Chapter One: Introductory Concepts

1.1 Definition

An integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. A typical form of an integral equation in $u(x)$ is of the form:

$$cu(x) = f(x) + \int_a^{b(x)} k\big(x,t,u(t)\big)dt \qquad ...(1.1)$$

where the forcing function *f(x)* and the kernel function *k(x,t)* are prescribed, while *u(x)* is the unknown function to be determined, and *c* is constant. The parameter λ is often omitted; it is, however, of importance in certain theoretical investigations (e.g. stability) and the eigenvalue problem.

1.2 Classification of Linear Integral Equations

Definition (1.1):

The integral equation (1.1) is called ***linear integral equation*** if the kernel *k(x,t,u(t))=k(x,t)u(t)*, otherwise it is called ***nonlinear***.

i.e. $cu(x) = f(x) + \int_a^{b(x)} k(x,t)u(t)dt$ (linear integral equation)

$cu(x) = f(x) + \int_a^{b(x)} k\big(x,t,u(t)\big)dt$ (nonlinear integral equation)

Definition (1.2):

The linear integral equation (1.1) is called ***homogeneous***, if $f(x) \equiv 0$, otherwise it is called ***nonhomogeneous***.

i.e. $cu(x) = \int_a^{b(x)} k(x,t)u(t)dt$ (homogeneous integral equation)

$cu(x) = f(x) + \int_a^{b(x)} k(x,t)u(t)dt$ (nonhomogeneous integral equation)

Definition (1.3):

The integral equation (1.1) is said to be an equation of the ***first kind*** if c=0

i.e. $f(x) = \int_a^{b(x)} k(x,t)u(t)dt$

Definition (1.4):

The integral equation (1.1) is said to be an equation of the ***second kind*** if c=1

i.e. $u(x) = f(x) + \int_a^{b(x)} k(x,t)u(t)dt$

Definition (1.5):

The integral equation (1.1) is called ***Volterra integral equation (VIE)*** when *b(x)=x.*

i.e. $u(x) = f(x) + \int_a^{x} k(x,t)u(t)dt$

Definition (1.6):

The integral equation (1.1) is called ***Fredholm integral equation (FIE)***, if *b(x)=b*, where *b* is constant such that $b \geq a$.

i.e. $u(x) = f(x) + \int_a^{b} k(x,t)u(t)dt$

Definition (1.7):

An ***integro-differential equation*** is an equation that involves one (or more) of an unknown function *u(x)*, together with differential and integral operations on *x*.

The following are examples of integro-differential equations:

1. $u''(x) = -x + \int_0^{x}(x-t)u(t)dt$, $u(0) = 0, u'(0) = 1$, (2^{nd} order Volterra integro-differential equation)

2. $u'(x) = 1 - \frac{1}{3}x + \int_0^1 xtu(t)dt \;\;, u(0) = 1,$ (1^{st} order Fredholm integro-differential equation)

Definition (1.8):

the integral equation is called ***singular*** if the lower limit, the upper limit or both limits of integration are *infinite*. In addition, the integral equation is also called a **singular integral equation** if the kernel $K(x, t)$ becomes *infinite* at one or more points in the domain of integration.
Examples of the second kind of *singular* integral equations are given by:

$$u(x) = 2x + 6\int_0^\infty \sin(x-t)u(t)dt$$

$$u(x) = x + \frac{1}{3}\int_{-\infty}^0 \cos(x+t)u(t)dt$$

$$u(x) = 1 + x^2 + \frac{1}{6}\int_{-\infty}^\infty (x+t)u(t)dt$$

Examples of the first kind of *singular* integral equations are given by:

$$x^2 = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt$$

$$x = \int_0^x \frac{1}{(x-t)^\alpha}u(t)dt \quad 0 < \alpha < 1$$

1.3 Special Types of Kernels

The following special cases of the kernel of an integral equation are of main interest:

Definition (1.9):

The kernel $k(x,t)$ is called ***difference kernel***, if $k(x,t)=k(x-t)$. And the linear integral equation is called ***an integral equation of convolution type***.

i.e. $u(x) = f(x) + \int_a^b k(x-t)u(t)dt$

***Definition (1.10)*:**

The kernel *k(x,t)* is called ***the separable*** or ***degenerate kernel of rank n*** if it is of the form:

$$k(x,t) = \sum_{j=1}^{n} a_j(x) b_j(t)$$

where *n* is finite and the functions $\{a_j\}$ *and* $\{b_j\}$ are sufficiently smooth functions.

