نظرية الحلقات

المحاضرة الثانية

العام الدراسي 2024-2025

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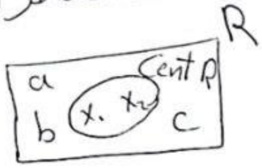
Def:- Let R be a ring. A set $\text{Cent}\, R = \{x \in R \mid r\cdot x = x\cdot r \ \forall\, r \in R\}$ is called Center of a ring R.

عبارة أخرى إن مركز الحلقة هو مجموعة كل العناصر للحلقة R التي لها خاصية التبادل مع بقية العناصر لـ R .



$x_1 a = a x_1$

- جميعهم يتبادلون مع بعضهم ومع أنفسهم .

Remark:-

1. $\text{Cent}\, R \neq \emptyset$ since $\exists\, 0 \in R$ s.t $0\cdot a = a\cdot 0 = 0$.
2. if R is comm. then $\text{Cent}\, R = R$.
3. $\text{Cent}\, R$ is $\subseteq R$ (H.W)
4. $C(y) = \{r \in R \mid r\cdot y = y\cdot r\}$

   $C(y) \neq \emptyset$ since $y\cdot ? = ?\cdot y$

   $y\cdot y = y\cdot y$.
5. $\text{Cent}\, R \leq R$

(H.W) $C(y) \leq R$.

الجداء المباشر للحلقات "The direct product of Rings"

<u>Def.:-</u> Let R and R' be two rings. Define $R\times R'$ as follows

$R\times R' = \{(a,b) \mid a\in R \text{ and } b\in R'\}$

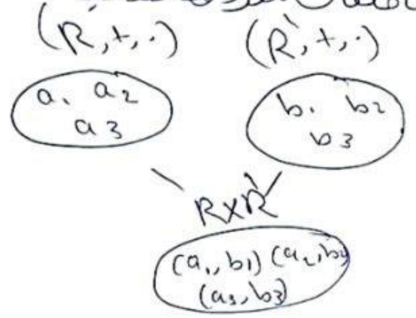
and def $\oplus$ & $\odot$ on $R\times R'$ as follows:-

$\forall\ (a,b)\ (c,d) \in R\times R'$

$(a,b)+(c,d)=(a+c,\ b+d)$

$(a,b)\cdot(c,d)=(ac,\ bd)$

- هدف الجداء المباشر هو ايجاد حلقات جديدة من حلقات معروفة سابقاً.



<u>Examples:-</u>

① $Z_2\times Z$ $\quad Z_2=\{\bar{0},\bar{1}\}$, $Z=\{0,\mp1,\mp2,\dots\}$

$Z_2\times Z=\{(a,b) \mid a\in Z_2,\ b\in Z\}$

$=\{(\bar{0},\bar{0}),(\bar{0},\bar{1}),(\bar{0},\bar{2}),\dots,(\bar{1},\bar{0}),(\bar{1},\bar{1}),\dots\}$

$\boxed{2}$ $Z_3 \times Z_4$, $Z_3 = \{\bar{0}, \bar{1}, \bar{2}\}$, $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

$Z_3 \times Z_4 = \{(a,b) \mid a \in Z_3 , b \in Z_4\}$

$Z_3 \times Z_4 = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3}), (\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2}), (\bar{1},\bar{3})$
$(\bar{2},\bar{0}), (\bar{2},\bar{1}), (\bar{2},\bar{2}), (\bar{2},\bar{3})\}$.

$R \times R' \neq \phi$ since
$\exists (0, 0') \in R \times R'$

Remark:- 1- IF R & R' are comm. ring Then So is $R \times R'$.
2- IF R & R' are with 1 So of $R \times R'$.
3- IF R & R' has no zero divisor element Then $R \times R'$ may has zero divisor element.
(In particular if $R \times R'$ are I.D So $R \times R'$ a ring not I.D).

Proof ①:- Let $(a,b), (c,d) \in R \times R'$ T.P $(a,b)(c,d) = (c,d).(a,b)$.

$\Vert$ by def (.) on $R \times R'$

$(ac . bd)$
$ac \in R$ and R is comm.
$ac = ca$
also, $bd \in R'$, R' is comm.
$bd = db$
~~(ac, bd)~~
$\therefore (ca, db)$ By def (.) on $R \times R'$ $(c,d)(a,b)$
$\therefore R \times R'$ is comm.

ooF ②: Let $(a,b) \in R \times R'$ To Find $\underline{\underline{?}} \in R \times R'$

$\Rightarrow (a,b) \cdot ? = ?(a,b) = (a,b)$.

