Chapter Three: Fredholm Integral Equations

3.1 Introduction

In this chapter, we shall be concerned with the nonhomogeneous Fredholm integral equations of the second kind of the form**:**

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \quad , a \leq x \leq b \tag{3.1}$$

where $K(x,t)$ is the kernel of the integral equation, and λ is a parameter. A considerable amount of discussion will be directed toward the various methods and techniques that are used for solving this type of equation starting with the most recent methods that proved to be highly reliable and accurate. To do this we will naturally focus our study on the *degenerate* or *separable* kernels all through this chapter. The standard form of the *degenerate* or *separable* kernel is given by**:**

$$K(x,t) = \sum_{j=1}^{n} g_j(x)h_j(t) \tag{3.2}$$

The expressions $x - t$, $x + t$, xt, $x^2 - 3xt + t^2$, etc. are examples of separable kernels. For other well-behaved non-separable kernels, we can convert them to separable in the form (3.2) simply by expanding these kernels using Taylor's expansion.

Definition (3.1)

The kernel $K(x, t)$ is defined to be ***square integrable*** in both x and t in the square $a \leq x \leq b$, $a \leq t \leq b$ if the following *regularity condition***:**

$$\int_a^b \int_a^b K(x,t)dx\, dt < \infty \tag{3.3}$$

is satisfied.

This condition gives rise to the development of the solution of the Fredholm integral equation (3.1). It is also convenient to state, without proof, the so-called *Fredholm Alternative Theorem* that relates the solutions of homogeneous and nonhomogeneous Fredholm integral equations.

3.1.1 Fredholm Alternative Theorem

The nonhomogeneous Fredholm integral equation (3.1) has one and only one solution if the only solution to the homogeneous Fredholm integral equation**:**

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt \tag{3.4}$$

is the trivial solution $u(x) = 0$.

We end this section by introducing the necessary condition that will guarantee a unique solution to the integral equation (3.11) in the interval of discussion. Considering (3.2), if the kernel $K(x, t)$ is real, continuous, and bounded in the square $a \leq x \leq b$ and $a \leq t \leq b$, i.e. if**:**

$$|K(x,t)| \leq M \quad , a \leq x \leq b \text{ and } a \leq t \leq b \tag{3.5}$$

and if $f(x) \neq 0$, and continuous in $a \leq x \leq b$, then the necessary condition that will guarantee that (3.1) has only a unique solution is given by**:**

$$|\lambda| M(b-a) < 1 \qquad (3.6)$$

It is important to note that a continuous solution to Fredholm integral equation may exist, even though the condition (3.6) is not satisfied. This may be seen by considering the equation:

$$u(x) = -4 + \int_0^1 (2x + 3t) u(t) dt \qquad (3.7)$$

In this example, $\lambda = 1$, $|K(x, t)| \leq 5$ and $(b - a) = 1$; therefore :

$$|\lambda| M(b-a) = 5 \nless 1 \qquad (3.8)$$

Accordingly, the necessary condition (3.6) fails to hold, but the integral equation (3.7) has an exact solution given by:

$$u(x) = 4x \qquad (3.9)$$

and this can be justified through direct substitution.

In the following, we will discuss several methods that handle successfully the Fredholm integral equations of the second kind.

3.2 The Adomian Decomposition Method

Adomian developed the so-called Adomian decomposition method or simply the ***decomposition method*** **(ADM).** The method proved to be reliable and effective for a wide class of equations, differential and integral equations, and linear and nonlinear models. The method was applied mostly to ordinary and partial differential equations and was rarely used for integral equations.

