Chapter Four: Volterra Integral Equations

Volterra integral equations arise in many scientific applications such as population dynamics, the spread of epidemics, and semiconductor devices. It was also shown in chapter two that Volterra integral equations can be derived from initial value problems. We will study Volterra integral equations of the second kind given:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \qquad (4.1)$$

The unknown function $u(x)$, which will be determined, occurs inside and outside the integral sign. The kernel $K(x, t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. In what follows we will present the methods that will be used.

4.1 The Adomian Decomposition Method

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (4.2)$$

where the components $u_n(x)$, $n \geq 0$ are to be determined recursively. The decomposition method concerns itself with finding the components u_0, u_1, u_2, . . . individually. The determination of these components can be achieved easily through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (4.2) into the Volterra integral equation (4.1) to obtain:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^x K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt \qquad (4.3)$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x)$, $j \geq 1$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

$$u_0(x) = f(x) \qquad (4.4)$$

$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt \quad , n \geq 0 \qquad (4.5)$$

Example 4.1. Solve the following Volterra integral equation:

$$u(x) = 1 - \int_0^x u(t)dt \qquad (4.6)$$

We notice that $f(x) = 1$, $\lambda = -1$, $K(x, t) = 1$. Recall that the solution $u(x)$ is assumed to have a series form given in (4.2). Substituting the decomposition series (4.2) into both sides of (4.6) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \left(\sum_{n=0}^{\infty} u_n(t) \right) dt$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x) = 1$$

$$u_1(x) = -\int_0^x u_0(t)dt == -\int_0^x 1dt = -x$$

$$u_2(x) = -\int_0^x u_1(t)dt = \int_0^x tdt = \frac{1}{2!}x^2$$

$$u_3(x) = -\int_0^x u_2(t)dt = -\int_0^x t^2dt = -\frac{1}{3!}x^3$$

$$u_4(x) = -\int_0^x u_3(t)dt = \int_0^x t^3dt = \frac{1}{4!}x^4$$

and so on. Using (4.2) gives the series solution:

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

that converges to the closed form solution:

$$u(x) = e^{-x}$$

Example 4.2. Solve the following Volterra integral equation:

$$u(x) = 1 + \int_0^x (t - x)u(t)dt \qquad (4.7)$$

We notice that $f(x) = 1$, $\lambda = 1$, $K(x, t) = t - x$. Substituting the decomposition series (4.2) into both sides of (4.7) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x (t - x) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + \int_0^x (t - x)(u_0(t) + u_1(t) + u_2(t) + \cdots)dt$$

Proceeding as before we set the following recurrence relation:

$$u_0(x) = 1$$

$$u_k(x) = \int_0^x (t-x)u_{k-1}(t)dt \quad , k \geq 1$$

that gives

$$u_0(x) = 1$$

$$u_1(x) = \int_0^x (t-x)u_0(t)dt = \int_0^x (t-x)dt = -\frac{1}{2!}x^2$$

$$u_2(x) = \int_0^x (t-x)u_1(t)dt = \int_0^x (t-x)\left(-\frac{1}{2!}t^2\right)dt = \frac{1}{4!}x^4$$

$$u_3(x) = \int_0^x (t-x)u_2(t)dt = \int_0^x (t-x)\left(\frac{1}{4!}t^4\right)dt = -\frac{1}{6!}x^6$$

$$u_4(x) = \int_0^x (t-x)u_3(t)dt = \int_0^x (t-x)\left(-\frac{1}{6!}t^6\right)dt = \frac{1}{8!}x^8$$

and so on. The solution in a series form is given by:

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

and in a closed form by:

$$u(x) = \cos x$$

obtained upon using the Taylor expansion for cos x.

Example 4.3. Solve the following Volterra integral equation:

$$u(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)u(t)dt \qquad (4.8)$$

We notice that $f(x) = 1 - x - \frac{1}{2}x^2$, λ = -1 , $K(x, t) = t - x$. Substituting the decomposition series (4.2) into both sides of (4.8) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$= 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)(u_0(t) + u_1(t) + u_2(t) + \cdots)dt$$

This allows us to set the following recurrence relation:

$$u_0(x) = 1 - x - \frac{1}{2}x^2$$

$$u_k(x) = \int_0^x (t-x)u_{k-1}(t)dt \quad ,k \geq 1$$

that gives:

$$u_0(x) = 1 - x - \frac{1}{2}x^2$$

$$u_1(x) = \int_0^x (t-x)u_0(t)dt = \int_0^x (t-x)\left(1 - t - \frac{1}{2}t^2\right)dt = \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4$$

$$u_2(x) = \int_0^x (t-x)u_1(t)dt = \int_0^x (t-x)\left(\frac{1}{2!}t^2 - \frac{1}{3!}t^3 - \frac{1}{4!}t^4\right)dt$$

$$= \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6$$

$$u_3(x) = \int_0^x (t-x)u_2(t)dt = \int_0^x (t-x)\left(\frac{1}{4!}t^4 - \frac{1}{5!}t^5 - \frac{1}{6!}t^6\right)dt$$

$$= \frac{1}{6!}x^6 - \frac{1}{7!}x^7 - \frac{1}{8!}x^8$$

and so on. The solution in a series form is given by:

$$u(x) = 1 - x - \frac{1}{2}x^2 + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 + \cdots$$

$$= 1 - (x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots)$$

and in a closed form by:

$$u(x) = 1 - \sinh x$$

obtained upon using the Taylor expansion for $\sinh x$.

