المحاضرة الثانية

P- Groups

Definition 1: Let p be a prime number. A group G is said to be a **P-group** if the order of each element of G is some power of p (not necessarily the same power).

Examples 2:

1. o(S3) = 6 = 2.3. It is not P-group, because S3 cannot be written as p^k for k = 0, 1, 2, …).

2. If O(G)=16, then G is P-group. Why?

3. G = {e}, then o(G)= p^0 for each prime number p.

Proposition 3: Every subgroup of P-group is a P-group.

Proof: Let H be a subgroup of a P-group G. Then $o(G) = p^k$, $k = 1,2,3, \dots$

By Lagrange's theorem, $o(H)/O(G)$. So, $O(H) = p^r$,$0 \leq r \leq k$. Hence, H is a P-group.

2. Let G be a P-group and $f: G \rightarrow G'$ be a homomorphism group. Then, f(G) is P-group.

Proof:
Since G is p- group, then $G= p^k$, $k=0, 1, 2, 3, \dots$) and kerf subgroup of G, then kerf is P-group, so that $o(kerf) = p^r$, $0 \leq r \leq k$). By 1st iso th. , $f(G) \cong G\backslash kerf$. So, $o(f(G)) = o(G)O(kerf) = P^{k-r}$ and cleary $0 \leq k-r \leq k$. This implies that f(G) is a P-group.

Corollary 4: Let H be a normal subgroup of G. If G is P-group, then both H and G/H are P-groups.

The converse of the previous proposition is not
true. The following proposition gives the necessary condition for the converse.

Proposition 5: If G is a finite group and H normal subgroup of G and both H and G\H are P-groups, then G is P-group.

Proof: Since $O(G) = O(H). [G: H] = O(H) . O(G/H)$
Assume that $O(H)= p^r$ and $O(G/H)= p^s$. This implies that $O(G) = p^{r+s}$ and $r+s = 0, 1, 2, \dots$. Then G is P-group.

From Corollary 9 and Proposition 10, we deduce the following.

Theorem 6: Let G be a finite group and H is a normal subgroup of G then both H and G\H are P-groups if and only if G is P-group.

Remake 7: If G_1, G_2 are P-group, then so is $G_1 x G_2$.

Proof: It is obvious

Theorem 8: Let G be a finite group. Then G is P-group if and only if the order of each element of G is a power of p.

Proof: For the First direction, Let x∈G. Then o(x) = o((x)) ((x) = is the cyclic group generated by x).
Now, since (x) subgroup of G, then (x) is a P-group (by Remark *).
Each element of G is a power of p.
Conversely, suppose that each element of G is a power of p, and to prove that G is a P-group. Suppose G is not P-group. So, there is a prime number q, q ≠ p; such that q divides o(G). By Cauchy theorem, G contains a subgroup of order q, and hence G contains an element of order q ≠ p C! (since each element of G is a power of p).

Examples 14:

1. If G = S3, then o(S3) = 6 = 2.3. So the order of each element is not power of p. Hence S3 is not P-group.

2. If G = 8. Then o(8)= 23. The subgroup of 8 is { }, { , }, { , }and 8. Hence each element of 8 is a power of 2.

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