) = \int_0^x \int_0^x f(t)dt\,dt$$

Exercises 2.2. Prove that $\int_0^x \int_0^x \int_0^x f(t)dt\,dt\,dt = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt$

Example 2.7. Convert the following quadruple integral:

$$I(x) = \int_0^x \int_0^x \int_0^x \int_0^x u(t)dt\,dt\,dt\,dt$$

to a single integral.
Using the formula (2.2), noting that $n = 4$, we find:

$$\boldsymbol{I(x) = \frac{1}{3!}\int_0^x (x-t)^3 u(t)dt}$$

Returning to the main goal of this section, we discuss the technique that will be used to convert an initial value problem to an equivalent Volterra integral equation. Without loss of generality, and for simplicity reasons, we apply this technique to a third-order initial value problem given by:

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = g(x) \qquad (2.7)$$

subject to the initial conditions:

$$y(0) = \alpha\,, y'(0) = \beta\,, y''(0) = \gamma \quad ,\alpha, \beta \text{ and } \gamma \text{ are constant} \qquad (2.8)$$

The coefficient functions $p(x)$, $q(x)$, and $r(x)$ are analytic functions by assuming that these functions have Taylor expansions about the origin. Besides, we assume that $g(x)$ is continuous through the interval of discussion. To transform (2.7) into an equivalent Volterra integral equation, we first set:

$$y'''(x) = u(x) \qquad (2.9)$$

where $u(x)$ is a continuous function on the interval of discussion. Based on (2.9), it remains to find other relations for y and its derivatives as single integrals involving $u(x)$. This can be simply performed by integrating both sides of (2.9) from 0 to x where we find:

$$y''(x) - y''(0) = \int_0^x u(t)dt$$

or equivalently

$$y''(x) = \gamma + \int_0^x u(t)dt \qquad (2.10)$$

To obtain $y'(x)$ we integrate both sides of (2.10) from 0 to x, to find that:

$$y'(x) = \beta + \gamma x + \int_0^x \int_0^x u(t)dt\,dt \qquad (2.11)$$

Similarly, we integrate both sides of (2.11) from 0 to x to obtain:

$$y(x) = \alpha + \beta x + \frac{1}{2}\gamma x^2 + \int_0^x \int_0^x \int_0^x u(t)dt\,dt\,dt \qquad (2.12)$$

respectively. Substituting (2.9), (2.10), (2.11), and (2.12) into (2.7) leads to the following Volterra integral equation of the second kind:

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = g(x)$$

$$\Rightarrow u(x) + p(x)\left[\gamma + \int_0^x u(t)dt\right] + q(x)\left[\beta + \gamma x + \int_0^x \int_0^x u(t)dt\,dt\right] +$$

$$r(x)\left[\alpha + \beta x + \frac{1}{2}\gamma x^2 + \int_0^x \int_0^x \int_0^x u(t)dt\,dt\,dt\right] = g(x)$$

$$\Rightarrow u(x)+p(x)\left[\gamma+\int_0^x u(t)dt\right]+q(x)\left[\beta+\gamma x+\int_0^x (x-t)u(t)dt\right]$$
$$+\ r(x)\left[\alpha+\beta x+\frac{1}{2}\gamma x^2+\frac{1}{2}\int_0^x (x-t)^2dt\right]=g(x)$$
$$\Rightarrow u(x)=\left(g(x)-\left\{p(x)\gamma+q(x)(\beta+\gamma x)+r(x)\left(\alpha+\beta x+\frac{1}{2}\gamma x^2\right)\right\}\right)$$
$$+\int_0^x \left[-p(x)-(x-t)q(x)-\frac{1}{2}(x-t)^2r(x)\right]u(t)dt$$
$$\Rightarrow u(x)=F(x)+\int_0^x K(x,t)u(t)dt$$