Example 4.4. Solve the following Volterra integral equation:

$$u(x) = 5x^3 - x^5 + \int_0^x tu(t)dt \qquad (4.9)$$

We notice that $f(x) = 5x^3 - x^5$, $\lambda = 1$, $K(x, t) = t$. Substituting the decomposition series (4.2) into both sides of (4.9) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 5x^3 - x^5 + \int_0^x t\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 5x^3 - x^5 - \int_0^x t(u_0(t) + u_1(t) + u_2(t) + \cdots)dt$$

This allows us to set the following recurrence relation:

$$u_0(x) = 5x^3 - x^5$$

$$u_k(x) = \int_0^x t u_{k-1}(t)dt \quad , k \geq 1$$

that gives:

$$u_0(x) = 5x^3 - x^5$$

$$u_1(x) = \int_0^x t u_0(t)dt = \int_0^x t(5t^3 - t^5)dt = x^5 - \frac{1}{7}x^7$$

$$u_2(x) = \int_0^x t u_1(t)dt = \int_0^x t\left(t^5 - \frac{1}{7}t^7\right)dt = \frac{1}{7}x^7 - \frac{1}{63}x^9$$

$$u_3(x) = \int_0^x t u_2(t)dt = \int_0^x t\left(\frac{1}{7}t^7 - \frac{1}{63}t^9\right)dt = \frac{1}{63}x^9 - \frac{1}{693}x^{11}$$

The solution in a series form is given by:

$$u(x) = (5x^3 - x^5) + \left(x^5 - \frac{1}{7}x^7\right) + \left(\frac{1}{7}x^7 - \frac{1}{63}x^9\right) + \left(\frac{1}{63}x^9 - \frac{1}{693}x^{11}\right) + \cdots$$

We can easily notice the appearance of identical terms with opposite signs. Such terms are called **noise terms** which will be discussed later. Canceling the identical terms with opposite signs gives the exact solution:

$$u(x) = 5x^3$$

Example 4.5. We finally solve the Volterra integral equation:

$$u(x) = 2 + \tfrac{1}{3}\int_0^x x t^3 u(t)dt \qquad (4.10)$$

Proceeding as before, we set the recurrence relation:

$$u_0(x) = 2$$

$$u_k(x) = \frac{1}{3}\int_0^x x t^3 u_{k-1}(t)dt \quad , k \geq 1$$

This in turn gives:

$$u_0(x) = 2$$

$$u_1(x) = \frac{1}{3}\int_0^x x t^3 u_0(t)dt = \frac{2}{3}\int_0^x x t^3 dt = \frac{1}{6}x^5$$

$$u_2(x) = \frac{1}{3}\int_0^x x t^3 u_1(t)dt = \frac{1}{3}\int_0^x x t^3\left(\frac{1}{6}t^5\right)dt = \frac{1}{162}x^{10}$$

$$u_3(x) = \frac{1}{3}\int_0^x x t^3 u_2(t)dt = \frac{1}{3}\int_0^x x t^3\left(\frac{1}{162}t^{10}\right)dt = \frac{1}{6804}x^{15}$$

and so on. The solution in a series form is given by:

$$u(x) = 2 + \frac{1}{6}x^5 + \frac{1}{162}x^{10} + \frac{1}{6804}x^{15} + \cdots$$

It seems that an exact solution is not obtainable. The obtained series solution can be used for numerical purposes. The more components that we determine the higher the accuracy level that we can achieve.

Exercises 4.1. solve the following Volterra integral equations by using the *Adomian decomposition method*:

1. $u(x) = 6x - 3x^2 + \int_0^x u(t)dt$
2. $u(x) = 1 + x + \int_0^x (x-t)u(t)dt$
3. $u(x) = 1 + x^2 + \int_0^x (x-t+1)^2 u(t)dt$

4.2 The Modified Decomposition Method

To give a clear description of the technique, we recall that the standard Adomian decomposition method admits the use of the recurrence relation:

$$u_0(x) = f(x)$$
$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt \quad , n \geq 0 \qquad (4.11)$$

where the solution $u(x)$ is expressed by an infinite sum of components defined before by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (4.12)$$

In view of (4.11), the components $u_n(x)$, $n \geq 0$ can be easily evaluated. The modified decomposition method introduces a slight variation to the recurrence relation (4.11) that will lead to the determination of the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x) \qquad (4.13)$$

In view of (4.13), we introduce a qualitative change in the formation of the recurrence relation (4.11). To minimize the size of calculations, we identify the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ among other terms. In other words, the modified decomposition method introduces the modified recurrence relation:

$$u_0(x) = f_1(x)$$
$$u_1(x) = f_2(x) + \lambda \int_a^x K(x,t)u_0(t)dt$$
$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt \quad , n \geq 1 \qquad (4.14)$$

Example 4.6. Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = \sin x + (e^1 - e^{\cos x}) - \int_0^x e^{\cos t}u(t)dt \qquad (4.15)$$