In the decomposition method, we usually express the solution $u(x)$ of the integral equation (3.1) in a series form defined by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (3.10)$$

Substituting the decomposition (3.10) into both sides of (3.1) yields:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t) \left(\sum_{n=0}^{\infty} u_n(t)\right) dt \qquad (3.11)$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_a^b K(x,t) u_0(t) dt + \lambda \int_a^b K(x,t) u_1(t) dt + \lambda \int_a^b K(x,t) u_2(t) dt + \cdots \qquad (3.12)$$

The components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... of the unknown function $u(x)$ are completely determined in a recurrent manner, if we set:

$$u_0(x) = f(x) \qquad (3.13)$$

$$u_1(x) = \lambda \int_a^b K(x,t) u_0(t) dt \qquad (3.14)$$

$$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt \qquad (3.15)$$

$$u_3(x) = \lambda \int_a^b K(x,t)u_2(t)dt \qquad (3.16)$$

and so on. The above-discussed scheme for the determination of the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... of the solution $u(x)$ of Eq. (3.1) can be written recursively by:

$$u_0(x) = f(x) \qquad (3.17)$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt \quad ,n \geq 0 \qquad (3.18)$$

In view of (3.17) and (3.18), the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... follow immediately. With these components determined, the solution $u(x)$ of (3.1) is readily determined in a series form using the decomposition (3.10). It is important to note that the obtained series for $u(x)$ converges to the exact solution in a closed form if such a solution exists as will be seen later. However, for concrete problems, where the exact solution cannot be evaluated, a truncated series $\sum_{n=0}^{k} u_n(x)$ is usually used to approximate the solution $u(x)$ and this can be used for numerical purposes. We point out here that a few terms of the truncated series usually provide a higher accuracy level of the approximate solution if compared with the existing numerical techniques.
In the following, we discuss some examples that illustrate the decomposition method outlined above.

Example 3.1. We first consider the Fredholm integral equation of the second kind

$$u(x) = \frac{9}{10}\boldsymbol{x^2} + \int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x^2 t^2 u(t) dt} \qquad \mathbf{(3.19)}$$

It is clear that $f(x) = \frac{9}{10}\boldsymbol{x^2}$, $\lambda = 1$, . To evaluate the components $u_0(x)$, $u_1(x)$, $u_2(x)$, ... of the series solution, we use the recursive scheme (3.17) and (3.18) to find**:**

$$u_0(x) = f(x) = \frac{9}{10}\boldsymbol{x^2} \qquad \mathbf{(3.20)}$$

$$u_1(x) = \lambda \int_a^b K(x,t)u_0(t)dt = \int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x^2 t^2}\left(\frac{9}{10}\boldsymbol{t^2}\right)\boldsymbol{dt} = \int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{9}}{\mathbf{20}}\boldsymbol{x^2 t^4 dt} = \frac{\mathbf{9}}{\mathbf{100}}\boldsymbol{x^2} \quad \mathbf{(3.21)}$$

$$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt = \int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x^2 t^2}\left(\frac{9}{100}\boldsymbol{t^2}\right)\boldsymbol{dt} = \int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{9}}{\mathbf{200}}\boldsymbol{x^2 t^4 dt} = \frac{\mathbf{9}}{\mathbf{1000}}\boldsymbol{x^2} \mathbf{(3.22)}$$

and so on. Noting that:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \qquad \mathbf{(3.23)}$$

We can easily obtain the solution in a series form given by:

$$u(x) = \frac{9}{10}\boldsymbol{x^2} + \frac{\mathbf{9}}{\mathbf{100}}\boldsymbol{x^2} + \frac{\mathbf{9}}{\mathbf{1000}}\boldsymbol{x^2} + \cdots \qquad \mathbf{(3.24)}$$

so that the solution of (3.19) in a closed form:

$$u(x) = \boldsymbol{x^2} \qquad \mathbf{(3.25)}$$

follows immediately upon using the formula for the sum of the infinite geometric series.

Example 3.2. Consider the Fredholm integral equation:

$$u(x) = cosx + 2x + \int_0^{\pi} xtu(t)dt \qquad \textbf{(3.26)}$$

Proceeding as in example 3.1, we set:

$$u_0(x) = cosx + 2x \qquad \textbf{(3.27)}$$

$$u_1(x) = \int_0^{\pi} xt(cost + 2t)dt = \left(-2 + \frac{2}{3}\pi^3\right)x \qquad \textbf{(3.28)}$$

$$u_2(x) = \int_0^{\pi} xt\left(-2 + \frac{2}{3}\pi^3\right)tdt = \left(-\frac{2}{3}\pi^3 + \frac{2}{9}\pi^6\right)x \qquad \textbf{(3.29)}$$

