Chapter Five Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modeling of the Spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely:

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt \qquad (5.1)$$

or,

$$u(x) = f(x) + u(x) = f(x) + \lambda \int_a^x \int_a^b k(x,t)u(t)dt \qquad (5.2)$$

where *f*(*x*) and *K*(*x*, *t*) are analytic functions. It is interesting to note that (5.1) contains disjoint Volterra and Fredholm integrals, whereas (5.2) contains mixed Volterra and Fredholm integrals. Moreover, the unknown functions $u(x)$ appear inside and outside the integral signs. This is a characteristic feature of the second kind of integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of the first kind.

In this chapter, we will study some of the reliable methods that will be used for the analytic treatment of the Volterra-Fredholm integral equations of the form (5.1).

This type of equation will be handled by using the Taylor series method and the Adomian decomposition method combined with the noise terms phenomenon or the modified decomposition method.

5.1 The Series Solution Method:

The series solution method was examined before. A real function u(x) is called analytic if it has derivatives of all orders such that the generic form of the Taylor series at x = 0 can be written as:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (5.3)$$

In this section, we will apply the series solution method, which stems mainly from the Taylor series for analytic functions, for solving Volterra-Fredholm integral equations. We will assume that the solution u(x) of the Volterra-Fredholm integral equation (5.1) is analytic, and therefore possesses a Taylor series of the form given in (5.3), where the coefficients an will be determined recurrently. In this method, we usually substitute the Taylor series (5.3) into both sides of (5.1) to obtain:

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda_1 \int_a^x k_1(x,t)\left(\sum_{n=0}^{\infty} a_n x^n\right)dt + \lambda_2 \int_a^b k_2(x,t)\left(\sum_{n=0}^{\infty} a_n x^n\right)dt \qquad (5.4)$$

where T (f(x)) is the Taylor series for f(x). The Volterra-Fredholm integral equation (5.1) will be converted to a regular integral in (5.4) where instead of integrating the unknown function u(x), terms of the form $t^n, n \geq 0$, will be integrated. Notice that because we are seeking a series solution, then if f(x) includes elementary functions such as trigonometric functions, exponential functions, etc., Taylor expansions for functions involved in f(x) should be used.

We first integrate the right side of the integrals in (5.4) and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a recurrence relation in $a_j, j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geq 0$. Having determined the coefficients $a_j, j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (5.3). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher the accuracy level we achieve.

Example 5.1

Solve the Volterra-Fredholm integral equation by using the series solution method:

$$u(x) = -5 - x + 12x^2 - x^3 - x^4 + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt \quad (5.5)$$

Substituting u(x) by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (5.5) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = -5 - x + 12x^2 - x^3 - x^4 + \int_0^x (x-t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt + \int_0^1 (x+t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integrals at the right side, using a few terms from both sides, and collecting the coefficients of like powers of x, we find:

$$\begin{aligned}(a_0 + a_1 x + a_2 x^2 + \cdots)\\ &= -5 + \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 + \frac{1}{6}a_4\\ &+ \left(-1 + a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 + \frac{1}{5}a_4\right)x + \left(12 + \frac{1}{2}a_0\right)x^2 + \left(-1 + \frac{1}{6}a_1\right)x^3\\ &+ \left(-1 + \frac{1}{12}a_2\right)x^4 + \cdots\end{aligned}$$

Equating the coefficients of like powers of x on both sides of the above equation and solving the resulting system of equations, we obtain:

$$a_0 = 0, a_1 = 6, a_2 = 12, a_3 = a_4 = a_5 = \cdots = 0$$

the exact solution is therefore given by:

$$u(x) = 6x + 12x^3$$

Example 5.2

Solve the Volterra-Fredholm integral equation by using the series solution method:

$$u(x) = 2 - x - x^2 - 6x^3 + x^5 + \int_0^x tu(t)dt + \int_{-1}^1 (x+t)u(t)dt \quad (5.6)$$

Substituting u(x) by the series:

$$u(x)=\sum_{n=0}^{\infty}a_nx^n$$

into both sides of Eq. (5.6) leads to:

$$\sum_{n=0}^{\infty}a_nx^n=2-x-x^2-6x^3+x^5+\int_0^x t\left(\sum_{n=0}^{\infty}a_nt^n\right)dt+\int_{-1}^{1}(x+t)\left(\sum_{n=0}^{\infty}a_nt^n\right)dt$$