We first split $f(x)$ given by:

$$f(x) = \sin x + (e^1 - e^{\cos x})$$

into two parts, namely

$$f_1(x) = \sin x \quad and \quad f_2(x) = (e^1 - e^{\cos x})$$

We next use the modified recurrence formula (4.14) to obtain:

$$u_0(x) = f_1(x) = \sin x$$

$$u_1(x) = (e^1 - e^{\cos x}) - \int_0^x e^{\cos t} u_0(t)dt = (e^1 - e^{\cos x}) - \int_0^x e^{\cos t}(\sin t)dt = 0$$

$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt = 0, n \geq 1$$

It is obvious that each component of u_j, $j \geq 1$ is zero. This in turn gives the exact solution by:

$$u(x) = \sin x$$

Example 4.7. Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = \sec x \tan x + (e^{\sec x} - e^1) - \int_0^x e^{\sec t} u(t)dt \quad , x < \frac{\pi}{2} \qquad (4.16)$$

Proceeding as before we split $f(x)$ into two parts:

$$f_1(x) = \sec x \tan x \quad and \quad f_2(x) = (e^{\sec x} - e^1)$$

We next use the modified recurrence formula (4.14) to obtain:

$$u_0(x) = f_1(x) = \sec x \tan x$$

$$u_1(x) = (e^{\sec x} - e^1) - \int_0^x e^{\sec t} u_0(t)dt = (e^{\sec x} - e^1) - \int_0^x e^{\sec t}(\sec t \tan t)dt = 0$$

$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt = 0, n \geq 1$$

It is obvious that each component of u_j, $j \geq 1$ is zero. This in turn gives the exact solution by:

$$u(x) = \sec x \tan x$$

Example 4.8. Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = 1 + x^2 + \cos x - x - \frac{1}{3}x^3 - \sin x + \int_0^x u(t)dt \qquad (4.17)$$

Proceeding as before we split $f(x)$ into two parts:

$$f_1(x) = 1 + x^2 + \cos x \quad and \quad f_2(x) = -\left(x + \frac{1}{3}x^3 + \sin x\right)$$

We next use the modified recurrence formula (4.14) to obtain:

$$u_0(x) = f_1(x) = 1 + x^2 + \cos x$$

$$u_1(x) = -\left(x + \frac{1}{3}x^3 + \sin x\right) + \int_0^x u_0(t)dt$$
$$= -\left(x + \frac{1}{3}x^3 + \sin x\right) + \int_0^x (1 + t^2 + \cos t)dt = 0$$

$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt = 0, n \geq 1$$

It is obvious that each component of u_j, $j \geq 1$ is zero. This in turn gives the exact solution by:

$$u(x) = 1 + x^2 + \cos x$$

Exercises 4.2. Use the *modified decomposition method* to solve the following Volterra integral equations:

1. $u(x) = \sinh x + \cosh x - 1 - \int_0^x u(t)dt$
2. $u(x) = 2x + \left(1 - e^{-x^2}\right) - \int_0^x e^{-x^2+t^2} u(t)dt$

4.3 The Successive Approximations Method

The *successive approximations method* also called the *Picard iteration method* provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations
to the solution by starting with an initial guess, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence in relation to determining the other approximations. The successive approximations method introduces the recurrence relation:

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)u_{n-1}(t)dt \qquad , n \geq 1 \qquad (4.18)$$

We always start with an initial guess for $u_0(x)$, mostly we select 0, 1, x for $u_0(x)$, and by using (4.18), several successive approximations u_k , $k \geq 1$ will be determined as:

$$u_1(x) = f(x) + \lambda \int_a^x K(x,t)u_0(t)dt$$
$$u_2(x) = f(x) + \lambda \int_a^x K(x,t)u_1(t)dt$$
$$\vdots$$
$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)u_{n-1}(t)dt$$

The successive approximations method or the Picard iteration method will be illustrated by the following examples.

Example 4.9. Solve the Volterra integral equation by using the successive approximations method:

$$u(x) = 1 - \int_0^x (x-t)u(t)dt \qquad (4.19)$$

The method of successive approximations admits the use of the iteration formula:

$$u_n(x) = 1 - \int_0^x (x-t)u_{n-1}(t)dt \quad , n \geq 1 \qquad (4.20)$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = 1 \qquad (4.21)$$

Substituting (4.21) into (4.20), we obtain:

$$u_1(x) = 1 - \int_0^x (x-t)u_0(t)dt = 1 - \int_0^x (x-t)dt = 1 - \frac{1}{2!}x^2$$

$$u_2(x) = 1 - \int_0^x (x-t)u_1(t)dt = 1 - \int_0^x (x-t)\left(1 - \frac{1}{2!}t^2\right)dt = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

$$u_3(x) = 1 - \int_0^x (x-t)u_2(t)dt = 1 - \int_0^x (x-t)\left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4\right)dt$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

Consequently, we obtain:

$$u_{n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$$

The solution $u(x)$ of (4.19):

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \cos x$$

Example 4.10. Solve the Volterra integral equation by using the successive approximations method:

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt \qquad (4.22)$$

The method of successive approximations admits the use of the iteration formula:

$$u_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_{n-1}(t)dt \quad , n \geq 1 \qquad (4.23)$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = 0 \qquad (4.24)$$

Substituting (4.24) into (4.23), we obtain:

$$u_1(x) = 1 + x + \frac{1}{2!}x^2$$

$$u_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_1(t)dt = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$\vdots$$

and so on. The solution $u(x)$ of (4.22) is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = e^x$$