Exercises 1.1.

Classify each of the following integral equations:

1. $u(x) = x + \int_0^1 xtu(t)dt$
2. $u(x) = 1 + x^2 + \int_0^x (x - t)u(t)dt$
3. $u(x) = e^x + \int_0^x (tu^2(t)dt$
4. $u(x) = \int_0^1 (x - t)^2 u(t)dt$
5. $u(x) = \frac{2}{3}x + \int_0^1 xtu(t)dt$
6. $u(x) = 1 + \frac{x}{4}\int_0^1 \frac{1}{x+t}\frac{1}{u(t)} dt$
7. $u'(x) = 1 + \int_0^x e^{-2t} u^3(t)dt \quad , u(0) = 0$
8. $u'''(x) = -\frac{1}{12}x^4 + \int_0^x e^{x-t} u(t)dt \quad , u(0) = u'(0) = 0 , u''(0) = 2$

1.4 Solution of an Integral Equation

A solution of an integral equation or an integro-differential equation on the interval of integration is a function *u*(*x*) such that it satisfies the given equation. In other words, if the given solution is substituted on the right-hand side of the equation, the output of this direct substitution must yield on the left-hand side, i.e. we should verify that the given function *u*(*x*) satisfies the integral equation or the integro-differential equation under discussion. This important concept will be illustrated first by examining the following examples.

Example 1.1. Show that $u(x) = e^x$ is a solution of the Volterra integral equation:

$$u(x) = 1 + \int_0^x u(t)dt$$

Substituting $u(x) = e^x$ in the right-hand side (RHS) of the above integral equation yields**:**

RHS=$1 + \int_0^x u(t)dt$=$1 + \int_0^x e^t dt$=$1 + \int_0^x [e^t]_0^x$=$1+e^x - e^0$=$e^x = u(x) = LHS$

Example 1.2. Show that $u(x) = x$ is a solution of the following Fredholm integral equation:

$$u(x) = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)u(t)dt$$

RHS= $\frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)u(t)dt$

$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)tdt$

$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\left[\frac{xt^2}{2} + \frac{t^3}{3}\right]_0^1$

$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\left[\frac{x}{2} + \frac{1}{3}\right]$ = x = u(x)=LHS

Exercises 1.2.

verify that the given function is a solution of the corresponding integral equation:

1. $u(x) = \frac{2}{3}x + \int_0^1 xtu(t)dt \quad u(x) = x$
2. $u(x) = x - \int_0^x (x-t)u(t)dt \quad u(x) = \sin x$

1.5 Taylor Series

In this section, we will introduce a brief idea on the Taylor series. Recall that the Taylor series exists for analytic functions only. Let $f(x)$ be a function that is infinitely differentiable in an interval $[b, c]$ that contains an interior point a. The Taylor series of $f(x)$ generated at $x = a$ is given by the sigma notation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

which can be written as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The Taylor series of the function $f(x)$ at $a = 0$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

1.6 Infinite Geometric Series

A *geometric series* is a series with a constant ratio between successive terms. The standard form of an infinite geometric series is given by:

$$S_n = \sum_{k=0}^{n} a_1 r^k$$

An *infinite geometric series* converges if and only if $|r| < 1$, otherwise it diverges. The the sum of infinite geometric series, for $|r| < 1$, is given by:

$$S_n = \frac{a_1}{1-r}$$

Example 1.3. Find the sum of the infinite geometric series:

$$1 + \frac{3}{5} + \frac{9}{25} + \frac{27}{125} + \cdots$$

It is obvious that the first term is $a_1 = 1$ and the common ratio is $r = \frac{3}{5}$. The sum is therefore given by:

$$S = \frac{1}{1 - \frac{3}{5}} = \frac{5}{2}$$

Example 1.4. Find the sum of the infinite geometric series:

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$$

It is obvious that the first term is $a_1 = 1$ and the common ratio is $r = -\frac{1}{3}$, $|r| < 1$. The sum is therefore given by:

$$S = \frac{1}{1 + \frac{1}{3}} = \frac{3}{4}$$