نظرية الحلقات

المحاضرة الثالثة

العام الدراسي 2024-2025

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Remark:- In any ring R There is Two trivial ideals which are $\{0\}$ and R it self.

- في اي حلقة R يوجد مثاليين تافهين هما {0} و R نفسها.

Pro. Let R be a ring with 1 And Let I be an ideal of R. Then :-

① if $1 \in I$ Then $I = R$ $\quad I \subseteq R$, $R \subseteq I$ (i.e if $I \subsetneq R$ Then $1 \notin I$)

② if $a \in I$, with a is unit element Then $I = R$

Proof 1

Suppose $1 \in I$ To Prove That $I = R$ it's clear $I \subseteq R$ ... ⓘ

Let $r \in R$ $\quad \because 1 \in I \longrightarrow 1 \cdot r \in I$ $\quad (ra \in I \ \forall a \in R, \forall a \in I)$

$\because I$ is ideal of R

$\Rightarrow 1 \cdot r = r \cdot 1 = r \in I$

$\therefore R \subseteq I$ ... ⓘⓘ

From ⓘ & ⓘⓘ we get $I = R$.

Proof 2

$\because a$ is unit element

$\therefore a\,a^{-1} \in I$ By def of inv. $\{a a^{-1} = a^{-1} a\}$

$1 \in I$ $\qquad \qquad \overset{\shortparallel}{I}$

$\therefore I = R$ $\qquad$ I is ideal.

كل هناك فعلي لا يمكن أن يحوي على عنصر محايد أو عنصر قابل للعكس.

virtually everythink can not have a neutral or reversible element.

جوابه : النقطتين أكواص مع البرهان.

Def. An ideal I of $\mathbb{R}$ is called proper (فعلي) if $I \subseteq R$ & $I \neq R$ (i.e $I \underset{\text{فعلي}}{\subsetneq} R$)

(0) is proper ideal of R. Yes

R is proper ideal of R. No

Prop. Let R be a ring and let I and J two ideals of R Then $I \cap J$ is ideal of R. Does $I \cup J$ is also ideal of R. $I \cap J = \{a \mid a \in I \text{ \& } a \in J\}$ $\neq \emptyset$ since $\exists\, 0 \in I \cap J$).

H.W I & J is ideals of R Than $I \cap J$ is also ideal of R.

Def. The union of Two ideals not nessesarly ideal for examplie : both $2\mathbb{Z}$ & $3\mathbb{Z}$ are ideals of $\mathbb{Z}$ but $2\mathbb{Z} \cup 3\mathbb{Z}$ is not ideal of R since $\exists\, 2 \in 2\mathbb{Z}$ & $3 \in 3\mathbb{Z}$ but $3-2 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$.

ملاحظة. تقاطع مثاليين يكون مثالي.
اتحاد مثاليين مو بالضرورة يكون مثالي.
يكون اتحاد مثاليين مثالي اذا كانت R (chain).

equationo.- when The union of Two ideals of R is also ideal.

ans. if R is chain (i.e. for any Two ideals I & J of R, either $I \subseteq J$ or $J \subseteq I$)

Proof. if $I \subseteq J$ or $J \subseteq R$ Then $I \cup J$ is ideal.

if $I \subseteq J$ Than $I \cup J = J$ whice is ideal of R.

$\therefore I \cup J$ is ideal.

If R is chan Than the unio of any Two ideal is also ideal.

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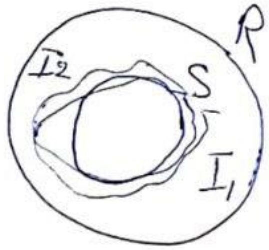
Principal ideal:- المثالي الرئيسي

نقص تعريف
cyclic group.

Def.: Let R be a ring and Let S be a nonempty subset of R A set:

$(S) = \cap\{ I \mid \text{where } I \text{ is an ideal of } R \text{ containg } S \}$ تحوي

is Called ideal generated by S



$\phi = S \subseteq R$

generated by

$\langle S \rangle = \cap \left\{ I / \begin{array}{l} I \text{ is ideal of } R \\ I \supseteq S \end{array} \right\}$

Prop. ① $(S) \neq \phi$ (since $0 \in (S)$) and $S \subseteq \langle S \rangle$)

② is The smallest ideal Containg S.

