

Chapter 5: Numerical Solution of Integral Equations

5.1 Classification of Integral Equations:

An integral equation is an equation in which the unknown function $u(x)$ appears under an integral sign. The most general linear integral equation in $u(x)$ can be presented as:

$$h(x)u(x) = f(x) + \int_a^{b(x)} k(x, t)u(t)dt \quad (5.1)$$

where $k(x, t)$ is a function of two variables called the **kernel** of the integral equation.

This equation is called a **Volterra integral equation** when $b(x)=x$,

$$h(x)u(x) = f(x) + \int_a^x k(x, t)u(t)dt \quad (5.2)$$

when $h(x)=0$ it is called a **Volterra equation of the first kind**,

$$-f(x) = \int_a^x k(x, t)u(t)dt \quad (5.3)$$

and is called a **Volterra equation of the second kind** when $h(x)=1$,

$$u(x) = f(x) + \int_a^x k(x, t)u(t)dt \quad \dots(5.4)$$

The integral equation (5.1) is called a **Fredholm integral equation** when $b(x)=b$, where b constant,

$$h(x)u(x) = f(x) + \int_a^b k(x, t)u(t)dt \quad \dots(5.5)$$

It is also called a **Fredholm equation of the first and second kinds** when $h(x)=0$ and $h(x)=1$, respectively:

$$-f(x) = \int_a^b k(x, t)u(t)dt \quad \dots(5.6)$$

$$u(x) = f(x) + \int_a^b k(x, t)u(t)dt \quad \dots(5.7)$$

5.2 Numerical Solution of Volterra Integral Equations:

Let us consider the Volterra equation of the second kind:

$$u(x) = f(x) + \int_a^x k(x, t)u(t)dt$$

we will subdivide the interval of integration (a, x) into n equal subintervals of width $h=(x_n - a)/n$, $n \geq 1$, where x_n is the end point we choose for x , we shall set $t_0=a$ and $t_j=a+jh$. Note that the particular value $u(x_0)=f(a)$, so if we use the trapezoidal rule with n subintervals to approximate the integral in the Volterra integral equation of the second kind (5.4), we have:

$$\int_a^x k(x, t)u(t)dt \approx \frac{h}{2} \left[k(x, t_0)u(t_0) + 2k(x, t_1)u(t_1) + \cdots + 2k(x, t_{n-1})u(t_{n-1}) + k(x, t_n)u(t_n) \right] \quad (5.8)$$

and the integral equation (5.4) is then approximated by the sum:

$$u(x) = f(x) + \frac{h}{2} \left[k(x, t_0)u(t_0) + 2 \sum_{j=1}^{n-1} k(x, t_j)u(t_j) + k(x, t_n)u(t_n) \right] \quad (5.9)$$

If we consider $n+1$ sample values of $u(x)$, $u(x_i), i=0, 1, \dots, n$, equation (5.9) will become a set of $n+1$ equations in $u(x_i)$ (or u_i) [note that $u(x_0)=f(x_0)$ since the integral in (5.4) vanishes for $x=x_0=a$].

$$\left. \begin{aligned} u_0 &= f_0 \\ u_i &= f_i + \frac{h}{2} [k_{i0}u_0 + 2 \sum_{j=1}^{i-1} k_{ij}u_j + k_{in}u_n], \\ i &= 1, 2, \dots, n, k_{ij} = k(x_i, t_j), \quad j \leq i \end{aligned} \right\} \quad (5.10)$$

which are $n+1$ equations in u_i , the approximation to the solution $u(x)$ of (5.4) at $x_i=a+ih$ for $i=0, 1, \dots, n$.

Example 5.1: Use trapezoidal method to find an approximate values to the solution for the following Volterra integral equation $u(x) = x - \int_0^x (x-t)u(t)dt$ at $x=0, 1, 2, 3$, and 4.