$\therefore R$ & R' are with 1.

$1 \in R$, $1' \in R'$

We claim that $(1, 1')$ is That id. of $R \times R'$

$$(a,b) \cdot (1,1) = (1a, b1)$$

// By def id.

$$(a,b)$$

$$//$$

$$(1,1)(a,b)$$

③ Examplies: ـ تناقض نعطي مثال لإثبات العكس

$R = \mathbb{Z}$ has no Zero divisor element.

Tak $R' = R = \mathbb{Z}$.

$0 \neq (5,0)$ & $(0,3) \in \mathbb{Z} \times \mathbb{Z}$
$\quad 0 \neq$

$(5,0)(0,3) = (0,0)$

$\therefore$ both $(5,0)(3,0)$ are Zero diver. element

examplie: R has no Zero diver element
$R \times \{0\}$ also has no Zero diver element.

عدد المرات

Def:- Let R be a ring and let $a \in R$, $n \in Z_+$ Then

① $na = a + a + a - - - n$ -times.

② $(-n)a = (-a) + (-a) + - - - - - n$-times.

③ $a^n = a \cdot a \cdot a - - - - - - - n$-times.

④ $a^{-n} = a^{-1} \cdot a^{-1} \cdot a^{-1} - - - - - - n$-times.

If $a \in R$ s.t a is unit — بشرط

اذا لم يتوفر هذا الشرط تعتبر العبارة (4) خاطئة.

Pro. Let R be a ring and let $a, b \in R$, $n, m \in Z_+$ Then

① $(n+m)a = na + ma$.

② $(nm)a = n(ma)$.

③ $n(a+b) = na + nb$.

④ $n(ab) = (na)b$.

⑤ $(na)(mb) = (nm)(ab)$.

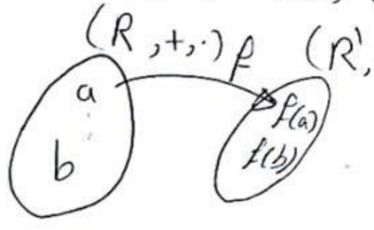
Ring homomorphism التشاكل الحلقي :-

Def:- Let $(R, +, \cdot)$ and $(R', +', \cdot')$ be two rings and let $f: R \to R'$ be a map (Function). f is called ring homo: (simply homo.) if the following condition are satisfying:-

① $f(a+b) = f(a) +' f(b)$

② $f(a \cdot b) = f(a) \cdot' f(b)$ $\Big\} \forall a, b \in R$.

الصورة تكون للجهة المقابلة



Examples:- (1) The Function $f: \mathbb{Z} \to \mathbb{Q}$ which is defined by $f(n) = n$ is homo. $\forall n \in \mathbb{Z}$.

T.P f is homo.

Let $a, b \in \mathbb{Z}$

① $f(a+b) \overset{?}{=} f(a) + f(b)$

$a + b = a + b$

② $f(a \cdot b) \overset{?}{=} f(a) \cdot f(b)$

$a \cdot b = a \cdot b$ $\therefore f$ is homo.

$\boxed{2}$ $g(x) = 2x$, not homo.

T.P f is not homo.

Let $a, b \in \mathbb{R}$

$f(a+b) \overset{?}{=} f(a) + f(b)$

$2ab \quad 2a \quad 2b$

$\neq \quad 4ab$ $\therefore$ not homo.

$\boxed{3}$ $g(x) = 2^x$ not homo.

Let $a, b \in \mathbb{R}$

$f(a+b) \overset{?}{=} f(a) + f(b)$

$2^{a+b} \overset{?}{=} 2^a + 2^b$

$2^a \cdot 2^b \neq 2^a + 2^b$ $\therefore$ not homo.

[4] $g(x) = x+1$

Let $a, b \in R$

$f(a+b) \overset{?}{=} f(a) + f(b)$

$a+b+1 \overset{?}{=} a+1 + b+1$

$a+b+1 \neq a+b+2$ ∴ not homo.

<u>Def:-</u> Let $f: R \to R'$ be a homo.

(a) if f is 1-1 Then f is mon.

(b) if f is onto then f is epi.

(c) if f is 1-1 and onto Then f is iso.

<u>Def:-</u> iso.:- Two rings R & R' is called iso. if $\exists f: R \to R'$ s.t f is iso.