③ S is ideal $\longleftrightarrow (S) = S$

فائدة هذا التعريف يعني انه بالامكان تكوين مثالي من اي مجموعة غير خالية S.

Def. if S is finit set, say $S = \{a_1, a_2, \dots a_n\}$
Then (S) is called **Finitely generated** (simple f.g)

اذا كانت S التي بدأت بها منتهية موشرط يكون المثالي المتولد منها منتهياً.
أن المثالي المنتهي للتولد لا يكون بالضرورة منتهياً مثلاً.

Z is an ideal of Z
which is f.g but it is not finite.

Def. Let R be a ring and let $S = \{a\}$, then $\langle S \rangle$ is Principal ideal (ضابطي) That is if $\langle S \rangle$ Containing one element say Then $(S) \equiv \langle a \rangle$ is called **Principle ideal**
and $(a) = \{ra \mid r \in R\}$

* صوز للي المثالي المتولد بواسطة عنصر واحد.

if $w \in (a) \rightarrow w = ra, r \in R$

Principle $\Longrightarrow$ Finitely
$\Longleftarrow$ والعاكس غير صحيح

$w \in (a)$
$= \{a^0, a^1, a^2 \dots\}$
$= \{0, a, a+a \dots\}$

Examples $R = Z \oplus Z$
$= (1,0), (0,1)$
R is f.g but not Prime.

xamplie :

$(Z_4, +, \cdot)$, $I = \langle \bar{2} \rangle$

$\langle \bar{2} \rangle = \{ \bar{2}^0, \bar{2}^1, \bar{2}^2, \bar{2}^3 \dots \}$

$= \{ \bar{0}, \bar{2}, \bar{2}+\bar{2}, \bar{2}+\bar{2}+\bar{2} \dots \}$

$= \{ \bar{0}, \bar{2} \}$

ساحة مثالي رئيسي   حلقة مثالي رئيسي

Give example about P.I.D and not P.I.R

$I \cdot D \longrightarrow I \cdot R$

$\not\longleftarrow$

proof :- $\Leftarrow$ every ideal of R is Pincipel and R is I.D $\rightarrow$ R is P.I.D

$\Leftarrow$ العكس لا يصح

For example :- Z_6 is P.I.D but not P.I.R since Z_6 is not I.D in fact, $\exists$ zero diviers elemant, what is $\bar{2}$.

Def: Let R be a ring in whic every ideal is Principal Than R is called (Principal ideal ring simply (P.I.R).

And if R is integral domain R is called Principle ideal domain simply (P.I.D)

* اذا كانت R حلقة كل مثالي فيها رئيسي فان R تسمى حلقة مثالي رئيسي.
* اذا كانت R ساحة فان R تسمى ساحة مثالي رئيسي.

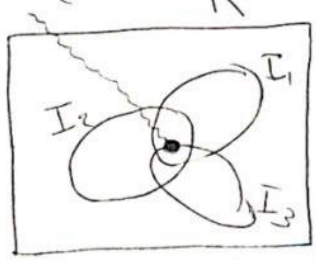
$\mathbb{Z} = (1)$

$2\mathbb{Z} = (2)$ the genar

$3\mathbb{Z} = (3)$ $n\mathbb{Z} = (n)$.

id تتقاطع جميعا ب

R



كل مثالياتها رئيسية نقول عنها حلقة مثالي رئيسي.

P.I.R

مولدة بواسطة عنصر واحد.

* عند دمج الحلقة الحسن السابقة والحلقة الحسن P.I.R تصبح

P.I.D

Def: Let I_1 and I_2 be two ideals of R A set:-

$$I_1 + I_2 = \{a+b \mid a \in I_1 \text{ and } b \in I_2\}$$

is Called The sum of I_1 & I_2

Note:

It is clear That ① $I_1 + I_2 \neq \phi$ since $\exists\, 0 + 0 \in I_1 + I_2$

② if both I_1 & I_2 are ideals $\rightarrow$ So is $I_1 + I_2$.