Where $F(x)=\left(g(x)-\left\{p(x)\gamma+q(x)(\beta+\gamma x)+r(x)\left(\alpha+\beta x+\frac{1}{2}\gamma x^2\right)\right\}\right)$
and $\left[K(x,t)=-p(x)-(x-t)q(x)-\frac{1}{2}(x-t)^2r(x)\right]$
The following examples will be used to illustrate the above-discussed technique.
Example 2.8. Convert the following initial value problem

$$y'''-3y''-6y'+5y=0$$

Subject to the initial conditions: $y(0)=y'(0)=y''(0)=1$
to an equivalent Volterra integral equation.
As indicated before, we first set:

$$y'''(x)=u(x) \qquad (2.13)$$

Integrating both sides of (2.13) from 0 to x and using the initial condition $y''(0) = 1$, we find:

$$y''(x)=1+\int_0^x u(t)dt \qquad (2.14)$$

And $y'(x)=1+x+\int_0^x\int_0^x u(t)dt\,dt$

$$y'(x)=1+x+\int_0^x (x-t)u(t)dt \qquad (2.15)$$

And $y(x)=1+x+\frac{1}{2}x^2+\int_0^x\int_0^x\int_0^x u(t)dt\,dtdt$

$$y(x)=1+x+\frac{1}{2}x^2+\frac{1}{2}\int_0^x (x-t)^2u(t)dt \qquad (2.16)$$

Substituting (2.13), (2.14), (2.15), and (2.16) into the IVP, we find:

$$y'''-3y''-6y'+5y=0$$

$$\Rightarrow u(x)-3\left[1+\int_0^x u(t)dt\right]-6\left[1+x+\int_0^x (x-t)u(t)dt\right]+5\left[1+x+\frac{1}{2}x^2+\frac{1}{2}\int_0^x (x-t)^2u(t)dt\right]=0$$

$$\Rightarrow u(x) = 3\left[1 + \int_0^x u(t)dt\right] + 6\left[1 + x + \int_0^x (x-t)u(t)dt\right] - 5\left[1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt\right]$$

$$\Rightarrow u(x) = 4 + x - \frac{5}{2}x^2 + \int_0^x \left[3 + 6(x-t) - \frac{5}{2}(x-t)^2\right] u(t)dt$$

the equivalent Volterra integral equation.

Example 2.9. Find the equivalent Volterra integral equation to the following initial value problem:

$$y''(x) + y(x) = cosx \ , y(0) = 0, y'(0) = 1$$

As indicated before, we first set:

$$y''(x) = u(x) \qquad (2.17)$$

Integrating both sides of (2.17) from 0 to x and using the initial condition $y'(0) = 1$, we find:

$$y'(x) = 1 + \int_0^x u(t)dt$$

$$y(x) = x + \int_0^x (x-t)u(t)dt$$

And $y''(x) + y(x) = cosx \Rightarrow u(x) + x + \int_0^x (x-t)u(t)dt = cosx$

$$\Rightarrow u(x) = cosx - x - \int_0^x (x-t)u(t)dt$$

the equivalent Volterra integral equation.

Exercises 2.3.

Convert each of the following first-order initial value problems to a Volterra integral equation:

1. $y' + y = \sec 2x$, $y(0) = 0$
2. $y'' - \sin x\, y' + e^{xy} = x$, $y(0) = 1$, $y'(0) = -1$
3. $y''' - y'' - y' + y = 0$, $y(0) = 2$, $y'(0) = 0$, $y''(0) = 2$

2.3 Converting BVP to Fredholm Equation

So far we have discussed how an initial value problem can be transformed to an equivalent Volterra integral equation. In this section, we will present the technique that will be used to convert a boundary value problem to an equivalent Fredholm integral equation. The technique is similar to that discussed in the previous section with some exceptions that are related to the boundary conditions. It is important to point out here that the procedure of reducing the boundary value problem to the Fredholm integral equation is complicated and rarely used. The method is similar to the technique discussed above, which reduces the initial value problem to Volterra integral equation, with the exception that we are given boundary conditions.

Special attention should be taken to define $y'(0)$ since it is not always given, as will be seen later. This can be easily determined from the resulting equations. It seems useful and practical to illustrate this method by applying it to an example rather than proving it.

