نظرية الحلقات

المحاضرة الخامسة

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Theorm:- Let $f: F \to F'$ be a ring homo. from F to F'

Then either f is mono (متباينة) or f is trivial (تافهة) homo.

التشاكل بين حقول يكون اما 1-1 or صفر.

Proof:-

$f: F \to F'$ (ring homo)

mono 1-1 or triveal صفر

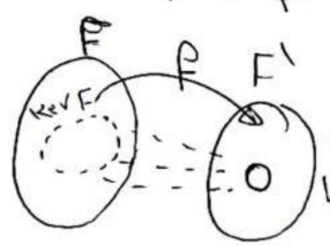
Ker f is an ideal of F

$\because$ F is Field is f has only ideals (0) & F.

$\therefore$ We have Two Cases:

either $\ker f = (0)$ By f is mono $\Longleftrightarrow \ker f = (0)$ $\Longrightarrow$ f is mono

or $\ker f = F$ $\xrightarrow{\text{By def of triveal homo.}}$ f is trival



التشاكل هو مجموعة
كل العناصر التي صورتها
تساوي صفر.

Prop. $f(I \cap J) \subseteq f(I) \cap f(J)$ اقل

if $\ker f \subseteq I$ or J Then $f(I) \cap f(J) \subseteq f(I \cap J)$.

Certain types of ideals:- أنواع معينة من المثاليات:

① maximal ideal.
② Prime ideal.
③ Primary.
④ semi Prime ideal.

- هنالك أنواع معينة من المثاليات التي لها دور في نظرية الحلقات منها المثالي الأعظم والمثالي الأولي والمثالي الابتدائي والمثالي شبه الأولي.

المثالي الأعظم:

Def:- Let R be a ring and let I by an ideal of R. Then I is maximal if it is satisfying the following:-

① $I \neq R$ (i.e I is Proper ideal of R).

② if J be an ideal of R s.t $J \supsetneq I$ Then $J = R$.

(i.e) $\nexists J \subsetneq R \Rightarrow I \subsetneq J$

* R لا تكون أعظم في نفسها

* لا يوجد مثالي اكبر منه يحويه بشكل فعلي في R وان وجد فيجب ان يكون كل R.

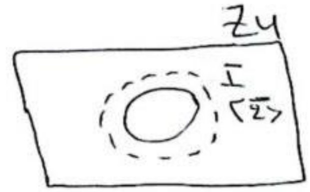
$$I \text{ is maximal if } \begin{cases} I \neq R \; (I \subsetneq R) \\ \text{if } J \subsetneq R \Rightarrow I \subsetneq J \\ \Rightarrow J = R \end{cases}$$

مثالي

Examples:

① $R = \mathbb{Z}_4$, $I = \langle \bar{2} \rangle = \{\bar{0}, \bar{2}\}$

$\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

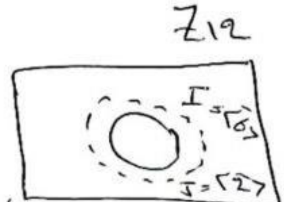


(i) $\langle \bar{2} \rangle \neq \mathbb{Z}_4$

(ii) $\nexists J \trianglelefteq \mathbb{Z}_4 \Rightarrow \langle \bar{2} \rangle \subsetneq J$

$\therefore$ $I = \langle \bar{2} \rangle$ is maximal.

② $\mathbb{Z}_{12} = R$, $I = \langle \bar{6} \rangle = \{\bar{0}, \bar{6}\}$



(i) $\langle \bar{6} \rangle \subsetneq \mathbb{Z}_{12}$ or $\langle \bar{6} \rangle \neq \mathbb{Z}_{12}$

(ii) $\exists J = \langle \bar{2} \rangle$ in $R = \mathbb{Z}_{12}$ s.t. $\langle \bar{6} \rangle \subsetneq \langle \bar{2} \rangle$ $\Big\{$ $\mathbb{Z}_{12}$ تحوي مثاليات $\langle\bar{2}\rangle$, $\langle \bar{6} \rangle$, $\langle \bar{3} \rangle$.

$\therefore$ not maximal since we find ideal contians $I = \langle \bar{2} \rangle$ proper.