H.W Let I_1 & I_2 be two ideals T.P $I_1 + I_2$ is ideals.

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Example :-

$R = Z_{10}$, $I_1 = \langle \bar{2} \rangle$, $I_2 = \langle \bar{5} \rangle$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\}$, $\langle \bar{5} \rangle = \{\bar{0}, \bar{5}\}$

$$I_1 + I_2 = \left\{ \begin{array}{l} (\bar{0},\bar{0}), (\bar{0},\bar{5}), (\bar{2},\bar{0}), (\bar{2},\bar{5}), (\bar{4},\bar{0}), (\bar{4},\bar{5}) \\ (\bar{6},\bar{0}), (\bar{6},\bar{5}), (\bar{8},\bar{0}), (\bar{8},\bar{5}) \end{array} \right\}$$

$$= \{\bar{0}, \bar{2}, \bar{4}, \bar{5}, 6, \bar{8}, 9\} = Z_{10}$$

H.W Find $\langle \bar{3} \rangle$ in Z_{10}, $\langle \bar{5} \rangle$ in Z_8 , $\langle \bar{3} \rangle$ in Z_{12}

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Def: Let R be a ring and let I and J be ideals

$$I \cdot J = \left\{ \sum_{\text{Finite}} a_i \cdot b_i \;/\; a_i \in I \text{ and } b_i \in J \right\}$$

توضيح

$$\left.\begin{array}{l} w \in IJ \Rightarrow w = \sum_{i=1}^{n} a_i b_i \\ h \in IJ \Rightarrow h = \sum_{i=1}^{m} a_i b_i \end{array}\right\}$$

كتابة عنصر موجود
في الضرب

is called The Product of I and J

Note:

① $IJ \neq \phi$

② IJ is ideal of R.

*

if $I = J$ so $IJ = I^2$ Particular case!

In general: if $I_1 = I_2 = \cdots I_n = I \cdot I \cdots I^n$

(42)

Def.: Let I be an ideal of a ring R. I is called nil ideal (مثالي عديم القوة أو مثالي صفري) if every element of I is nilpotent. In other word $I^n = 0$ for some $n \in \mathbb{Z}_+$

Example: In the ring $\mathbb{Z}_8$, the ideal $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is nil ideal since every element of I is nilpotent element

$(0)^4 = 0$, $(\bar{2})^3 = 0$, $(\bar{4})^2 = 0$, $(\bar{6})^3 = 0$, $(\bar{4})^3 = 0$

So, $I^3 = \bar{0}$ أصغر عدد صحيح يصفر الجميع هو 3

Example:

$R = \mathbb{Z}_6$, $I = \langle \bar{3} \rangle \leq \mathbb{Z}_6$ is nil ideal

ans. No, since

$\langle \bar{3} \rangle = \{\bar{0}, \bar{3}\}$

$(\bar{3})^n \neq 0 \ \forall n \in \mathbb{Z}_6$

$\therefore I$ is not nil ideal

$\bar{J} = \{\bar{0}, \bar{4}\}$

$0^4 = 0$

$4^2 = 0$ $\quad \therefore J$ is nilideal.

* I is nilideal if ever element of I is nilpotent تكون المثالي صفري أو عديم القوة اذا كانت كل عناصر المثالي عديمة القوة.

*لا يعني ان المثالي يساوي صفر.

I

$\boxed{\begin{matrix} a & b \\ c & d \end{matrix}}$

$a^n = 0$, $b^m = 0$, $c^k = 0$ wher $n, m, k =$ عدد صحيح موجب

اذا $d^n \neq 0$

تكون nilideal $\neq I$

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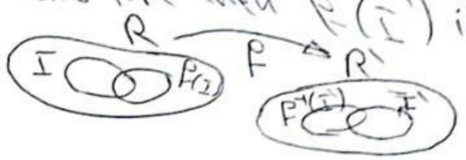
Theorem :- Let $f: R \to R'$ be a ring epimorphism

نبرهن Then:

① if I is an ideal of R then $f(I)$ is an ideal of R'.