Example 2.10. We want to derive an equivalent Fredholm integral equation to the following boundary value problem**:** $y''(x) + y(x) = x \ , 0 < x < \pi$
subject to the boundary conditions**:** $y(0) = 1, y(\pi) = \pi - 1$

We first set**:** $$y''(x) = u(x) \qquad (2.18)$$

Integrating both sides of the above equation from 0 to x gives:

$$\int_0^x y''(t)dt = \int_0^x u(t)dt \Rightarrow \mathbf{y'(x) = y'(0) + \int_0^x u(t)dt} \qquad (2.19)$$

As indicated earlier, $y'(0)$ is not given in this boundary value problem. However, $y'(0)$ will be determined later by using the boundary condition at $x = \pi$. Integrating both sides of the last equation from 0 to x and using the given boundary condition at $x= 0$, we find**:**

$$\boldsymbol{y(x) = y(0) + y'(0)x} + \int_0^x \int_0^x u(t)dt\,dt \Rightarrow \boldsymbol{y(x) = 1 + y'(0)x} + \int_0^x (x-t)u(t)dt \qquad (2.20)$$

upon converting the resulting double integral to a single integral as discussed before. It remains to evaluate $y'(0)$, and this can be obtained by substituting $x = \pi$ on both sides of the last equation and using the boundary condition at $x = \pi$, hence, we find**:**

$$\boldsymbol{\pi y'(0) = y(\pi) - 1} - \int_0^\pi (\pi - t)u(t)dt$$

$$\Rightarrow \boldsymbol{y'(0) = \frac{1}{\pi}}\left[\boldsymbol{\pi - 2} - \int_0^\pi (\pi - t)u(t)dt\right]$$

Substituting $y'(0)$ into (2.20) yields:

$$\boldsymbol{y(x) = 1 + x\left[\frac{1}{\pi}\left[\pi - 2 - \int_0^{\pi}(\pi - t)u(t)dt\right]\right] + \int_0^{x}(x - t)u(t)dt} \qquad (2.21)$$

Substituting (2.18) and (2.21) into BVP, we get:

$$y''(x) + y(x) = x$$

$$\Rightarrow u(x) + \mathbf{1} + \frac{\boldsymbol{x}}{\boldsymbol{\pi}}\left[\boldsymbol{\pi} - \mathbf{2} - \int_0^{\pi}(\pi - t)u(t)dt\right] + \int_0^{x}(x - t)u(t)dt = x$$

$$\Rightarrow u(x) = x - \mathbf{1} - \frac{\boldsymbol{x}}{\boldsymbol{\pi}}\left[\boldsymbol{\pi} - \mathbf{2} - \int_0^{\pi}(\pi - t)u(t)dt\right] - \int_0^{x}(x - t)u(t)dt$$

$$\Rightarrow u(x) = x - \mathbf{1} - \frac{\boldsymbol{x}}{\boldsymbol{\pi}}\left[\boldsymbol{\pi} - \mathbf{2} - \int_0^{x}(\pi - t)u(t)dt - \int_x^{\pi}(\pi - t)u(t)dt\right]$$

$$- \int_0^{x}(x - t)u(t)dt$$

or equivalently, after performing simple calculations and adding integrals with similar limits:

$$u(x) = \frac{\boldsymbol{2x - \pi}}{\boldsymbol{\pi}} - \int_0^{x}\frac{\boldsymbol{t(x - \pi)}}{\boldsymbol{\pi}}\boldsymbol{u(t)dt} - \int_x^{\pi}\frac{\boldsymbol{x(t - \pi)}}{\boldsymbol{\pi}}\boldsymbol{u(t)dt}$$

Consequently, the desired Fredholm integral equation of the second kind is given by

$$u(x) = \frac{\boldsymbol{2x - \pi}}{\boldsymbol{\pi}} - \int_{\mathbf{0}}^{\boldsymbol{\pi}}\boldsymbol{K(x, t)u(t)dt}$$

where the kernel $K(x, t)$ is defined by:

$$\boldsymbol{K(x, t)} = \begin{cases} \dfrac{\boldsymbol{t(x - \pi)}}{\boldsymbol{\pi}} & \boldsymbol{, for\ 0 \le t \le x} \\ \dfrac{\boldsymbol{x(t - \pi)}}{\boldsymbol{\pi}} & \boldsymbol{, for\ x \le t \le \pi} \end{cases}$$

Exercises 2.4.

Derive the equivalent Fredholm integral equation for the following boundary value problems:

$y'' + 4y = \sin x$, $0 < x < 1$, $y(0) = y(1) = 0$