المحاضرة الخامسة

Finite Nilpotent Groups and Solvable Groups

First: Finite Nilpotent Groups

Definition: Let G be a finite group. G is said to be **a Nilpotent group** if every Sylow group of G is a normal subgroup of G.
Examples.

Proposition:

Every finite abelian group is Nilpotent.

Proof:
let G be a finite abelian group and H be a subgroup of G.
$H \Delta G$ (every subgroup of an abelian group is normal)
Every Sylow subgroup of an abelian group is normal.
G is a Nilpotent group (by definition of a Nilpotent group).

Remark: Every P-group G is Nilpotent (In fact G contains only one Sylow group G and hence $G \Delta G$).

Exercise:

- 10 is Nilpotent group.
- S3 is not Nilpotent group.

Proposition: A finite group G is Nilpotent iff for all prime number p \ o(G), then G contains a unique Sylow P-group.
Proposition: Every Nilpotent group can be represented as an internal direct product of its Sylow P-group.
(i.e. if S1, S2, …, Sn are Sylow P-group of the group G, then G = S1 S2 … Sn).

Remarks:

1. If both G1 and G2 are P-group, then G1 x G2 is Nilpotent group.
2. The center of Nilpotent group is not trivial.

3. Every subgroup of the Nilpotent group is Nilpotent.

Definition: Let G be a group and H be a subgroup of G. then H is said to be the **maximal** subgroup if whenever H ≤ K ≤ G, then either H = K or G = K.

Remark:

- If G is a finite group, then G contains a maximal subgroup
- 2. Every subgroup is contained in one (or more one) maximal subgroups.

Proposition: Let G be a finite group and S Sylow P-group. Let N0(S) = N. If H is a subgroup of H such that N ≤ H ≤ G, then NG(H) = H.

Proof: Let x G such that x NG(H). So xHx-1 = H. since each of S and xSx-1 is a Sylow subgroup of H, then by the second Sylow theorem, S and xSx-1 are a conjugate group. Hence there is h H such that h-1x-1Sxh = S. That implies xh N(S) ≤ H, then xh H. Hence x H.

Proposition: Let G be a finite group. If every maximal subgroup of G is normal, then G is the Nilpotent group.

Remark: If G is a finite Nilpotent group, then every maximal subgroup of G is normal.

Second: Solvable Groups

Definition: Let G be a group. Then G is said to be **a solvable group** if there exists a finite set {G0, G1, G2, …, Gr}of subgroups of G that satisfy the following conditions:

a) $G = G_0, G_1, G_2, \ldots, G_r = \{e\}$

b) Every subgroup G_i is a normal subgroup in Gi-1

c) A quotient group is an abelian group for all i, $0 \leq i \leq r - 1$.

Examples:

- Every abelian group is solvable.
- Every P-group is solvable.

Proposition: Every subgroup of a solvable group is solvable.

Proof: Let H be a subgroup of a solvable group G and let $\{G = G_0, G_1, G_2, \ldots, G_r = \{e\}\}$ be a set of subgroups of G satisfying the conditions of solvable group. Put $H_i = G_i \cap H$. then the set $\{ H = H_0, H_1, H_2, \ldots, H_r = \{e\}\}$satisfy the following conditions

1. $H = H_0\ H_1\ H_2 \ldots H_n = \{e\}$
2. Since $H \Delta H$ and $G_i \Delta G_{i-1}$, then $Hi \Delta H_{i-1}$
3. H.W.

Proposition: Every finite Nilpotent group is solvable.

Remark: The group S_3 is solvable not Nilpotent group.

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