نب ② if I' is an ideal of R' then $f^{-1}(I')$ is an ideal of R

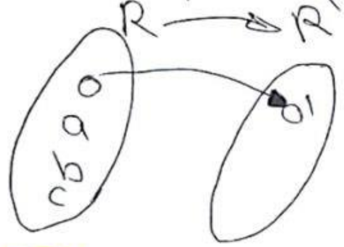


Proof ① H.W

(45)

Def.: Let $f: R \to R'$ be a ring homo, A set $\subseteq R$: $\operatorname{Ker} f = \{x \in R \mid f(x) = 0'\}$ where $0'$ is the id. of $(R', +)$ is called Kernal of f.

$\operatorname{Ker} f \neq \emptyset$ (since $0 \in \operatorname{Ker} f$).



نأخذ العناصر التي صورها = صفر تكون KerF

Not.: ① $\operatorname{Ker} f$ is an ideal of R.

Proof ① $\operatorname{Ker} f \neq \emptyset$ $\left\{\{\text{since } \exists\, 0 \in \operatorname{Ker} f\}\ f(0) = 0'\right\}$

$\operatorname{Ker} f = \{r \in R \mid f(r) = 0\}$

$f^{-1}\{0\} = \{h \in R \mid f(h) = 0\}$

$\{(0')\}$ is trival ideal of R'

Than $f^{-1}\{0'\}$ is ideal of R' (By Th. if I' is ideal of R' than $f^{-1}(I')$ is an ideal of R)

but $f^{-1}(0) = \{h \in R \mid f(h) = 0$

$\operatorname{Ker} f = \{r \in R \mid f(r) = 0$

$\therefore \operatorname{Ker} f$ is ideal of R.

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Prop.: Let $f: R \to R'$ be a ring homo ~~...~~ اذا ساوت الصور تساوي العناصر

Then f is monom. (متباينة) iff $\ker f = \{0\}$

$$f \text{ is mono.} \longleftrightarrow \ker f = \{0\}$$

- نقول عن الدالة متباينة اذا وفقط اذا كان النواة تساوي صفر .
- يتم فحص ~~...~~ التشاكل فيما اذا كان متباين او لا بواسطة النواة .

Proof: H.W

الى هنا امتحان الشهر الاول :-

External direct prod. الجداء المباشر الخارجي للحلقات
(تم شرحه سابقاً).

Internal direct sum. جمع مباشر داخلي

Def: Let $I_1, I_2, \ldots, I_n$ be ideals of ring R. R is called the internal direct sum of $I_1, I_2, \ldots, I_n$ & denoted by (يرمزله) $R = I_1 \oplus I_2 \oplus \cdots \oplus I_n$

if:

① $R = I_1 + I_2 + \cdots + I_n$

مجرد رمز

② $I_i \cap (I_1 + I_2 + \cdots + I_{i-1} + I_{i+1} + \cdots + I_n) = \{0\}$

أحدهما تقاطع جميع العناصر المجموعة ماعداه يكون الناتج التقاطع صفر.

Examples: ① $n = 2$, $R = I_1 \oplus I_2$ if:

Ans (i) $I_1 + I_2 = R$

(ii) $I_1 \cap I_2 = 0$

② $n = 3$, $R = I_1 \oplus I_2 \oplus I_3$ if:

(i) $I_1 + I_2 + I_3 = R$

(ii) $I_1 \cap (I_2 + I_3) = 0$

$I_2 \cap (I_1 + I_3) = 0$

$I_3 \cap (I_1 + I_2) = 0$

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③ $n=4$, $R = I_1 \oplus I_2 \oplus I_3 \oplus I_4$ if

(i) $I_1 + I_2 + I_3 + I_4 = R$

(ii) $I_1 \cap (I_2 + I_3 + I_4) = 0$

$I_2 \cap (I_1 + I_3 + I_4) = 0$

$I_3 \cap (I_1 + I_2 + I_4) = 0$

$I_4 \cap (I_1 + I_2 + I_3) = 0$

④ $n = 5$ H.W

H.W: ① is $Z_{15} = \langle \bar{3} \rangle \oplus \langle \bar{5} \rangle$

$\langle \bar{3} \rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \bar{12}\}$, $\langle \bar{5} \rangle = \{\bar{0}, \bar{5}, \bar{10}\}$

② Find all ~~ideals~~ submodeals of Z_{15}.