Consequently, the solution of (3.26) in a series form is given by

$$u(x) = cosx + 2x + \left(-2 + \frac{2}{3}\pi^3\right)x + \left(-\frac{2}{3}\pi^3 + \frac{2}{9}\pi^6\right)x + \left(-\frac{2}{9}\pi^6 + \frac{2}{27}\pi^9\right)x + \cdots \qquad \textbf{(3.30)}$$

and in a closed form:

$$u(x) = cosx \qquad \textbf{(3.31)}$$

Example 3.3. We consider here the Fredholm integral equation**:**

$$u(x) = e^x - 1 + \int_0^1 tu(t)dt \qquad \textbf{(3.32)}$$

Applying the decomposition technique as discussed before, we find**:**

$$u_0(x) = e^x - 1 \qquad \textbf{(3.33)}$$

$$u_1(x) = \int_0^1 t(e^t - 1)dt = \frac{1}{2} \qquad \textbf{(3.34)}$$

$$u_2(x) = \int_0^1 \frac{1}{2}tdt = \frac{1}{4} \qquad \textbf{(3.35)}$$

The determination of the components (3.33)-(3.35) yields the solution of the equation (3.32) in a series form given by**:**

$$u(x) = e^x - 1 + \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{4} + \cdots) \qquad \textbf{(3.36)}$$

where we can easily obtain the solution in a closed form given by**:**

$$u(x) = e^x \qquad \textbf{(3.37)}$$

Example 3.4. Solve the following Fredholm integral equation**:**

$$u(x) = 1 + \frac{1}{2}\int_0^{\frac{\pi}{4}} sec^2(x)u(t)dt \qquad \textbf{(3.38)}$$

Applying the decomposition technique as discussed before, we find**:**

$$u_0(x) = 1 \qquad \textbf{(3.39)}$$

$$u_1(x) = \frac{1}{2}\int_0^{\frac{\pi}{4}} sec^2(x)dt = \frac{\pi}{8}sec^2(x) \qquad \textbf{(3.40)}$$

$$u_2(x) = \frac{1}{2}\int_0^{\frac{\pi}{4}} sec^2(x)\left(\frac{\pi}{8}sec^2(t)\right)dt = \frac{\pi}{16}sec^2(x) \qquad \textbf{(3.41)}$$

The determination of the components (3.39)-(3.41) yields the solution of the equation (3.38) in a series form given by**:**

$$u(x) = 1 + \frac{\pi}{8}sec^2(x) + \frac{\pi}{16}sec^2(x) + \frac{\pi}{32}sec^2(x) + \cdots \qquad \textbf{(3.42)}$$

where we can easily obtain the solution in a closed form given by**:**

$$u(x) = 1 + \frac{\pi}{4}sec^2(x) \qquad \textbf{(3.43)}$$

Exercises 3.1. Solve the following Fredholm integral equations by using the *Adomian* *decomposition method*:

1. $u(x) = sinx - x + \int_0^{\frac{\pi}{2}} xtu(t)dt$
2. $u(x) = e^{x+2} - 2\int_0^1 e^{x+t}u(t)dt$
3. $u(x) = xsinx - \frac{1}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} u(t)dt$

3.3. The Modified Decomposition Method

As stated before, the Adomian decomposition method provides the solutions in an infinite series of components. The components u_j, $j \geq 0$ are easily computed if the inhomogeneous term $f(x)$ in the Fredholm integral equation:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$

consists of a polynomial of one or two terms. However, if the function $f(x)$ consists of a combination of two or more polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of the components u_j, $j \geq 0$ requires more work.