Here, $f(x)=x$, $k(x, t)=t-x$ for $t \leq x=4$ and is zero for $t > x=4$, and $a=0$ with $u(0)=0$. We also have $n=4$ and hence $h=(4-0)/4=1$. So using (5.10) to obtain:

$$u_0=f_0=0$$

$$u_1=f_1+\frac{h}{2} [k_{10}u_0 + k_{11}u_1]=1+\frac{1}{2} [(0-1)(0) + (1-1)u_1]=1$$

$$u_2=f_2+\frac{h}{2} [k_{20}u_0 + 2k_{21}u_1 + k_{22}u_2]$$

$$= 2 + \frac{1}{2} [(0-2)(0) + 2(1-2)(1) + (2-2)u_2]=1$$

$$u_3 = f_3 + \frac{h}{2} [k_{30}u_0 + 2k_{31}u_1 + 2k_{32}u_2 + k_{33}u_3] = 3 + \frac{1}{2} [(0-3)(0) + 2(1-3)(1) + 2(2-3)(1) + (3-3)u_3] = 3 + \frac{1}{2} [-4 - 2] = 0$$

$$u_4 = f_4 + \frac{h}{2} [k_{40}u_0 + 2k_{41}u_1 + 2k_{42}u_2 + 2k_{43}u_3 + k_{44}u_4] = 4 + \frac{1}{2} [(0-4)(0) + 2(1-4)(1) + 2(2-4)(1) + 2(3-4)(0) + (4-4)u_4] = 4 + \frac{1}{2} [-6 - 4] = -1$$

x_k	0	1	2	3	4
u_k	0	1	1	0	-1

5.3 Numerical Solution of Fredholm Integral Equations:

Let us consider the Fredholm equation of the second kind:

$$u(x) = f(x) + \int_a^b k(x, t)u(t)dt \quad (5.11)$$

we will subdivide the interval of integration (a,b) into n equal subintervals of width $h=(b-a)/n$, $n \geq 1$, we shall set $t_0=a, t_n=b$ and $t_j=a+jh$. Note that the particular value , so if we use the trapezoidal rule with n subintervals to approximate the integral in the Fredholm integral equation of the second kind (5.11), we have:

$$\int_a^b k(x, t)u(t)dt \approx \frac{h}{2} \left[k(x, t_0)u(t_0) + 2k(x, t_1)u(t_1) + \dots + 2k(x, t_{n-1})u(t_{n-1}) + k(x, t_n)u(t_n) \right] \quad (5.12)$$

and the integral equation (5.11) is then approximated by the sum:

$$u(x) = f(x) + \frac{h}{2} [k(x, t_0)u(t_0) + 2 \sum_{j=1}^{n-1} k(x, t_j)u(t_j) + k(x, t_n)u(t_n)] \quad (5.13)$$

If we consider n+1 sample values of $u(x)$, $u(x_i), i=0, 1, \dots, n$, equation (5.13) will become a set of n+1 equations in $u(x_i)$ (or u_i).

$$\left. \begin{aligned} u_i &= f_i + \frac{h}{2} [k_{i0}u_0 + 2 \sum_{j=1}^{n-1} k_{ij}u_j + k_{in}u_n], \\ i &= 1, 2, \dots, n, k_{ij} = k(x_i, t_j), \quad j \leq i \end{aligned} \right\} \quad (5.14)$$

which are n+1 equations in u_i , the approximation to the solution $u(x)$ of (5.11) at $x_i=a+ih$ for $i=0, 1, \dots, n$.

Example 5.2: Use trapezoidal method to find an approximate values to the solution for the integral equation $u(x)=x^2 + \frac{1}{4} - \frac{1}{3}x + \int_0^1 (x-t)u(t)dt$ with $h=0.25$ notice that the real solution is $u(x)=x^2$

We have $f(x)=x^2 + \frac{1}{4} - \frac{1}{3}x$ and $k(x,t)=x-t$.