<u>Prop:-</u> Let $f: R \to R'$ be a ring homo. Then:-

(1) $f(0) = 0$ where (0) is id. of R & $0'$ is id. of R'

(2) $f(-a) = -f(a)$ $\forall a \in R$.

(3) if both R & R' have id. Then $f(1) = 1'$ قەدکوی دنو 13 یداد لەنوری.

(4) $f(a^{-1}) = (f(a))^{-1}$ if a is unit.

Excercies:

① Let R be a ring with 1 and has no Zero divisor element

if $a^2 = a$ then $a = 1$

*المتحايد دائماً موجود بالحلقة (مثلاً الصفر) وإذا كانت R تمتلك متحايد فالواحد أيضاً يكون عنصر متحايد.

متحايد

② An element a in ring R is called **idempotent element** if $a^2 = a$ / an element a in R is called **nilpotent element** (عديم القوة) if $a^n = 0$ for some $n \in \mathbb{Z}_+$.

<u>examples:</u> ① $R = \mathbb{Z}_6$

$\exists\ \bar{3} \in \mathbb{Z}_6$ s.t $(\bar{3})^2 = \bar{3} \cdot \bar{3} = \bar{9} = \bar{3}$ $\therefore \bar{3}$ is idemp.

② $\exists\ \bar{2} \in \mathbb{Z}_6$ s.t $(\bar{2})^2 = 4$ $\therefore \bar{2}$ is not idemp.

nilPotente تعريف آخر لل An element $a \in R$ is called nilp. if $\exists\, n \in \mathbb{Z}_+ \Rightarrow a^n = 0$.

<u>examples:</u>

① $R = \mathbb{Z}_4$

$\exists\ \bar{2} \in \mathbb{Z}_4$, s.t $(\bar{2})^2 = \bar{2} \cdot \bar{2} = 4 = 0$, $n \in \mathbb{Z}_+$

$\therefore \bar{2}$ is nilpotente.

② $R = \mathbb{Z}_6$

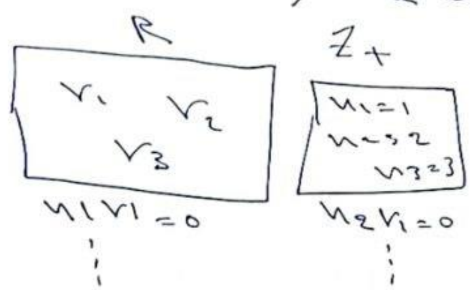
$\exists\ \bar{3} \in \mathbb{Z}_6$ s.t $(\bar{3})^2 = \bar{9} = \bar{3}$ or $(\bar{3})^3 = 27$ or $(\bar{3})^4 = 18$

$\therefore \bar{3}$ is not nilpotent. since $\forall n \in \mathbb{Z}_+$ Then $(\bar{3})^n \neq 0$

characteristic of a ring (مميز الحلقة)

Def:- Let R be a ring, If There exisit a Positive integer n s.t $na = 0$ (يتبع العملية الأولى) $\forall a \in R$, Then The smallest Positive integer (أصغر عنصر يصفر جميع عناصر المجموعة) with this Property is Called char. of the ring R. If no such Positive exisits Then R is said to be of char. zero we denote by char R.

وإذا لم يوجد إذن نقول مميز الحلقة R = صفر.



(أصغر عدد صحيح يصفر جميع عناصر المجموعة هو المميز وإذا لم يوجد إذن مميز الحلقة صفر.)

Examplie:-

$(P(X), \Delta, \cap)$

Sol: Let $A \in P(X)$

$1A = A$

$2A = A \Delta A = (A-A) \cup (A-A)$

$\phi \cup \phi = \phi$

$\therefore 2A = \phi$

$\therefore (P(X), \Delta, \cap)$ have char. 2.

(All of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ char. zero.)

- البرهنة الآتية تعطي طريقة أبسط من التعريف إعلاه لحساب مميز الحلقة بشرط ان تكون الحلقة R ذات عنصر محايد .

Theorem: ① If R a ring with 1, Then $\operatorname{char} R = n > 0$ iff n is the least positive integer s.t $n1 = 0$

* في حالة كون الحلقة R غير منتهية نحتاج لهذه البرهنة إعلاه (نتفحص المحايد إذا وجد أقل عدد موجب يصفر المحايد اذن يكون هو المميز واذا لم يوجد فالمميز صفر.

Example: $Z_8 = 8$

Sol. $Z_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

char. is 8 since $\bar{8} \cdot \bar{1} = \bar{8} = 0$ لأن فقط 8 هنا تصفر جميع عناصر المجموعة .