H.W: is $\mathbb{Z} \times \{0\}$ is maximal in $\mathbb{Z} \times \mathbb{Z}$?

* المثالي الأعظم لا يعني انه اكبر مثالي مثالي في الحلقة
وايضاً لا يعني انه الوحيد فيها كما موضح في الأمثلة الآتية :-

Examples:-

① $R = \mathbb{Z}_6$, $I = \langle \bar{2} \rangle$ is maximal?

$\{\bar{0}, \bar{2}, \bar{4}\}$

Yes, since ⓘ $\langle \bar{2} \rangle \subsetneqq \mathbb{Z}_6$

ⓘⓘ $\nexists J \leq \mathbb{Z}_6 \Rightarrow \langle \bar{2} \rangle \subsetneqq J$

Also $I = \langle \bar{3} \rangle$ is maximal

since $\langle \bar{3} \rangle \subsetneqq \mathbb{Z}_6$ and $\nexists J \leq \mathbb{Z}_6$

$\Rightarrow \langle \bar{3} \rangle \subsetneqq J$

② $R = \mathbb{Z}$

$2\mathbb{Z} \supsetneqq 4\mathbb{Z} \supsetneqq 8\mathbb{Z} \supsetneqq \cdots$

$3\mathbb{Z} \supsetneqq 6\mathbb{Z} \supsetneqq 6\mathbb{Z} \supsetneqq \cdots$

$4\mathbb{Z}$ is maximal in $2\mathbb{Z}$ but not maximal in $\mathbb{Z}$.

$3\mathbb{Z}$ 〃 〃 〃 〃 〃 〃 〃 〃.

$6\mathbb{Z}$ 〃 〃 〃 $3\mathbb{Z}$ 〃 〃 〃 〃 〃.

*(الصفر) مرات يكون مثالي أعظم او لا حسب الحلقة التي تحويه.

(0) is maximal in $\mathbb{Z}$ (since $0 \subsetneqq \mathbb{Z}$ and $(0) \subsetneqq 2\mathbb{Z}$)

(0) is maximal in F $\left(\begin{array}{l} \text{since } (0) \subsetneqq F \text{ and} \\ \nexists J \leq F \Rightarrow (0) \subsetneqq F \end{array}\right)$.

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The following Theorm Give anther charactionZution of maximal ideal :-

Theorm :- Let I be a proper ideal of R Then I is maximal if and only if $(I, a) = R$ $\forall a \in R/I$

دورية Cycli · $a = (ra \mid r \in R)$.

Note :- ① from now to on every ring is Comm. with 1

② if I is an ideal of R and $a \in R$, Then
$(I, a) = \{ i + ra \mid i \in I, r \in R \}$.

$a \notin I$

$I \subsetneq R$ is max $\Longleftrightarrow (I, a) = R \quad \forall a \in R / \bar{I}$

يعني $\bar{I}$ كبيرة لدرجة لاتستطيع اضافة شيئا اخر (a) الا

<u>Proof</u> :- $\Rightarrow$ We have I is maximal and T.P

$(I, a) = R \quad \forall a \in R / I$.

مثالي

It is clear that : $I \subsetneq \underbrace{(I, a)}_{J}$ $\qquad (I,a) = \{i + ra \mid i \in I, r \in R\}$

But J is maximal $\Rightarrow (I, a) = R$

$\Leftarrow$ We have $(I, a) = R$ and T.P I is maximal.

(i) $I \subsetneq R$ (véro)

(ii) Let J be an ideal of $R \Rightarrow J \supsetneq I \xrightarrow{T.P} J = R$

$\Rightarrow \exists a \in J \,/\, a \notin I$

By assum. $(I, a) = R$

But $1 \in R \Rightarrow 1 \in (I, a) \Rightarrow 1 = \underset{I}{i} + \underset{J}{r_0 a}$, $r_0 \in R$

لأنها حلقة ذات محايد.

$\Rightarrow 1 \in J \Rightarrow J \in R$ $\qquad$ $\underbrace{I + J}_{J}$ (since J is an ideal).

(if $1 \in K$ Then $K = R$)

$\therefore$ J is maximal ideal (By def of maximal ideal).