Since $h=0.25$, we have $x_0=t_0=0, x_1=t_1=0.25, x_2=t_2=0.5, x_3=t_3=0.75$ and $x_4=t_4=1$

for $i=0,1,2,3$ and 4, we have:

$$u_0 = f_0 + \frac{h}{2} [k_{00}u_0 + 2k_{01}u_1 + 2k_{02}u_2 + 2k_{03}u_3 + k_{04}u_4]$$

$$u_1 = f_1 + \frac{h}{2} [k_{10}u_0 + 2k_{11}u_1 + 2k_{12}u_2 + 2k_{13}u_3 + k_{14}u_4]$$

$$u_2 = f_2 + \frac{h}{2} [k_{20}u_0 + 2k_{21}u_1 + 2k_{22}u_2 + 2k_{23}u_3 + k_{24}u_4]$$

$$u_3 = f_3 + \frac{h}{2} [k_{30}u_0 + 2k_{31}u_1 + 2k_{32}u_2 + 2k_{33}u_3 + k_{34}u_4]$$

$$u_4 = f_4 + \frac{h}{2} [k_{40}u_0 + 2k_{41}u_1 + 2k_{42}u_2 + 2k_{43}u_3 + k_{44}u_4]$$

therefore, we hence:

$$u_0 = 0.25 + \frac{0.25}{2} [(0-0)u_0 + 2(0-0.25)u_1 + 2(0-0.5)u_2 + 2(0-0.75)u_3 + (0-1)u_4]$$

$$u_1 = 0.22917 + \frac{0.25}{2} [(0.25-0)u_0 + 2(0.25-0.25)u_1 + 2(0.25-0.5)u_2 + 2(0.25-0.75)u_3 + (0.25-1)u_4]$$

$$u_2 = 0.33333 + \frac{0.25}{2} [(0.5-0)u_0 + 2(0.5-0.25)u_1 + 2(0.5-0.5)u_2 + 2(0.5-0.75)u_3 + (0.5-1)u_4]$$

$$u_3 = 0.5625 + \frac{0.25}{2} [(0.75 - 0)u_0 + 2(0.75 - 0.25)u_1 + 2(0.75 - 0.5)u_2 + 2(0.75 - 0.75)u_3 + (0.75 - 1)u_4]$$

$$u_4 = 0.91667 + \frac{0.25}{2} [(1 - 0)u_0 + 2(1 - 0.25)u_1 + 2(1 - 0.5)u_2 + 2(1 - 0.75)u_3 + (1 - 1)u_4]$$

then,

$$8u_0 + 0.5u_1 + u_2 + 1.5u_3 + u_4 = 2$$

$$-0.25u_0 + 8u_1 + 0.5u_2 + u_3 + 0.75u_4 = 1.8333$$

$$-0.5u_0 - 0.5u_1 + 8u_2 + 0.5u_3 + 0.5u_4 = 2.6667$$

$$-0.75u_0 - u_1 - 0.5u_2 + 8u_3 + 0.25u_4 = 4.5$$

$$-u_0 - 1.5u_1 - u_2 - 0.5u_3 + 8u_4 = 7.3333$$

solving this system, we get:

$$u = [-0.010417 \quad 0.052083 \quad 0.23958 \quad 0.55208 \quad 0.98958]^T$$

x_k	u_k	$u(x_k)$
0	-0.010417	0
0.25	0.052083	0.0625
0.5	0.23958	0.25
0.75	0.55208	0.5625
1	0.98958	1

Exercise:

1. Use trapezoidal method to find an approximate values to the solution for the integral equation $u(x) = x - \frac{x^3}{3} + \int_0^x tu(t)dt$, $x \in [0,1]$, with $h=0.25$. (note that $u(x)=x$)
2. Use trapezoidal method to find an approximate values to the solution for the integral equation $u(x) = e^x - xe^1 + x + \int_0^1 xu(t)dt$, with $h=0.5$ (note that $u(x)=e^x$).