المحاضرة الخامسة

Second and Third Isomorphism Theorem

Before giving the Second Isomorphism Theorem, we need the following definition.

Definition (1): If H and K are subgroups of a finite group G, then then the Product Formula is:

$$|HK||H \cap K| = |H||K|$$

Theorem (2): (Second Isomorphism Theorem)

If H and K are subgroups of a group G with H is normal in G, then HK is a subgroup of G and $H \cap K$ is normal in K. Moreover:

$$K/(H \cap K) \cong HK/H$$

Proof:

We begin by showing first that HK/H makes sense, and then describing its elements. Since H is normal subgroup of G, then HK is a subgroup of G. Normality of H in HK follows from a more general fact: if $H \subseteq S \subseteq G$ and if H is normal in G, then H is normal in S.

We can easily show that each coset $xH \in HK/H$ has the form kH for some $k \in K$. It follows that the function $f: K \to HK/H$, given by $f(k) = kH$, is surjective. Moreover, f is a homomorphism, for it is the restriction of the natural map $\pi: G \to G/H$. Since $\ker \pi = H$, it follows that $\ker f = H \cap K$ and so $H \cap K$ is a normal subgroup of K. The first isomorphisim theorem gives:

$$K/(H \cap K) \cong HK/H$$

Remark (3):

The second isomorphism theorem gives the product formula in the special case when one of the subgroups is normal: if $K/(H \cap K) \cong HK/H$, then:
$|K/(H \cap K)| = |HK/H|$, and so $|HK||H \cap K| = |H||K|$.

Third Isomorphism Theorem

In the following lecture we study the third important theorem of fundamental isomorphism theorem.

<u>Theorem (1):</u> (Third Isomorphism Theorem)

If H and K are normal subgroups of a group G with $K \leq H$, then H/K is normal in G/K and

$$(G/K)/(H/K) \cong G/H$$

<u>Proof:</u>

Define $f: G/K \to G/H$ by $f(a K) = a H$. Note that f is a (well- defined function, for if $\bar{a}$ G = b K, then $a^{-1}b \in K$ But $K \subseteq H$, thus $a^{-1}b \in H$, and so a H= b H , and we are doneIt is easy to see that f is an epimorphism.

Now ker f = H/K. Also clearly H/K is a normal subgroup of G/K. Since f is monomorphism, so by the first isomorphism theorem we have:

$$(G/K)/(H/K) \cong G/H$$

The third isomorphism theorem is easy to remember: the K's in the fraction (G/K)/(H/K) can be canceled. One can better appreciate the first isomorphism theorem after having proved the third one. The quotient group (G/K)/(H/K) consists of cosets (of H/K) whose representatives are themselves cosets (of G/K).

The direct product of groups

There are two types of direct products:

1. External direct product
2. Internal direct product

Firstly, we study the external direct product.

Here is another construction of a new group from two given groups.

Definition (2): If H and K are groups, then their direct product, denoted by HxK, is the set of all ordered pairs (h, k) equipped with the following operation:

$$(h, k)(h_1, k_1) = (hh_1, kk_1)$$

It is routine to check that HXK is a group [the identity element is (e, e_1) and $(h, k)^{-1} = (h^{-1}, k^{-1})$.

Remark (3): let G and h be groups. Then HxK is abelian if and only if both H and K are abelian.

We end the tenth lecture by the following example.

Example (4):

Zx2Z is the direct product between (Z, +) and (2Z, +) groups.

The identity element is (0, 0), and the inverse element of (a, b) is (-a, -b).

References

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