Example 4.11. Solve the Volterra integral equation by using the successive approximations method:

$$u(x) = -1 + e^x + \tfrac{1}{2}x^2e^x - \tfrac{1}{2}\int_0^x tu(t)dt \qquad (4.25)$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = 0 \qquad (4.26)$$

We next use the iteration formula:

$$u_{n+1}(x) = -1 + e^x + \tfrac{1}{2}x^2e^x - \tfrac{1}{2}\int_0^x tu_n(t)dt \ \ , \ n \geq 0 \qquad (4.27)$$

Substituting (4.26) into (4.27), we obtain:

$$u_1(x) = -1 + e^x + \frac{1}{2}x^2e^x$$

$$u_2(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x tu_1(t)dt$$

$$= -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x t\left(-1 + e^t + \frac{1}{2}t^2e^t\right)dt$$

$$= -3 + \frac{1}{4}x^2 + e^x\left(3 - 2x + \frac{5}{4}x^2 - \frac{1}{4}x^3\right)$$

$$u_3(x) = x\left(1 + x + \frac{1}{2!}x^2\right)$$

Example 4.12. Solve the Volterra integral equation by using the successive approximations method:

$$u(x) = 1 - x\sin x + x\cos x + \int_0^x tu(t)dt \qquad (4.28)$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = x \qquad (4.29)$$

We next use the iteration formula:

$$u_{n+1}(x) = 1 - x\sin x + x\cos x + \int_0^x tu_n(t)dt\,, \ n \geq 0 \qquad (4.30)$$

Substituting (4.29) into (4.30), we obtain:

$$u_1(x) = 1 + \frac{1}{3}x^3 - x\sin x + x\cos x$$

$$u_2(x) = 3 + \frac{1}{2}x^2 + \frac{1}{15}x^3 - (2 + 3x - x^2)\sin x - (2 - 3x - x^2)\cos x$$

$$u_3(x) = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7\right) + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6\right)$$
$$\vdots$$
$$u_{n+1}(x) = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{(2k+1)!} + \sum_{k=0}^{n}(-1)^k \frac{x^{2k}}{(2k)!}$$

Notice that we used the Taylor expansion for sin x and cos x to determine the approximations $u_3(x)$, $u_4(x)$, ... The solution $u(x)$ of (4.28) is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \sin x + \cos x$$

Exercises 4.3.

Use the *successive approximations method* to solve the following Volterra integral equations:

1. $u(x) = x + \int_0^x (x-t)u(t)dt$
2. $u(x) = x\cosh x - \int_0^x tu(t)dt$
3. $u(x) = 1 - x\sin x + \int_0^x tu(t)dt$
4. $u(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \int_0^x u(t)dt$

4.4 The Laplace Transform Method

The *Laplace transform method* is a powerful technique that can be used for solving initial value problems and integral equations as well. The details and properties of the Laplace method can be found in ordinary differential equations texts.

Before we start applying this method, we summarize some of the concepts presented in Section 1.3. In the convolution theorem for the Laplace transform, it was stated that if the kernel $K(x, t)$ of the integral equation:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt$$

depends on the difference $x-t$, then it is called a ***difference kernel***. Examples of the difference kernel are e^{x-t}, $\cos(x-t)$, and $x-t$. The integral equation can thus be expressed as:

$$u(x) = f(x) + \lambda \int_a^x K(x-t)u(t)dt \qquad (4.31)$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by:

$$\mathcal{L}\{f_1(x)\} = F_1(s)$$
$$\mathcal{L}\{f_2(x)\} = F_2(s)$$

The *Laplace convolution product* of these two functions is defined by:

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x)$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by:

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s)$$

Based on this summary, we will examine specific Volterra integral equations where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform using the following Table :

f(x)	F(s)=$\mathcal{L}\{f(x)\}$
C	$\frac{c}{s}, s > 0$
X	$\frac{1}{s^2}, s > 0$
x^n	$\frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}, S > 0, Re\,(n) > -1$
e^{ax}	$\frac{1}{s-a}, s > a$
$\sin ax$	$\frac{a}{s^2+a^2}$
$\cos ax$	$\frac{s}{s^2+a^2}$
$sin^2 ax$	$\frac{2a^2}{s(s^2+4a^2)}, Re\,(s) > \lvert Im(a)\rvert$
$cos^2 ax$	$\frac{s^2+2a^2}{s(s^2+4a^2)}, Re\,(s) > \lvert Im(a)\rvert$
x $\sin ax$	$\frac{2as}{(s^2+a^2)^2}$
x $\cos ax$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh ax$	$\frac{a}{s^2-a^2}, s > \lvert a\rvert$

$\cosh ax$	$\frac{s}{s^2-a^2}$, $s>\vert a\vert$
$sinh^2 ax$	$\frac{2a^2}{s(s^2-4a^2)}$, $Re\ (s) > \vert Im(a)\vert$
$cosh^2 ax$	$\frac{s^2-2a^2}{s(s^2-4a^2)}$, $Re\ (s) > \vert Im(a)\vert$
x $\sinh ax$	$\frac{2as}{(s^2-a^2)^2}$, $s>\vert a\vert$
x $\cosh ax$	$\frac{s^2+a^2}{(s^2-a^2)^2}$, $s>\vert a\vert$
$x^n e^{ax}$	$\frac{n!}{(s-a)^{n+1}}$, $s>a$, n is a positive integer
$e^{ax}\sin bx$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{ax}\cos bx$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$
$e^{ax}\sinh bx$	$\frac{b}{(s-a)^2-b^2}$, $s>a$
$e^{ax}\cosh bx$	$\frac{s-a}{(s-a)^2-b^2}$, $s>a$