The modified decomposition method depends mainly on splitting the function $f(x)$ into two parts, therefore it cannot be used if the $f(x)$ consists of only one term. The modified decomposition method will be briefly outlined here,

The standard Adomian decomposition method employs the recurrence relation:

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt \quad , n \geq 0 \quad \textbf{(3.44)}$$

where the solution $u(x)$ is expressed by an infinite sum of components defined by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad \textbf{(3.45)}$$

The modified decomposition method presents a slight variation to the recurrence relation (3.44) to determine the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set:

$$f(x) = f_1(x) + f_2(x) \qquad (3.46)$$

Because of (3.46), we introduce a qualitative change in the formation of the recurrence relation (3.44). The modified decomposition method identifies the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ that exists in the standard recurrence relation. The modified decomposition method admits the use of the modified recurrence relation:

$$u_0(x) = f_1(x)$$
$$u_1(x) = f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt$$
$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt \quad , n \geq 1 \qquad \textbf{(3.47)}$$

Example 3.5. Solve the Fredholm integral equation by using the modified decomposition method.

$$u(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4) + \int_0^1 tu(t)dt$$

We first decompose $f(x)$ given by

$$f(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4)$$

into two parts, namely

$$f_1(x) = 3x + e^{4x} \ , \quad f_2(x) = -\frac{1}{16}(17 + 3e^4)$$

We next use the modified recurrence formula (3.47) to obtain:

$$u_0(x) = 3x + e^{4x}$$
$$u_1(x) = -\frac{1}{16}(17 + 3e^4) + \int_0^1 t(3t + e^{4t})dt = 0$$
$$u_{n+1}(x) = \int_0^{b1} tu_n(t)dt = 0, n \geq 1$$

It is obvious that each component of $u_j, j \geq 1$ is zero. This in turn gives the exact solution by: $u(x) = 3x + e^{4x}$

Example 3.6. Solve the Fredholm integral equation by using the modified decomposition method.

$$u(x) = \frac{1}{1 + x^2} - 2sinh\frac{\pi}{4} + \int_{-1}^1 e^{\tan^{-1} t}u(t)dt$$

We first decompose $f(x)$ given by

$$f(x) = \frac{1}{1 + x^2} - 2sinh\frac{\pi}{4}$$

into two parts, namely

$$f_1(x) = \frac{1}{1 + x^2} \ , \quad f_2(x) = -2sinh\frac{\pi}{4}$$

We next use the modified recurrence formula (3.47) to obtain:

$$u_0(x) = \frac{1}{1 + x^2}$$
$$u_1(x) = -2sinh\frac{\pi}{4} + \int_{-1}^1 e^{\tan^{-1} t}\left(\frac{1}{1+t^2}\right)dt = 0$$
$$u_{n+1}(x) = \int_0^{b1} e^{\tan^{-1} t}u_n(t)dt = 0, n \geq 1$$

It is obvious that each component of $u_j, j \geq 1$ is zero. This in turn gives the exact solution by: $u(x) = \frac{1}{1+x^2}$

Exercises 3.2. Use the *modified decomposition method* to solve the following Fredholm integral equations:

1. $u(x) = sinx - x + x\int_0^{\frac{\pi}{2}} tu(t)$
2. $u(x) = e^x + 12x^2 + (3 + e^1)x - 4 - \int_0^1 (x - t)u(t)dt$

3.4 The Successive Approximations Method

The *successive approximations method* or the *Picard iteration method* provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess as $u_0(x)$, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence in relation to determining the other approximations. The most commonly used values for the zeroth approximations are 0, 1, or x. Of course, other real values can be selected as well. Given Fredholm integral equation of the second kind: $u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt$

where $u(x)$ is the unknown function to be determined, $K(x, t)$ is the kernel, and λ is a parameter. The successive approximations method introduces the recurrence relation:

$u_0(x)$ = any selective real-valued function,

$$u_{n+1}(x) = f(x) + \lambda\int_a^b K(x,t)u_n(t)dt \quad ,n \geq 0 \qquad \textbf{(3.48)}$$

the solution is determined by using the limit:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) \qquad \textbf{(3.49)}$$

3.4.1 The difference between The successive approximations method and the Adomian method can be summarized as follows:

1. The successive approximations method gives successive approximations of the solution $u(x)$, whereas the Adomian method gives successive components of the solution $u(x)$.
2. The successive approximations method admits the use of a selective real-valued function for the zeroth approximation u_0, whereas the Adomian decomposition method assigns all terms that are not inside the integral sign for the zeroth component $u_0(x)$. Recall that this assignment was modified when using the modified decomposition method.
3. The successive approximations method gives the exact solution, if it exists, by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)$$

However, the Adomian decomposition method gives the solution as infinite series of components by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

This series solution converges rapidly to the exact solution if such a solution exists.