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Example: $R = \mathbb{Z}_6$, $I_1 = \{\bar{0}, \bar{3}\}$, $I_2 = \{\bar{0}, \bar{2}, \bar{4}\}$

does $R = I_1 \oplus I_2$?

Sol. In order $R = I_1 \oplus I_2$ we must satisfy def of the internal direct sum

(i) $\mathbb{Z}_6 \stackrel{?}{=} I_1 + I_2$

$I_1 + I_2 = \{\bar{0}, \bar{3}\} + \{\bar{0}, \bar{2}, \bar{4}\}$

$= \{\bar{0}, \bar{2}, \bar{4}, \bar{3}, \bar{5}, \bar{1}\} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} = \mathbb{Z}_6$

(ii) $I_1 \cap I_2 \stackrel{?}{=} 0$

Yes, $I_1 \cap I_2 = 0 \quad \therefore \mathbb{Z}_6 = I_1 \oplus I_2$

Example: $R = \mathbb{Z}_{10}$, $I = \langle \bar{2} \rangle$, $J = \langle \bar{5} \rangle$

is $\mathbb{Z}_{10} = \langle \bar{2} \rangle \oplus \langle \bar{5} \rangle$

(i) $I + J \stackrel{?}{=} \mathbb{Z}_{10}$

$\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\} + \{\bar{0}, \bar{5}\} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{5}, \bar{7}, \bar{9}, \bar{1}, \bar{3}$

$= \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}\} = \mathbb{Z}_{10}$

(ii) $I \cap J = 0$

$\therefore \mathbb{Z}_{10} = I \oplus J$

(50)

Example: is $\mathbb{Z}_{10} = \langle \bar{6} \rangle \oplus \langle \bar{2} \rangle$

Ans. No, since.

$$I_1 + I_2 = \mathbb{Z}_{10}$$

$$\{\bar{0}, \bar{6}\} + \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\} \neq \mathbb{Z}_{12}$$

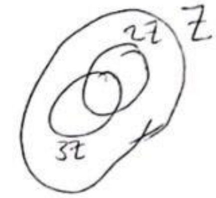
Example: $R = \mathbb{Z}$, $I = 2\mathbb{Z}$, $J = 3\mathbb{Z}$ does $\mathbb{Z} = I \oplus J$?

① $2\mathbb{Z} + 3\mathbb{Z} \overset{?}{=} \mathbb{Z}$

$$2\mathbb{Z} + 3\mathbb{Z} = \{0, \mp 2, \mp 4, \mp 6, \mp 8, \dots\dots, \mp 3, \mp 6, \mp 9 \dots \; 1, -1\}$$

Since $1 \in 2\mathbb{Z} + 3\mathbb{Z}$

$\therefore 2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$



(ii) $I \cap J \overset{?}{=} 0$

$$\{0, \mp 2, \mp 4 \dots\dots, \mp 3, \mp 6 \dots\dots 1, -1\}$$

$$= \{\bar{0}, \bar{6}, 9 \dots\dots\} \neq 0, \; I \cap J \neq \{\bar{0}\}$$

$\therefore \mathbb{Z} \neq I \oplus J$. مو يعني مساوي صفر يعني بيها اكثر من صفر.

Giv another Charctra? of Internal direct sum?

- أعطيا تشخيص آخر للجمع المباشر الداخلي

- المبرهنة الآتية تعطي تعريفاً آخر للجمع المباشر الداخلي.

Theorm:- Let R be a ring and Let I_1 & I_2 be Two ideals of R then $R = I_1 \oplus I_2$ iff each element X of R is written in only one way

يكتب بطريقة واحدة فقط

$X = a + b$ For some $a \in I_1$, $b \in I_2$

* نقول عن R تمثل جمع مباشر داخلي اذا وفقط اذا أن كل عناصر الحلقة تكتب بطريقة واحدة فقط $X = a + b$.

Proof:- ترك