By taking Laplace transform of both sides of (4.31), we find:

$$U(s) = F(s) + \lambda K(s)U(s) \qquad (4.32)$$

Where

$$U(s) = \mathcal{L}\{u(x)\}, F(s) = \mathcal{L}\{f(x)\}\ , K(s) = \mathcal{L}\{K(x)\}$$

Solving (4.32) for $U(s)$ gives:

$$U(s) = \frac{F(s)}{1-\lambda K(s)}\ ,\ \lambda K(s) \neq 1 \qquad (4.33)$$

The solution $u(x)$ is obtained by taking the inverse Laplace transform of both sides of (4.33), where we find:

$$u(x) = \mathcal{L}^{-1}\left\{\frac{F(s)}{1-\lambda K(s)}\right\} \qquad (4.34)$$

Recall that the right side of (4.34) can be evaluated by using the above Table. The Laplace transform method for solving Volterra integral equations will be illustrated by studying the following examples.

Example 4.13. Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = 1 + \int_0^x u(t)dt \qquad (4.35)$$

Notice that the kernel $K(x-t) = 1$, $\lambda = 1$. Taking Laplace transform of both sides (4.35) gives:

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} + \mathcal{L}\{1 * u(x)\}$$

So that

$$U(s) = \frac{1}{s} + \frac{1}{s}U(s)$$

$$U(s) = \frac{1}{s-1}$$

By taking the inverse Laplace transform of both sides of the above equation, the exact solution is therefore given by:

$$u(x) = e^x$$

Example 4.14. Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = 1 - \int_0^x (x-t)u(t)dt \qquad (4.36)$$

Notice that the kernel $K(x-t)$ = x-t, λ = -1. Taking Laplace transform of both sides (4.36) gives:

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} - \mathcal{L}\{(x-t) * u(x)\}$$

So that

$$U(s) = \frac{1}{s} - \frac{1}{s^2}U(s)$$

$$U(s) = \frac{s}{s^2+1}$$

By taking the inverse Laplace transform of both sides of the above equation, the exact solution is therefore given by:

$$u(x) = \cos x$$

Example 4.15. Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = \frac{1}{3!}x^3 - \int_0^x (x-t)u(t)dt \qquad (4.37)$$

Taking Laplace transform of both sides (4.37) gives:

$$\mathcal{L}\{u(x)\} = \frac{1}{3!}\mathcal{L}\{x^3\} - \mathcal{L}\{(x-t) * u(x)\}$$

So that

$$U(s) = \frac{1}{3!}\frac{3!}{s^4} - \frac{1}{s^2}U(s)$$

$$U(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

By taking the inverse Laplace transform of both sides of the above equation, the exact solution is therefore given by:

$$u(x) = x - \sin x$$

Example 4.16. Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = \sin x + \cos x + 2\int_0^x \sin(x-t)\, u(t)dt \qquad (4.38)$$

Taking Laplace transform of both sides (4.38) gives:

$$\mathcal{L}\{u(x)\} = \frac{1}{3!}\mathcal{L}\{x^3\} - \mathcal{L}\{(x-t) * u(x)\}$$

So that

$$U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1}U(s)$$

$$U(s) = \frac{1}{s-1}$$

By taking the inverse Laplace transform of both sides of the above equation, the exact solution is therefore given by:

$$u(x) = e^x$$

Exercises 4.4.

Use the *Laplace transform method* to solve the Volterra integral equations:

1. $u(x) = 1 - x - \int_0^x (x-t)u(t)dt$
2. $u(x) = \cos x - \sin x + 2\int_0^x \cos(x-t)u(t)dt$
3. $u(x) = e^x - \cos x - 2\int_0^x e^{x-t}u(t)dt$
4. $u(x) = 1 - \int_0^x ((x-t)^2 - 1)u(t)dt$
5. $u(x) = \sin x - \cos x + \cosh x - 2\int_0^x \cosh(x-t)u(t)dt$

4.5 The Series Solution Method

A real function *u*(*x*) is called analytic if it has derivatives of all orders such that the Taylor series at any point *b* in its domain

$$u(x) = \sum_{n=0}^{\infty} \frac{u^n(b)}{n!}(x-b)^n$$

converges to *u(x)* in a neighborhood of *b*. For simplicity, the generic form of the Taylor series at *x* = 0 can be written as:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (4.39)$$

In this section, we will present a useful method, that stems mainly from the Taylor series for analytic functions, for solving Volterra integral equations. We will assume that the solution *u*(*x*) of the Volterra integral equation:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \qquad (4.40)$$

is analytic, and therefore possesses a Taylor series of the form given in (4.40), where the coefficients a_n will be determined recurrently. Substituting (4.39) into both sides of (4.40) gives:

$$\sum_{n=0}^{\infty} a_n x^n = T\big(f(x)\big) + \lambda \int_a^x K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T\big(f(x)\big) + \lambda \int_a^x K(x,t)(a_0 + a_1 t + a_2 t^2 + \cdots)dt \qquad (4.41)$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (4.40) will be converted to a traditional integral in(4.41) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geq 0$ will be integrated. Notice that because we are seeking a series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (4.41) and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in a_j, $j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients a_j, $j \geq 0$. Having determined the coefficients a_j, $j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (4.39). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher the accuracy level we achieve.