The successive approximations method or iteration method will be illustrated by studying the following examples.

Example 3.7. Solve the Fredholm integral equation by using the successive approximations method:

$$u(x) = x + e^x - \int_0^1 xtu(t)dt$$

For the zeroth approximation *u*0(*x*), we can select:

$$u_0(x) = 0$$

The method of successive approximations admits the use of the iteration formula:

$$u_{n+1}(x) = x + e^x - \int_0^1 xtu_n(t)dt \quad , n \geq 0$$

Therefore, we obtain:

$$u_1(x) = x + e^x$$

$$u_2(x) = x + e^x - \int_0^1 xt(t + e^t)dt = e^x - \frac{1}{3}x$$

$$u_3(x) = x + e^x - \int_0^1 xt\left(e^t - \frac{1}{3}t\right)dt = e^x + \frac{1}{9}x$$

$$\vdots$$

$$u_{n+1}(x) = e^x + \frac{(-1)^n}{3^n}x$$

Consequently, the solution *u*(*x*) is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty}\left(e^x + \frac{(-1)^n}{3^n}x\right) = e^x$$

Example 3.8. Solve the Fredholm integral equation by using the successive approximations method:

$$u(x) = x + \lambda \int_{-1}^1 xtu(t)dt$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = x$$

The method of successive approximations admits the use of the iteration formula:

$$u_{n+1}(x) = x + \lambda \int_{-1}^{1} xtu_n(t)dt \quad , n \geq 0$$

Therefore, we obtain:

$$u_1(x) = x + \lambda \int_{-1}^{1} xt^2 dt \;= x + \frac{2}{3}\lambda x$$

$$u_2(x) = x + \lambda \int_{-1}^{1} xt\left(t + \frac{2}{3}\lambda t\right)dt \;= x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x$$

$$u_3(x) = x + \lambda \int_{-1}^{1} xt\left(t + \frac{2}{3}\lambda xt + \left(\frac{2}{3}\right)^2 \lambda^2 t\right)dt \;= x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x + \left(\frac{2}{3}\right)^3 \lambda^3 x$$

$$\vdots$$

$$u_{n+1}(x) = x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x + \left(\frac{2}{3}\right)^3 \lambda^3 x + \cdots + \left(\frac{2}{3}\right)^{n+1} \lambda^{n+1} x$$

The solution $u(x)$ is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)$$

$$= \lim_{n\to\infty}\left(x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x + \left(\frac{2}{3}\right)^3 \lambda^3 x + \cdots + \left(\frac{2}{3}\right)^{n+1} \lambda^{n+1} x\right)$$

$$= \frac{3x}{3 - 2\lambda} \quad , 0 < \lambda < \frac{3}{2}$$

obtained upon using the infinite geometric series for the right side of the above equation.

Example 3.9. Solve the Fredholm integral equation by using the successive approximations method:

$$u(x) = sinx + sinx \int_{0}^{\frac{\pi}{2}} costu(t)dt$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = 0$$

We next use the iteration formula

$$u_{n+1}(x) = sinx + sinx \int_{0}^{\frac{\pi}{2}} costu_n(t)dt \quad , n \geq 0$$

Therefore, we obtain:

$$u_1(x) = sinx \quad , \quad u_2(x) = \frac{3}{2} sinx$$
$$u_3(x) = \frac{7}{4} sinx \quad . \quad u_4(x) = \frac{15}{8} sinx$$
$$\vdots$$
$$u_{n+1}(x) = \frac{2^{n+1} - 1}{2^n} sinx = \left(2 - \frac{1}{2^n}\right) sinx$$

The solution $u(x)$ is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) sinx = 2sinx$$

Exercises 3.3. Use the *successive approximations method* to solve the following Fredholm integral equations:

1. $u(x) = 1 + x^3 + \lambda \int_{-1}^{1} xtu(t)dt$
2. $u(x) = x + \sec^2 x - \int_0^{\frac{\pi}{4}} xu(t)dt$

