المحاضرة الثالثة

Homomorphism

An important problem is determining whether two given groups G and H are somehow the same.

Definition (1): If (G, *) and (H, ∘) are groups, then a function f: G → H is a homomorphism if:

for all x , y ∈ G. If f is also a bijective, then f is called an isomorphism. We say that G and H are isomorphic, denoted by $G \cong H$, if there exists an isomorphism f: G → H.

Example (2):
Let be the group of all real numbers with operation addition, and let R^+ be the group of all positive real numbers with operation multiplication. The function f: $R \to R^+$, defined by f(x)=tx , where t is constant number, is a homomorphism; for if x , y ∈R, then

$$f(x + y) = t(x+y) = tx\, ty = f(x)\, f(y).$$

We now turn from isomorphisms to more general homomorphisms.

Lemma (3): Let f: G → H be a homomorphism.

(i) f (e) = e;
(ii) $f(x^{-1}) = f(x)^{-1}$;

Remark (4):

We can show that any two finite cyclic groups G and H of the same order m are isomorphic. It will then follow from that any two groups of prime order p are isomorphic.

Definition (5):
A property of a group G that is shared by every other group isomorphic to it is called an invariant of G. For example, the order, G, is an invariant of G, for isomorphic groups have the same order. Being abelian is an invariant [if a and b commute, then ab = ba and

$$f(a)\, f(b) = f(ab) = f(ba) = f(b)\, f(a);$$

hence, f (a) and f (b) commute]. Thus, M_{2x2} and GL(2,R) are not isomorphic, for is abelian and GL(2,R) is not.

Definition (6): If $f: G \rightarrow H$ is a homomorphism, define

$$\text{kernel } f = \{x \in G : f(x) = e\}$$

and

$$\text{image } f = \{h \in H : h = f(x) \text{ for some } x \in G\}$$

We usually abbreviate kernel f to ker f and image f to im f

So, if f: G → H is a homomorphism, and B is a subgroup of H then $f^{-1}(B)$ is a subgroup of G containing ker f .

Note: Kernel comes from the German word meaning "grain" or "seed" (corn comes from the same word).

Its usage here indicates an important ingredient of a homomorphism, we give it without proof.

Proposition (7): Let f: G → H be a homomorphism.

(i) ker f is a subgroup of G and im f is a subgroup of H .

(ii) If $x \in$ ker f and if $a \in G$, then $ax\,a^{-1} \in$ ker f.

(iii) f is an injection if and only if ker f = {e}.

Normal Subgroups

Definition (1): A subgroup K of a group G is called normal, if for each $k \in K$ and $g \in G$ imply $gkg^{-1} \in K$. that is $gKg^{-1} \subseteq G$ for every $g \in G$.

Definition (2):

Define the center of a group G, denoted by Z (G), to be

$$Z(G) = \{z \in G: zg = gz \text{ for all } g \in G\};$$

that is, Z (G) consists of all elements commuting with every element in G. (Note that the equation zg = gz can be rewritten as $z = gzg^{-1}$, so that no other elements in G are conjugate to z.

Remark (3):

Let us show that Z (G) is a subgroup of G. We can easily show that Z(G is subgroup of G. It is clear that Z(G)≠∅ since 1 ∈ Z (G), for 1 commutes with everything. Now, If y, z ∈ Z (G), then yg = gy and zg = gz for all g ∈ G. Therefore, (yz)g = y(zg) = y(gz) = (yg)z = g(yz), so that yz commutes with everything, hence yz ∈ Z (G). Finally, if z ∈ Z (G), then zg = gz for all g ∈ G; in particular, $zg^{-1} = g^{-1}z$. Therefore,

$$gz^{-1} = (zg^{-1})^{-1} = (g^{-1}z)^{-1} = z^{-1}g$$

(we are using $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$). So that Z(G) is subgroup pf G.

Clearly che center Z (G) is a normal subgroup; since if $z \in Z(G)$ and $g \in G$, then

$$gzg^{-1} = zgg^{-1} = z \in Z(G)$$

A group G is abelian if and only if Z (G) = G. At the other extreme are groups G for which Z (G) = {1}; such groups are called centerless. For example, it is easy to see that $Z(S_3) = \{1\}$; indeed, all large symmetric groups are centerless.

Proposition (4):

(i) If H is a subgroup of index 2 in a group G, then $g^2 \in H$ for every $g \in G$.

(ii) If H is a subgroup of index 2 in a group G, then H is a normal subgroup of G.

Proof:

(i) Since H has index 2, there are exactly two cosets, namely, H and a H, where $a \in G \backslash H$. Thus, G is the disjoint union $G = H \cup aH$. Take $g \in G$ with $g \notin H$. So that $g = ah$ for some $h \in H$. If $g^2 \notin H$, then $g^2 = ah_1$, where $h_1 \in H$. Hence,

$$g = g^{-1} g^2 = (ah)^{-1}a\, h_1 = h^{-1}a^{-1}a\, h_1 = h^{-1} h_1 \in H,$$

and this is a contradiction.

(ii) It suffices to prove that if $h \in H$, then the conjugate $ghg^{-1} \in H$ for every $g \in G$. Since H has index 2, there are exactly two cosets, namely, H and a H, where $a \notin H$. Now, either $g \in H$ or $g \in aH$. If $g \in H$, then $ghg^{-1} \in H$, because H is a subgroup. In the second case, write $g = ax$, where $x \in H$. Then $ghg^{-1} = a(x\,hx^{-1})a^{-1} = ah_Ia^{-1}$, where $h_I = x\,hx^{-1} \in H$ (for h_I is a product of three elements in H). If $ghg \notin H$, then $ghg^{-1} = ah_Ia^{-1} \in aH$; that is, $ah_Ia^{-1} = ay$ for some $y \in H$. Canceling a, we have $hIa^{-1} = y$, which gives the contradiction $a = y^{-1}h_I \in H$. Therefore, if $h \in H$, every conjugate of h also lies in H; that is, H is a normal subgroup of G.

Proposition(5): If K is a normal subgroup of a group G, then

$$bK = K\,b$$

for every $b \in G$.

Proof: We must show that $bK \subseteq Kb$ and $Kb \subseteq bK$. So if $bk \in bK$, then clearly $bK = bKb^{-1}b$.

Since $bKb^{-1} \in K$, then $bKb^{-1} = k_1$ for some $k_1 \in K$. This implies that $bK \in Kb$. Similarity for the other case. Thus $bK = Kb$.

References

1. D. M. Burton, Abstract and linear algebra, 1972.

2. Joseph J. Rotman, Advanced Modern Algebra, 2003.

3. John B. Fraleigh, A First Course in Abstract Algebra, Seventh Edition, 2002.

4. Joseph A. Gallian, Contemporary Abstract Algebra, 2010.