Example 4.17 Solve the Volterra integral equation by using the series solution method:

$$u(x) = 1 + \int_0^x u(t)dt \qquad (4.42)$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4. 42) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) dt$$

Evaluating the integral on the right side gives:

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1}$$

that can be rewritten as:

$$a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \cdots$$

Equating the coefficients of like powers of x on both sides of the above equation gives the recurrence relation:

$$a_0 = 1, a_n = \frac{1}{n} a_{n-1}, n \geq 1$$

where this result gives:

$$a_n = \frac{1}{n!} \ ,n \geq 0$$

Substituting this result into (4.39) gives the series solution:

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

that converges to the exact solution $u(x) = e^x$.

Example 4.18 Solve the Volterra integral equation by using the series solution method:

$$u(x) = x + \int_0^x (x-t)u(t)dt \qquad (4.43)$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4. 43) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = x + \int_0^x x\left(\sum_{n=0}^{\infty} a_n t^n\right) dt - \int_0^x \left(\sum_{n=0}^{\infty} a_n t^{n+1}\right) dt$$

Evaluating the integral on the right side gives:

$$\sum_{n=0}^{\infty} a_n x^n = x + \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)} a_n x^{n+2}$$

that can be rewritten as:

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} a_{n-2} x^n$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{12} a_2 x^4 + \cdots$$

Equating the coefficients of like powers of x on both sides of the above equation gives the recurrence relation:

$$a_0 = 0, a_1 = 1, a_n = \frac{1}{n(n-1)} a_{n-2} \ ,n \geq 2$$

where this result gives:

$$a_n = \frac{1}{(2n+1)!} \ ,n \geq 0$$

Substituting this result into (4.39) gives the series solution:

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

that converges to the exact solution $u(x) = \sinh x$.

Example 4.19 Solve the Volterra integral equation by using the series solution method:

$$u(x) = 1 - x\sin x + \int_0^x tu(t)dt \qquad (4.44)$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4. 44) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = 1 - x\sin x + \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integral on the right side gives:

$$(a_0 + a_1 x + a_2 x^2 + \cdots) = 1 - x\left(x - \frac{x^3}{3!} + \cdots\right) + \int_0^x t(a_0 + a_1 t + a_2 t^2 + \cdots)dt$$

Integrating the right side and collecting the like terms of x we find

$$(a_0 + a_1 x + a_2 x^2 + \cdots) = 1 + \left(\frac{1}{2}a_0 - 1\right)x^2 + \frac{1}{3}a_1 x^3 + \left(\frac{1}{6} + \frac{1}{4}a_2\right)x^4 + ..$$

Equating the coefficients of like powers of x on both sides of the above equation gives the recurrence relation:

$$a_0 = 1, a_1 = 0, a_2 = \left(\frac{1}{2}a_0 - 1\right) = -\frac{1}{2!}, a_3 = \frac{1}{3}a_1 = 0, a_4 = \left(\frac{1}{6} + \frac{1}{4}a_2\right) = \frac{1}{4!}, \ldots$$

and generally

$$a_{2n+1} = 0, a_{2n} = \frac{(-1)^n}{(2n)!}, n \geq 0$$

The solution in a series form is given by:

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

that converges to the exact solution $u(x) = \cos x$.

Example 4.20 Solve the Volterra integral equation by using the series solution method:

$$u(x) = 2e^x - 2 - x + \int_0^x (x - t)u(t)dt \qquad (4.45)$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4. 45) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = 2e^x - 2 - x + \int_0^x (x - t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integral on the right side gives:

$$(a_0 + a_1x + a_2x^2 + \cdots)$$
$$= x + \left(1 + \frac{1}{2}a_0\right)x^2 + \left(\frac{1}{3} + \frac{1}{6}a_1\right)x^3 + \left(\frac{1}{12} + \frac{1}{12}a_2\right)x^4 + \cdots$$

Equating the coefficients of like powers of *x* on both sides of the above equation gives the recurrence relation:

$$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = \frac{1}{2!}, a_4 = \frac{1}{3!}, \ldots$$

The solution in a series form is given by:

$$u(x) = x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)$$

that converges to the exact solution $u(x) = xe^x$

Exercises 4.5 Use the *series solution method* to solve the Volterra integral equations:

1. $u(x) = 1 + xe^x - \int_0^x tu(t)dt$
2. $u(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)dt$
3. $u(x) = \sec x + \tan x - \int_0^x \sec t\, u(t)dt$
4. $u(x) = 3 + x^2 - \int_0^x (x-t)u(t)dt$

4.6 The Variational Iteration Method

In this section, we will study the newly developed *variational iteration method* that proved to be effective and reliable for analytic and numerical purposes. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in the Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need for specific restrictions such as the so-called Adomian polynomials that we need for nonlinear problems. Moreover, the method gives the solution in a series form that converges to the closed-form solution if an exact solution exists. The obtained series can be employed for numerical purposes if an exact solution is not obtainable. In what follows, we present the main steps of the method.