3.5 The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain:

$$u(x) = \sum_{n=0}^{\infty} \frac{u^n(b)}{n!} (x - b)^n \qquad (3.50)$$

converges to $u(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (3.51)$$

The series solution method that stems mainly from the Taylor series for analytic functions, will be used for solving Fredholm integral equations. We will assume that the solution $u(x)$ of the Fredholm integral equations:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (3.52)$$

is analytic, and therefore possesses a Taylor series of the form given in (3.52), where the coefficients a_n will be determined recurrently. Substituting (3.51) into both sides of (3.52) gives:

$$\sum_{n=0}^{\infty} a_n x^n = T\big(f(x)\big) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt \qquad (3.53)$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T\big(f(x)\big) + \lambda \int_a^b K(x,t)(a_0 + a_1 t + a_2 t^2 + \cdots)dt \qquad (3.54)$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (3.52) will be converted to a traditional integral in (3.53) or (3.54) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geq 0$ will be integrated. Notice that because we are seeking a series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (3.53) or (3.54) and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in a_j, $j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients a_j, $j \geq 0$. Having determined the coefficients a_j, $j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (3.51). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher the accuracy level we achieve. It is worth noting that using the series solution method for solving Fredholm integral equations gives exact solutions if the solution $u(x)$ is a polynomial. However, if the solution is any other elementary function such as $\sin x$, e^x, etc, the series method gives the exact solution after rounding a few of the coefficients a_j, $j \geq 0$. This will be illustrated by studying the following examples.

Example 3.10. Solve the Fredholm integral equation by using the series solution method:

$$u(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)u(t)dt$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

leads to,

$$\sum_{n=0}^{\infty} a_n x^n = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integral on the right side gives:

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^2 + \cdots \\ = 1 + \left(2 + \frac{2}{3}a_1 + \frac{2}{5}a_3 + \frac{2}{7}a_5 + \frac{2}{9}a_7\right)x \\ + \left(1 + \frac{2}{3}a_0 + \frac{2}{5}a_2 + \frac{2}{7}a_4 + \frac{2}{9}a_6 + \frac{2}{11}a_8\right)x^2 \end{aligned}$$

Equating the coefficients of like powers of x on both sides gives:

$$a_0 = 1, a_1 = 6, a_2 = \frac{25}{9} \ , a_n = 0, n \geq 3$$

The exact solution is given by:

$$u(x) = 1 + 6x + \frac{25}{9}x^2$$

Example 3.11. Solve the Fredholm integral equation by using the series solution method:

$$u(x) = x^2 - x^3 + \int_0^1 (1 + xt)u(t)dt$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

leads to,

$$\sum_{n=0}^{\infty} a_n x^n = x^2 - x^3 + \int_0^1 (1 + xt)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integral on the right side, and equating the coefficients of like powers of x on both sides of the resulting equation we find

$a_0 = \frac{-29}{60}$, $a_1 = \frac{-1}{6}$, $a_2 = 1$, $a_3 = -1$, $a_n = 0$, $n \geq 4$.

Consequently, the exact solution is given by:

$$u(x) = \frac{-29}{60} - \frac{1}{6}x + x^2 - x^3$$

Example 3.12. Solve the Fredholm integral equation by using the series solution method:

$$u(x) = -x^4 + \int_{-1}^1 (xt^2 - x^2t)u(t)dt$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

leads to,

$$\sum_{n=0}^{\infty} a_n x^n == -x^4 + \int_{-1}^1 (xt^2 - x^2t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integral on the right side, and equating the coefficients of like powers of x on both sides of the resulting equation, we find:

$a_0 = 0$, $a_1 = \frac{-30}{133}$, $a_2 = \frac{20}{133}$, $a_3 = 0$, $a_4 = -1$, $a_n = 0$, $n \geq 5$.