Evaluating the integrals at the right side, using a few terms from both sides, and collecting the coefficients of like powers of x, we find:

$$\begin{aligned}(a_0+a_1x+a_2x^2+\cdots)\\ &=2+\frac{2}{3}a_1+\frac{2}{5}a_3+\left(-1+2a_0+\frac{2}{3}a_2+\frac{2}{5}a_4\right)x+\left(-1+\frac{1}{2}a_0\right)x^2\\ &+\left(-6+\frac{1}{3}a_1\right)x^3+\frac{1}{4}a_2x^4+\left(1+\frac{1}{5}a_3\right)x^5+\cdots\end{aligned}$$

Equating the coefficients of like powers of x on both sides of the above equation and solving the resulting system of equations, we obtain:

$$a_0=2,a_1=3,a_2=0,a_3=-5,a_4=a_5=\cdots=0$$

the exact solution is therefore given by:

$$u(x)=2+3x-5x^3$$

Example 5.3

Solve the Volterra-Fredholm integral equation by using the series solution method:

$$u(x)=e^x-1-x+\int_0^x u(t)dt+\int_0^1 xu(t)dt \qquad (5.7)$$

Using the Taylor polynomial for e^x, substituting u(x) by the Taylor polynomial

$$u(x)=\sum_{n=0}^{\infty}a_nx^n$$

into both sides of Eq. (5.7) leads to:

$$\sum_{n=0}^{\infty}a_nx^n=\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)-1-x+\int_0^x\left(\sum_{n=0}^{\infty}a_nt^n\right)dt+\int_0^1 x\left(\sum_{n=0}^{\infty}a_nt^n\right)dt$$

and proceeding as before leads to:

$$\begin{aligned}(a_0+a_1x+a_2x^2+\cdots)\\ &=\left(2a_0+\sum_{n=1}^{\infty}\frac{1}{n+1}a_n\right)x+\frac{1+a_1}{2!}x^2+\frac{(1+2!\,a_2)}{3!}x^3+\frac{1+3!\,a_3}{4!}x^4\\ &+\frac{(1+4!\,a_4)}{5!}x^5+\frac{(1+5!\,a_5)}{6!}x^6+\cdots\end{aligned}$$

Equating the coefficients of like powers of x on both sides of the above equation and solving the resulting system of equations, we obtain:

$$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = \frac{1}{2!}, a_4 = \frac{1}{3!}, a_5 = \frac{1}{4!}, \dots$$

the exact solution is therefore given by:

$$u(x) = xe^x$$

Example 5.4

Solve the Volterra-Fredholm integral equation by using the series solution method:

$$u(x) = 1 - \int_0^x (x-t)u(t)dt + \int_0^1 u(t)dt \qquad (5.8)$$

Using the Taylor polynomial for e^x, substituting u(x) by the Taylor polynomial

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

and proceeding as before we obtain that:

$$a_0 = 1, a_1 = a_3 = a_5 = a_7 = 0, \dots$$
$$a_2 = -\frac{1}{2!}, a_4 = \frac{1}{4!}, a_6 = -\frac{1}{6!}, \dots$$

the exact solution is therefore given by:

$$u(x) = \cos x$$

Exercises 5.1

Use the series solution method to solve the following Volterra-Fredholm integral equations:

1. $u(x) = 4 - x - 4x^2 - x^3 + \int_0^x (x-t+1)u(t)dt + \int_0^1 (x+t-1)u(t)dt$
2. $u(x) = 2 + x - 2\cos x - \int_0^x (x-t)u(t)dt - \int_0^{\frac{\pi}{2}} xu(t)dt$

5.2 The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced thoroughly in this text for handling independently Volterra and Fredholm integral equations. The method consists of decomposing the unknown function u(x) of any equation into a sum of an infinite number of components defined by the decomposition series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (5.9)$$

where the components $u_n(x), n \geq 0$ are to be determined recursively. To establish the recurrence relation, we substitute the decomposition series into the Volterra-Fredholm integral equation (5.1) to obtain:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt + \lambda_2 \int_a^b k_2(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, we set the recurrence relation:

$$u_0(x) = f(x) \quad (5.10)$$

$$u_{n+1}(x) = \lambda_1 \int_a^x k_1(x,t)u_n(t)dt + \lambda_2 \int_a^b k_2(x,t)u_n(t)dt \quad ,n \geq 0 \quad (5.11)$$

Having determined the components $u_0(x), u_1(x), u_2(x), \ldots$, the solution in a series form is readily obtained upon using (5.9). The series solution converges to the exact solution if such a solution exists. We point out here that the noise terms phenomenon and the modified decomposition method will be employed in this section. This will be illustrated by using the following examples.