Consider the differential equation:

$$\mathcal{L}u + \aleph u = g(t) \qquad (4.46)$$

where $\mathcal{L}$ and $\aleph$ are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term. The variational iteration method presents a correction functional for equation (4.46) in the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)\big(\mathcal{L}u_n(\psi) + \aleph\tilde{u}_n(\psi) - g(\psi)\big)d\psi \qquad (4.47)$$

where λ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n$ = 0, where δ is the variational derivative. The Lagrange multiplier λ can be identified optimally via the variational theory.

The determination of the Lagrange multiplier plays a major role in the determination of the solution to the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers, and its correction functional respectively:

$$(i)\begin{cases} u' + f\big(u(\psi), u'(\psi)\big) = 0, \lambda = -1 \\ u_{n+1} = u_n - \int_0^x [u_n' + f(u_n, u_n')]d\psi \end{cases}$$

$$(ii)\begin{cases} u'' + f\big(u(\psi), u'(\psi), u''(\psi)\big) = 0, \lambda = (\psi - x) \\ u_{n+1} = u_n + \int_0^x (\psi - x)[u_n'' + f(u_n, u_n', u_n'')]d\psi \end{cases}$$

$$(iii)\begin{cases} u''' + f\big(u(\psi), u'(\psi), u''(\psi), u'''(\psi)\big) = 0, \lambda = \frac{1}{2!}(\psi - x)^2 \\ u_{n+1} = u_n - \int_0^x \frac{1}{2!}(\psi - x)^2[u_n''' + f(u_n, u_n', u_n'', u_n''')]d\psi \end{cases}$$

and generally

$$\begin{cases} \boldsymbol{u^{(n)} + f\Big(u(\psi), u'(\psi), u''(\psi), \dots, u^{(n)}(\psi)\Big) = 0, \lambda = (-1)^n \frac{1}{(n-1)!}(\psi - x)^{(n-1)}} \\ \boldsymbol{u_{n+1} = u_n + (-1)^n \int_0^x \frac{1}{(n-1)!}(\psi - x)^{(n-1)}\Big[u_n^{(n)} + f\Big(u_n, u_n', u_n'', \dots, u_n^{(n)}\Big)\Big]d\psi} \end{cases} \boldsymbol{, for\ n \geq 1}$$

To use the variational iteration method for solving Volterra integral equations, it is necessary to convert the integral equation to an equivalent initial value problem or an equivalent integro-differential equation. As defined before, an integro-differential equation is an equation that contains differential and integral operators in the same equation.

Example 4.21 Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + \int_0^x u(t)dt \qquad (4.48)$$

Using the Leibnitz rule to differentiate both sides of (4.48)gives:

$$u'(x) - u(x) = 0 \qquad (4.49)$$

Substituting $x = 0$ into (4.48) gives the initial condition $u(0) = 1$.

<u>Using the variational iteration method</u>

The correction functional for equation (4.49)is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)[u'(\psi) - u(\psi)]d\psi \qquad (4.50)$$

Using the formula (i) given above leads to:

$$\lambda = -1$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (4.50) gives the iteration formula:

$$u_{n+1} = u_n - \int_0^x [u'(\psi) - u(\psi)]d\psi$$

As stated before, we can use the initial condition to select $u_0(x) = u(0) = 1$.
Using this selection into (4.50) gives the following successive approximations:

$$u_0 = 1$$

$$u_1 = 1 - \int_0^x [u_0'(\psi) - u_0(\psi)]d\psi = 1 + x$$

$$u_2 = 1 + x - \int_0^x [u_1'(\psi) - u_1(\psi)]d\psi = 1 + x + \frac{1}{2!}x^2$$

$$u_3 = 1 + x + \frac{1}{2!}x^2 - \int_0^x [u_2'(\psi) - u_2(\psi)]d\psi = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= \lim_{n\to\infty} 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

that gives the exact solution by: $u(x) = e^x$.

Example 4.22 Solve the Volterra integral equation by using the variational iteration method

$$u(x) = x + \int_0^x (x-t)u(t)dt \qquad (4.51)$$

Using the Leibnitz rule to differentiate both sides of (4.51) once with respect to x gives the integro-differential equation:

$$u'(x) = 1 + \int_0^x u(t)dt \qquad (4.52)$$

However, by differentiating (4.52) with respect to x we obtain the differential equation:

$$u''(x) - u(x) = 0 \qquad (4.53)$$

Substituting $x = 0$ into (4.51) and (4.52) gives the initial conditions $u(0) = 0$and u'(0)=1.
The resulting initial value problem, which consists of a second order ODE and initial conditions is given by:

$$u''(x) - u(x) = 0 \ , u(0) = 0 \text{and } u'(0) = 1 \qquad (4.54)$$

<u>Using the variational iteration method</u>

The correction functional for equation (4.54)is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)[u''(\psi) - \tilde{u}(\psi)]d\psi \qquad (4.55)$$

Using the formula (ii) given above leads to:

$$\lambda = \psi - x$$

Substituting this value of the Lagrange multiplier $\lambda = \psi - x$ into the functional (4.55) gives the iteration formula:

$$u_{n+1} = u_n - \int_0^x (\psi - x)[u'(\psi) - u(\psi)]d\psi \qquad (4.56)$$

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) = x$. Using this selection in (4.56) gives the following successive approximations:

$$u_0 = x$$

$$u_1 = x + \int_0^x (\psi - x)[u_0''(\psi) - u_0(\psi)]d\psi = x + \frac{1}{3!}x^3$$

$$u_2 = x + \frac{1}{3!}x^3 + \int_0^x (\psi - x)[u_1''(\psi) - u_1(\psi)]d\psi = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$u_3 = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \int_0^x (\psi - x)[u_2''(\psi) - u_2(\psi)]d\psi = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7$$

$$\vdots$$

$$u_n = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots + \frac{1}{(2n+1)!}x^{2n+1}$$