Consequently, the exact solution is given by:

$$u(x) = \frac{-30}{133}x + \frac{20}{133}x^2 - x^4$$

Example 3.13. Solve the Fredholm integral equation by using the series solution method:

$$u(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} u(t)dt$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

leads to,

$$\sum_{n=0}^{\infty} a_n x^n = -1 + \cos x + \int_0^{\frac{\pi}{2}} \left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

Evaluating the integral on the right side, and equating the coefficients of like powers of x on both sides of the resulting equation we find

$a_0 = 1$ $, a_{2j+1} = 0, a_{2j+2} = \frac{(-1)^j}{(2j)!}$ $\quad , j \geq 0$

Consequently, the exact solution is given by:

$$u(x) = cosx$$

Exercises 3.4. Use the *series solution method* to solve the following Fredholm integral equations:

1. $u(x) = 5x + \int_{-1}^{1}(1 - xt)u(t)dt$
2. $u(x) = \sec^2 x - 1 + \int_0^{\frac{\pi}{4}} u(t)dt$

3.6 The Direct Computation Method

In this section, the *direct computation method* will be applied to solve the Fredholm integral equations. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. It is important to point out that this method will be applied for the degenerate or separable kernels of the form:

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t) \qquad (3.55)$$

Examples of separable kernels are $x - t$, xt, $x^2 - t^2$, $xt^2 + x^2t$, etc.

The direct computation method can be applied as follows:

1. We first substitute (3.55) into the Fredholm integral equation of the form:

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt \qquad (3.56)$$

2. This substitution gives:

$$u(x) = f(x) + g_1(x)\int_a^b h_1(t)u(t)dt \; + g_2(x)\int_a^b h_2(t)u(t)dt \; + \cdots$$
$$+g_n(x)\int_a^b h_n(t)u(t)dt \qquad (3.57)$$

3. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (3.57) becomes:
$$u(x) = f(x) + \alpha_1 g_1(x) + \alpha_2 g_2(x) + \cdots + \alpha_n g_n(x) \qquad (3.58)$$
Where
$$\alpha_i = \int_a^b h_i(t)u(t)dt \quad 1 \le i \le n \qquad (3.59)$$
4. Substituting (3.58) into (3.59) gives a system of n algebraic equations that can be solved to determine the constants α_i , $1 \le i \le n$. Using the obtained numerical values of α_i into (3.59), the solution $u(x)$ of the Fredholm integral equation (3.56) is readily obtained.

Example 3.14 Solve the Fredholm integral equation by using the direct computation method $u(x) = 3x + 3x^2 + \frac{1}{2}\int_0^1 \boldsymbol{x^2 t u(t) dt}$ (3.60)

The kernel $K(x, t) = x^2 t$ is separable. Consequently, we rewrite (3.60) as:
$$u(x) = 3x + 3x^2 + \frac{1}{2}\boldsymbol{x^2}\int_0^1 \boldsymbol{t u(t) dt} \qquad (3.61)$$
The integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, Equation (3.61) can be rewritten as:
$$u(x) = 3x + 3x^2 + \frac{1}{2}\boldsymbol{\alpha x^2} \qquad \textbf{(3.62)}$$
Where $\boldsymbol{\alpha} = \int_0^1 \boldsymbol{t u(t) dt}$ **(3.63)**

To determine α, we substitute (3. 62) into (3.63) to obtain:
$$\boldsymbol{\alpha} = \int_0^1 \boldsymbol{t}\left(3t + 3t^2 + \frac{1}{2}\boldsymbol{\alpha t^2}\right)\boldsymbol{dt} \qquad \textbf{(3.64)}$$
Integrating the right side of (3.64) yields:
$$\boldsymbol{\alpha} = \frac{7}{4} + \frac{1}{8}\alpha$$
that gives $\boldsymbol{\alpha} = 2$

Substituting $\boldsymbol{\alpha} = 2$ into (3.62) leads to the exact solution: $u(x) = 3x + 4x^2$

Example 3.15 Solve the Fredholm integral equation by using the *direct computation method*
$$u(x) = \frac{1}{3}x + \sec x \tan x - \frac{1}{3}x\int_0^{\pi/3} u(t)dt$$

The integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, we can rewrite the above equation as:

$$u(x) = \frac{1}{3}x + \sec x \tan x - \frac{1}{3}\alpha x$$

Where $\alpha = \int_0^{\pi/3} u(t)dt = \int_0^{\pi/3} \left(\frac{1}{3}t + \sec t \tan t - \frac{1}{3}\alpha t\right) dt = 1 + \frac{1}{54}\pi^2 - \frac{1}{54}\alpha\pi^2$

that gives $\alpha = 1$. Therefore, the exact solution is: $u(x) = \sec x \tan x$

Example 3.16 Solve the Fredholm integral equation by using the *direct computation method*

$$u(x) = 11x + 10x^2 + x^3 - \int_0^1 (30xt^2 + 20x^2t)u(t)dt$$

The kernel $K(x, t) = 30xt^2 + 20x^2t$ is separable. Consequently, we rewrite the above equation as:

$$u(x) = 11x + 10x^2 + x^3 - 30x\int_0^1 t^2u(t)dt - 20x^2\int_0^1 tu(t)dt$$

Each integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, the above the equation can be rewritten as:

$u(x) = 11x + 10x^2 + x^3 - 30\alpha x - 20\beta x^2 = (11 - 30\alpha)x + (10 - 20\beta)x^2 + x^3$,

Where $\alpha = \int_0^1 t^2u(t)dt$ and $\beta = \int_0^1 tu(t)dt$

And then, we have:

$$\alpha = \int_0^1 t^2[(11 - 30\alpha)t + (10 - 20\beta)t^2 + t^3]dt = \frac{59}{12} - \frac{15}{2}\alpha - 4\beta$$

$$\beta = \int_0^1 t[(11 - 30\alpha)t + (10 - 20\beta)t^2 + t^3]dt = \frac{191}{30} - 10\alpha - 5\beta$$

Solving this system of algebraic equations gives:

$$\alpha = \frac{11}{30}, \qquad \beta = \frac{9}{20}$$

the exact solution is: $u(x) = x^2 + x^3$.

Example 3.17 Solve the Fredholm integral equation by using the *direct computation method*

$$u(x) = 4 + 45x + 26x^2 - \int_0^1 (1 + 30xt^2 + 12x^2t)u(t)dt$$

The kernel $K(x, t) = 1+30xt^2+12x^2t$ is separable. Consequently, we rewrite the above equation as:

$$u(x) = 4 + 45x + 26x^2 - \int_0^1 u(t)dt - 30x\int_0^1 t^2u(t)dt - 12x^2\int_0^1 tu(t)dt$$

Each integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, the above an equation can be rewritten as:

$$u(x) = (4 - \alpha) + (45 - 30\beta)x + (26 - 12\gamma)x^2$$

where $\alpha = \int_0^1 u(t)dt$, $\beta = \int_0^1 t^2u(t)dt$ and $\gamma = \int_0^1 tu(t)dt$.
And then, we have:

$$\alpha = \int_0^1 ((4 - \alpha) + (45 - 30\beta)t + (26 - 12\gamma)t^2)dt = \frac{211}{6} - \alpha - 15\beta - 4\gamma$$

$$\beta = \int_0^1 t^2((4 - \alpha) + (45 - 30\beta)t + (26 - 12\gamma)t^2)dt$$

$$= \frac{1067}{60} - \frac{1}{3}\alpha - \frac{15}{2}\beta - \frac{12}{5}\gamma$$

$$\gamma = \int_0^1 t((4 - \alpha) + (45 - 30\beta)t + (26 - 12\gamma)t^2)dt = \frac{47}{2} - \frac{1}{2}\alpha - 10\beta - 3\gamma$$

Solving this system of algebraic equations gives:
$\alpha = 3, \beta = \frac{43}{30}$ and $\gamma = \frac{23}{12}$, and the exact solution is : $u(x) = 1+2x+ 3x^2$

Exercises 3.5. Use the *direct computation method* to solve the following Fredholm integral equations:

1. $u(x) = 1 + 9x + 2x^2 + x^3 - \int_0^1 (20xt + 10x^2t^2)u(t)dt$
2. $u(x) = \left(\frac{2}{\sqrt{3}} - 1\right)x + \sec x \tan x - \int_0^{\pi/6} xu(t)dt$