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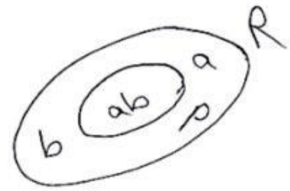
Prime ideal.

Def. Let R be a ring and Let P be an ideal of R. P is Called Prime if:

$\forall a, b \in R$ s.t $ab \in P$ then either $a \in P$ or $b \in P$

*متى يكون مثالي أولي يجب a or b أو كلاهما موجودة في p.

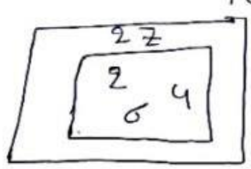


P is Prime ideal if $\forall a, b \in R \ni ab \in P \Rightarrow$ either $a \in P$ or $b \in P$.

Examples: ① $2\mathbb{Z}$ is Prime ideal of $\mathbb{Z}$?

Sol. $2\mathbb{Z} = \{0, \mp 2, \mp 4, \mp 6 \cdots\}$ مضاعفات الاثنين.

$R = \mathbb{Z}$



$2 <^{1}_{2}$ —

$4 <^{1}_{4}$ —

$4 <^{2}_{2}$ —

$6 <^{6}_{1}$ —

$6 <^{2}_{3}$ —

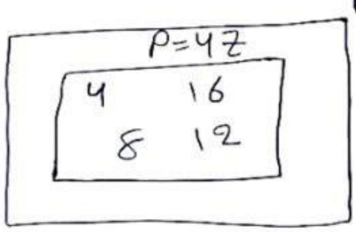
نحلل العدد الى عوامله الأولية يجب ان يكون احدى العوامل أو كلاهما ينتمي الى p.

$\therefore 2\mathbb{Z}$ Prime ideal since $\forall a, b \in R$ s.t $ab \in P \Rightarrow a \in P$ or $b \in P$.

② $P = 4\mathbb{Z}$, $R = \mathbb{Z}$ is Prime ideal.

Sol. $4\mathbb{Z} = \{0, \mp 4, \mp 8, \mp 12 \cdots\}$ مضاعفات 4.

$R = \mathbb{Z}$



$4 <^{1}_{4}$

$4 <^{2}_{2} \notin 4\mathbb{Z}$

اذا واحدة لا تنتمي يتوقف الكل.

$\therefore 4\mathbb{Z}$ is not Prime ideal so, $a = 2 \notin 4\mathbb{Z}$ and $b = 2 \notin 4\mathbb{Z}$

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Q: Does $\{0\}$ is Prime ideal in any ring R?

Sol. No:

example:- ① $\langle 0 \rangle$ is Prime ideal in $\mathbb{Z}$

$\forall a,b \in \mathbb{Z} \Rightarrow ab \in \langle 0 \rangle$ than either

$a \in \langle 0 \rangle$ or $b \in \langle 0 \rangle$

$a \in 0$ or $b \in 0$

Now $ab = 0$ ($\because \mathbb{Z}$ is I.D)

$\therefore$ either $a = 0 \Rightarrow a \in \langle 0 \rangle$

or $b = 0 \Rightarrow b \in \langle 0 \rangle$

② $\langle 0 \rangle$ is not Prime ideal in $\mathbb{Z}_6$

since:- $\exists\, 2, 3 \in \mathbb{Z}_6 \Rightarrow 2 \times 3 \in \langle 0 \rangle$

but $2 \neq 0$ & $3 \neq 0$

Remark:- $\langle 0 \rangle$ is Prime ideal in any I.D.

Proof: Let R is I.D

Let $a, b \in R \Rightarrow ab \in \langle 0 \rangle$

$\Rightarrow ab = 0 \quad \because R$ is I.D

$\therefore$ either $a = 0 \Rightarrow a \in \langle 0 \rangle$

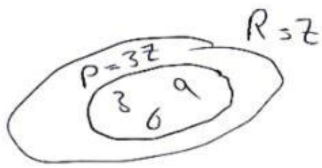
or $b = 0 \Rightarrow b \in \langle 0 \rangle$

$\therefore \langle 0 \rangle$ is Prime ideal.

