المحاضرة الرابعة

Quotient Group

Here is a fundamental construction of a new group from a given group.

<u>Theorem (1):</u> Let G/K denote the family of all the cosets of a a K bK = abK

<u>Definition (2):</u> The group G/K is called the quotient group; when G is finite, its order G/K is the index [G:K] (presumably, this is the reason quotient groups are so-called).

We can now prove the converse of Proposition 2.91(ii).

<u>Proposition (3):</u> Every normal subgroup K of a group G is the kernel of some homomorphism.

<u>Proof:</u>

Define the natural map $\pi: G \to G/K$ by $\pi(a) = a K$. With this notation, the formula a K bK =abK can be rewritten as $\pi(a)\pi(b) = \pi(ab)$; thus, π is a (surjective) homomorphism. Since K is the identity element in G/K,

$$\ker \pi = \{a \in G : \pi(a) = K\} = \{a \in G : a K = K\} = K$$

Isomorphism Theorems

The following theorem shows that every homomorphism gives rise to an isomorphism and that quotient groups are merely constructions of homomorphic images.

First Isomorphism Theorem

If $f: G \to H$ is a homomorphism, then:

$$G/\ker f \cong \text{im } f$$

Where im f = f(H). In more detail, if we put ker f = K , then the function ϕ : $G/K \to f(H)$ is given by:

ϕ: a K $r\to$ f (a) for each $a \in G$, is an isomorphism.

Proof:

It is clear that ker f is a normal subgroup of G, and we can easily show that ϕ is well-defined. Let us now see that ϕ is a homomorphism. Since f is a homomorphism and $\phi(aK) = f(a)$,

$$\phi(aK\,bK) = \phi(abK) = f(ab) = f(a)\,f(b) = \phi(aK)\phi(bK).$$

Also, ϕ is surjective and injective Therefore, $\phi: G/K \to \text{im } f$ is an isomorphism.

Remark (2):

1. Here is a minor application of the first isomorphism theorem. For any group G, the identity function $f: G \to G$ is a surjective homomorphism with ker f = $\{1\}$. By the first isomorphism theorem, we have

$$G/\{1\} \cong G$$

2. Given any homomorphism $f:G \to H$, one should immediately ask for its kernel and its image; the first isomorphism theorem will then provide an isomorphism

   $G/\ker f \cong \text{im } f$. Since there is no significant difference between isomorphic groups, the first isomorphism theorem also says that there is no significant difference between quotient groups and homomorphic images.

Proposition (3):

1. If H and K are subgroups of group G, and if one of them is a normal subgroup, then HK is a subgroup of G. Moreover, HK = KH.
2. If both H and K are normal subgroups, then HK is a normal subgroup.

Proof:

1. Assume first that K is normal in G. We claim that HK = KH. If $hk \in HK$, then:

$$hk = hkh^{-1}h = k_1 h \in KH$$

where $k_1 = hkh^{-1}$, then $k_1 \in K$, because K is normal subgroup

Hence, HK = KH. For the reverse inclusion, write $kh = hh^{-1}kh = hk_2 \in HK$, where $k_2 = h^{-1}kh$.

(Note that the same argument shows that HK = KH if H is normal subgroup of G.)

We now show that HK is a subgroup. Since $e \in H$ and $e \in K$, we have $e = e \cdot e \in HK$.

If $hk \in HK$, then $(hk)^{-1} = k^{-1} h^{-1} \in KH = HK$. If $hk, h_1k_1 \in HK$, then $h_1^{-1} kh_1 = ke \in K$ and

$$Hkh_1 k_1 = hh_1(h_1^{-1} kh_1)k_1 = (hh_1)(kek_1) \in HK.$$

Therefore, HK is a subgroup of G.

2. If $g \in G$, then:

$$ghkg^{-1} = (ghg^{-1})(gkg^{-1}) \in HK$$

Therefore, HK is normal in G.

References

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