Example 5.5

Use the Adomian decomposition method to solve the following Volterra-Fredholm integral equation:

$$u(x) = e^x + 1 + x + \int_0^x (x-t)u(t)dt - \int_0^x e^{x-t}u(t)dt \quad (5.12)$$

Using the decomposition series (5.9), and using the recurrence relation (5.10) and (5.11), we obtain:

$$u_0(x) = e^x + 1 + x$$

$$u_1(x) = \int_0^x (x-t)u_0(t)dt - \int_0^x e^{x-t}u_0(t)dt = -x - 1 + \frac{1}{2}x^2 + \cdots,$$

and so on. We notice the appearance of the noise terms ±1 and ±x between the components $u_0(x)$ and $u_1(x)$. By canceling these noise terms from $u_0(x)$, the non-canceled term of $u_0(x)$ gives the exact solution $u(x) = e^x$, that satisfies the given equation.

It is to be noted that the modified decomposition method can be applied here. Using the modified recurrence relation:

$$u_0(x) = e^x$$

$$u_1(x) = 1 + x + \int_0^x (x-t)u_0(t)dt - \int_0^x e^{x-t}u_0(t)dt = 0$$

The exact solution $u(x) = e^x$ follows immediately.

Example 5.6

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation:

$$u(x) = x^2 - \frac{1}{12}x^4 - \frac{1}{4} - \frac{1}{3}x + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt \quad (5.13)$$

Using the modified decomposition method gives the recurrence relation:

$$u_0(x) = x^2 - \frac{1}{12}x^4$$

$$u_1(x) = -\frac{1}{4} - \frac{1}{3}x + \int_0^x (x-t)u_0(t)dt + \int_0^1 (x+t)u_0(t)dt$$

$$= \frac{1}{12}x^4 - \frac{1}{360}x^6 - \frac{1}{60}x - \frac{1}{72}$$

and so on. We notice the appearance of the noise terms $\pm\frac{1}{12}x^4$ between the components $u_0(x)$ and $u_1(x)$. By canceling the noise term from the $u_0(x)$, the non-canceled term gives the exact solution $u(x) = x^2$, that satisfies the given equation.

Example 5.7

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation:

$$u(x) = \cos x - \sin x - 2 + \int_0^x u(t)dt + \int_0^\pi (x-t)u(t)dt \qquad (5.14)$$

Using the modified decomposition method gives the recurrence relation:

$$u_0(x) = \cos x$$

$$u_1(x) = -\sin x - 2 + \int_0^x u_0(t)dt + \int_0^\pi (x-t)u_0(t)dt = 0$$

Consequently, the exact solution is given by: $u(x) = \cos x$.

Example 5.8

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation:

$$u(x) = 3x + 4x^2 - x^3 - x^4 - 2 + \int_0^x tu(t)dt + \int_{-1}^1 tu(t)dt \qquad (5.15)$$

Using the modified decomposition method gives the recurrence relation:

$$u_0(x) = 3x + 4x^2 - x^3$$

$$u_1(x) = -x^4 - 2 + \int_0^x tu_0(t)dt + \int_{-1}^1 tu_0(t)dt = -\frac{2}{5} - \frac{1}{5}x^5 + x^3$$

Canceling the noise term $-x^3$ from $u_0(x)$ gives the exact solution $u(x) = 3x + 4x^2$

Exercises 5.2

Use the modified decomposition method to solve the following Volterra-Fredholm integral equations:

1. $u(x) = x - \frac{1}{3}x^3 + \int_0^x tu(t)dt + \int_{-1}^1 t^2u(t)dt$
2. $u(t) = \sec^2 x - \tan x - 1 + \int_0^x u(t)dt + \int_0^{\frac{\pi}{4}} u(t)dt$
3. $u(x) = x^3 - \frac{9}{20}x^5 - \frac{1}{4}x + \frac{1}{5} + \int_0^x (x+t)u(t)dt + \int_0^1 (x-t)u(t)dt$