Remark:- pZ is Prime ideal in Z, $\forall$ Prime no. (P)

where $P = \{5Z, 2Z, 3Z, 7Z ----\}$

$3Z = \{0, \mp 3, \mp 6, \mp 9 ----\}$



$3 <^{1}_{3}$ — , $6 <^{1}_{6}$ — , $6 <^{2}_{3}$ — , $9 <^{1}_{9}$ —

$9 <^{3}_{3}$ — $\therefore$ $3Z$ is Prime ideal of Z.

Theorm:- Every maximal ideal is Prime But the Converse is not true.

$$\max \longrightarrow \text{Prime}$$
$$\not\longleftarrow$$

$\Longrightarrow$ Every maximal ideal is Prime ideal.

Let I be a maximal ideal of R then by (I is max. iff R/I is field).

we get R/I is field.

$\therefore$ R/I is I.D (every field is I.D).

$\therefore$ I is Prime ideal. (I is Prime iff R/I is I.D)

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$\Leftarrow$ not every Prime ideal is max.

example:-

$R = \mathbb{Z}\times\mathbb{Z}$, $P = \mathbb{Z}\times\{0\}$

P is Prime ideal but not max. ideal.

* P is not max. since $\exists\ I = \mathbb{Z}\times 2\mathbb{Z} \leq \mathbb{Z}\times\mathbb{Z}$ and

$\mathbb{Z}\times\{0\} \leq \mathbb{Z}\times 2\mathbb{Z} \lneq \mathbb{Z}\times\mathbb{Z}$

** P is Prime ideal.

Let $(a,b)(c,d) \in \mathbb{Z}\times\mathbb{Z} \Rightarrow (a,b)(c,d) \in \mathbb{Z}\times\{0\}$

Now, $(a,b)(c,d) = (t,0)$, when $(t,0) \in \mathbb{Z}\times\{0\}$ T.P either $(a,b)\in\mathbb{Z}\times\{0\}$ or $(c,d) \in \mathbb{Z}\times\{0\}$

$(ac, bd) = (t,0)$

by def. of $(\cdot)$

$\left.\begin{array}{l} ac = t \\ bd = 0 \end{array}\right\}$ by def. of $(\cdot)$

b & $d \in \mathbb{Z}$ & $\mathbb{Z}$ is I.D $\Rightarrow$ either $b=0$ or $d=0$

If $b=0 \Rightarrow (a,b)=(a,0) \Rightarrow (a,b)\in\mathbb{Z}\times\{0\}$

If $d=0 \Rightarrow (c,d)=(c,0) \Rightarrow (c,d)\in\mathbb{Z}\times\{0\}$

$\therefore\ \mathbb{Z}\times\{0\}$ is Prime ideal.

Theorm:- Let R be a ring and Let I be an ideal of R then I is maximal iff $\frac{R}{I}$ is I.D.

Theorm:- Let R be a ring and I be ideal of R the ideal is Prime in R iff R/I is I.D.

Proof:- $\Rightarrow$ we have I is Prime ideal T.P R/I is I.D

Let $(a+I)$ & $(b+I) \in \frac{R}{I} \Rightarrow (a+I)(b+I) = I$

T.P either $a+I = I$ or $b+I = I$

Now, $(a+I)(b+I) = I$

$$\begin{aligned} (a+I)(b+I) &= I \\ ab + I &= I \quad (\text{By def. of } (\cdot)). \\ ab &\in I \quad (\text{since: } aH = H \Longleftrightarrow a \in H). \end{aligned}$$

$\because$ I is Prime ideal

$\therefore$ either $a \in I \Rightarrow a + I = I$

or $b \in I \rightarrow b + I = I$

$\therefore$ R/I is I.D.

$\Leftarrow$ We have R/I is I.D and T.P I is Prime ideal.

Let $a, b \in R \ni ab \in I$ T.P either $a \in I$ or $b \in I$

$ab \in I \Rightarrow ab + I = I \quad (aH = H \Leftrightarrow a \in H)$

$(a+I)(b+I) = I$ (by def. of $(\cdot)$ in R/I)

$\because R/I$ is I.D.

$\therefore$ either $a + I = I \Rightarrow a \in I$

or $b + I = I \Rightarrow b \in I$

$\therefore I$ is Prime ideal.

* تبتر البرهنة اعلاه مع البرهان تشخيص
المثالي الاولي .