The VIM admits the use of $u(x) = \lim_{n\to\infty} u_n(x)$

that gives the exact solution by: $u(x) = \sinh x$

Example 4.23 Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + x + \frac{1}{3!}x^3 - \int_0^x (x - t)u(t)dt \qquad (4.57)$$

Using the Leibnitz rule to differentiate both sides of (4.57) once with respect to *x* gives the integro-differential equation:

$$u'(x) = 1 + \frac{1}{2!}x^2 - \int_0^x u(t)dt \qquad (4.58)$$

However, by differentiating (4.58) with respect to *x* we obtain the differential equation:

$$u''(x) + u(x) = x \qquad (4.59)$$

Substituting $x = 0$ into (4.57) and (4.58) gives the initial conditions $u(0) = 1$and u'(0)=1. The resulting initial value problem, which consists of a second order ODE and initial conditions is given by:

$$u''(x) + u(x) = x \ , u(0) = 1 \text{and } u'(0) = 1 \qquad (4.60)$$

<u>Using the variational iteration method</u>

The correction functional for equation (4.60)is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)[u''(\psi) + \tilde{u}(\psi) - \psi]d\psi \qquad (4.61)$$

Using the formula (ii) given above leads to:

$$\lambda = \psi - x$$

Substituting this value of the Lagrange multiplier $\lambda = \psi - x$ into the functional (4.60) gives the iteration formula:

$$u_{n+1} = u_n - \int_0^x (\psi - x)[u'(\psi) + u(\psi) - \psi]d\psi \qquad (4.62)$$

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) = 1+x$. Using this selection in (4.62) gives the following successive approximations:

$$u_0 = 1 + x$$

$$u_1 = 1 + x + \int_0^x (\psi - x)[u_0''(\psi) + u_0(\psi) - \psi]d\psi$$

$$= 1 + x - \frac{1}{2!}x^2$$

$$u_2 = 1 + x - \frac{1}{2!}x^2 + \int_0^x (\psi - x)[u_1''(\psi) + u_1(\psi) - \psi]d\psi$$

$$= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

$$u_3 = 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \int_0^x (\psi - x)[u_2''(\psi) + (\psi) - \psi]d\psi$$

$$= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

$$\vdots$$

$$u_n = x + (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}$$

The VIM admits the use of $u(x) = \lim_{n\to\infty} u_n(x)$

that gives the exact solution by: $u(x) = x + \cos x$

Example 4.24 Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x - t)^2 u(t)dt \qquad (4.63)$$

Using the Leibnitz rule to differentiate both sides of (4.63) three times with respect to x gives the two integro-differential equations:

$$u'(x) = 1 + x + \int_0^x (x - t)u(t)dt \qquad (4.64)$$

$$u''(x) = 1 + \int_0^x u(t)dt \qquad (4.65)$$

However, by differentiating (4.65) with respect to x we obtain the differential equation:

$$u'''(x) - u(x) = 0 \qquad (4.66)$$

Substituting $x = 0$ into (4.63) , (3.64) and (4.65) gives the initial conditions:

$u(0) = u'(0) = u''(0) = 1.$

The resulting initial value problem, which consists of a third order ODE and initial conditions is given by:

$$u''(x) - u(x) = 0, u(0) = u'(0) = u''(0) = 1 \qquad (4.67)$$

Using the variational iteration method

The correction functional for equation (4.67)is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)[u'''(\psi) - \tilde{u}(\psi)]d\psi \qquad (4.68)$$

Using the formula (iii) given above leads to:

$$\lambda = -\frac{1}{2!}(\psi - x)^2$$

Substituting this value of the Lagrange multiplier $\lambda = -\frac{1}{2!}(\psi - x)^2$ into the functional (4.68) gives the iteration formula:

$$u_{n+1} = u_n - \frac{1}{2!}\int_0^x (\psi - x)^2[u'''(\psi) - \tilde{u}(\psi)]d\psi \qquad (4.69)$$

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) + \frac{x^2}{2}u''(0) = 1 + x + \frac{x^2}{2}$. Using this selection in (4.69) gives the following successive approximations:

$$u_0 = 1 + x + \frac{x^2}{2}$$

$$u_1 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$u_2 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + + \frac{x^6}{6!} + + \frac{x^7}{7!} + \frac{x^8}{8!}$$

$$\vdots$$

$$u_n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + + \frac{x^6}{6!} + + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$$

The VIM admits the use of $u(x) = \lim_{n\to\infty} u_n(x)$

that gives the exact solution by: $u(x) = e^x$

Exercises 4.6 Use the *variational iteration method* to solve the following Volterra integral equations:

1. $u(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5 - \int_0^x u(t)dt$
2. $u(x) = 2 + x - 2\cos x - \int_0^x (x - t + 2)u(t)dt$
3. $u(x) = 1 - x\sin x + x\cos x + \int_0^x tu(t)dt$
4. $u(x) = 1 - 2\sinh x + \int_0^x (x - t + 2)u(t)dt$