المحاضرة الرابعة

Sylow Theorems

First Sylow Theorem:

Let G be a finite group with o(G) = n and a prime number p \n , then G contains a Sylow P-group.

Example: If $o(G)=36=3^2.2^2$, then there exists a 3-Sylow subgroup H, with $O(H)=3^2=9$.

Second Sylow Theorem: Let G be a finite group, with o(G) = n and p be a prime number then:

1. All Sylow P-groups are conjugate.
2. If t is the number of Sylow P-group , then there exist $s\geq 0$, ; $t = 1 + s\,p$.
3. t divides O(G).

Proposition: A Sylow P-group H is a normal subgroup if and only if H is a unique P-group.

Remark: Any group G is said to be simple if it has no non-trivial normal subgroup.
(i.e. if $H \Delta G$, then either $H = \{e\}$ or $H = G$).

Some Applications of Sylow Theorems:

In this section, we give the main applications of the Sylow theorem in the group theory.

Example: There is no simple group with order 200.

Proof: Let G be a group with $o(G) = 200 = 5^2.2^3$.

G contains a Sylow 5-group say H and o(H) = 52 = 25 (by First Sylow Theorem).

Let t be the number of Sylow 5-group, then $t = 1 + 5k$ for $k \geq 0$.

Also, t | 200 (by Second Sylow Theorem). Now,
There exists a unique Sylow 5-group H and hence H is normal.
so G is not simple.
Now, group G contains a Sylow 2-group with o(K) = 23 = 8 (by First Sylow Theorem) and if r is the number of Sylow 2-group, then r = 1 + 2k for $k \geq 0$. Also, r \ 200 (by Second Sylow Theorem). Now,
r= {1, 5, 25}, this means there exists a three Sylow 2-group K and hence we cannot know if K is unique or not.
We cannot know if K is normal or not.

Remark:
In the previous example if group G is abelian, then K is normal and hence K is unique (by Second Sylow Theorem).

Example:

There is no simple group with order 30.

Proof: H.W

Definition. (Decomposable)
Let H and K be normal subgroups of the group G. Then G is said to be the internal direct product of H and K if :

1. $H \Delta G$ and $K \Delta G$
2. G = HK
3. $H \cap K = \{e\}$, then $G \approx H \times K$

Example: Every group G with order 35 is decomposable and cyclic.

Proof: Let G be a group with o(G) = 35. By Sylow Theorem, G contains a normal subgroup H with o(H) = 5 and a normal subgroup K with o(K) = 7. In the same way of the previous example, G H × K.

o(H) = 5 and o(K) = 7 (prime numbers), then each of H and K is cyclic.
Let H = (x) and K = (y) for x H and y K. Hence x5 = e and y7 = e.
Claim that H × K = (x, y). For that:
Since each of 5 and 7 is a prime number, then there is t and s such that 5t +7s = 1.
Let (xi, yj) H ×K ; $0 \leq i \leq 6$, $0 \leq j \leq 4$.
Since i – j = (5t + 7s)(i - j) → 5t(i - j) + j = 7s(j – i) + i = m.
xm = xst(i - j) + j= (x5)6-j × j = exi = xi and ym = yi.

(xi, yj) = (xm, ym) = (x, y)m → H ×K = (x, y) and hence H ×K is cyclic.

Remark: Let G be a finite group with order pn and n = 1, 2, 3, … then:

1. $o(Z(G)) \neq p^{n-1}$
2. Every subgroup H with $o(H) = p^{n-1}$ is normal.

Exercises

1. Every group of order 45 is not simple, abelian and decomposable.
2. Every group of order 63 is not simple.
3. Every group of order 77 is cyclic, abelian and decomposable.
4. Show that the group order 30 is not simple.

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