In general:-

$\operatorname{char} Z_n = n.$

$\operatorname{char} Z_n = n.$ دائماً

$\operatorname{char} Z_8$ is 16, No تأتي صح أو خطأ.

Theorm 2 Let R be a integral doman (حلقة). Then char R is either 0 or Primnomber.

Q: state and Prove Theorm in with char of integral doman be deter mind either 0 or Prim. no.

إذا كانت R حلقة: تمتلك محايد، ابدالية، غير قاسمة للصفر ⟹ المميز 0 or p , Prim. no.

If R is I.D Then char R $\begin{cases} 0 \\ \text{or} \\ p \end{cases}$

Proof: suppose That char R = $n > 0$ T.P n is P. Prime no. Assume that n is not Prime no.

$\Longrightarrow n = n_1 n_2$ --- (*) s.t $1 < n_1 < n_2 < n$

(تتحلل الى عوامله الأولية (يعني احدهما اصغر من الآخر).

$\because$ char R = n of R with 1 $\xrightarrow{\text{By Th.(1)}} n1 = 0$

$\xrightarrow{\text{By }(*)} (n_1(n_2)(1 \cdot 1) = 0$

$\xrightarrow{R \text{ is comm.}} (n_1 1)(n_2 1) = 0$

$\because$ R has no Zero divice element

$\Longrightarrow$ either $n_1 1 = 0$ C! or $n_2 1 = 0$ C! $\Big\}$ since: char R = 0, $n_1 < n$, n_2 n

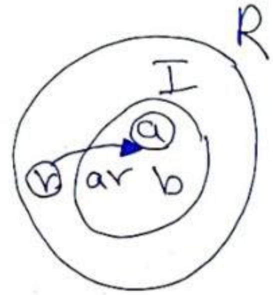
$\therefore$ n is Prime no.

Ideals :- المثالي (٢١)

Def :- Let R be a ring, and Let I be a non-empty Subset of R. I is Called an ideal of R if it Satisfying The following :-

1- $a-b \in I \quad \forall a,b \in I$

2- $\left.\begin{matrix} ra \in I \\ ar \in I \end{matrix}\right\} \begin{matrix} \forall r \in R \\ \forall a \in I \end{matrix}$



- هو حلقة جزئية محصنة لها خاصية سحب العناصر من R الى I.

Q :- Give Ration between The ideal and subring.

Sol. :- Every ideal $\longrightarrow$ subring

$\not\longleftarrow$

العكس لا يصح.

Proof:- $\Rightarrow$ Let I be ideals of $R \rightsquigarrow R$ is subring of I satisf The following:-

① $a-b \in I \quad \forall a,b \in I$

② $ra \in I \quad \forall r \in R, \forall a \in I$

In Particular when $r \in I$

$\therefore$ I is subring of R.

$\Leftarrow$ not every subring is ideal

Example:- Z is subring of Q

But Z is not ideals of Q since $\exists\ a = 3 \in Z$ & $r = \frac{1}{2} \in Q$ since $ar = 3 \cdot \frac{1}{2} \notin Z$.

Proof:- $\Rightarrow$ Let I be ideals of $R \rightsquigarrow R$ is subring of I satisf The following:-

① $a-b \in I \quad \forall a,b \in I$

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Example:- Z is subring of Q

But Z is not ideals of Q since $\exists\, a = 3 \in Z$ & $r = \frac{1}{2} \in Q$ since $ar = 3 \cdot \frac{1}{2} \notin Z$.

Examplie:

$R = \mathbb{Z}_8$, $\{\bar{0}, \bar{4}\} = (\bar{4}) = 4\mathbb{Z}_8$

$(\bar{4})$ is ideal of $\mathbb{Z}_8$ since $(\bar{4})$ is subring of $\mathbb{Z}_8$ and

$$\left.\begin{array}{c} \bar{0}\cdot\bar{0} \\ \bar{0}\cdot\bar{4} \\ \bar{1}\cdot\bar{0} \\ 1\cdot\bar{4} \\ \bar{2}\cdot\bar{0} \\ \bar{2}\cdot\bar{4} \\ \bar{3}\cdot\bar{0} \\ \bar{3}\cdot\bar{4} \\ \vdots \\ \bar{7}\cdot\bar{0} \\ \bar{7}\cdot 4 \end{array}\right\} \in (\bar{4})$$

$\therefore \bar{4}$ is ideal of $\mathbb{Z}_8$.