

# الاحتمالية الاحتمالية

## الفصل الثاني

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## Chapter Three

### "Random variables and Probability Distribution"

Def.: A random variable  $X$  is a function that maps all elements  $s \in S$  (all event in  $\zeta$ ) to a real numbers ( $Rx$ ) denoted by R. V.

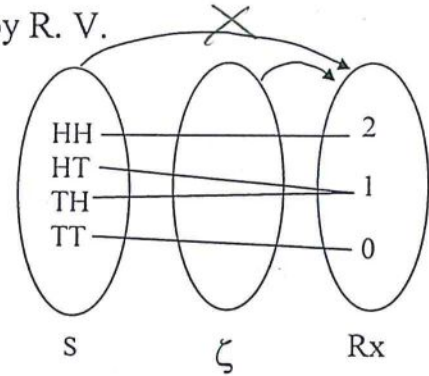
Ex 1.: Toss a coin twice.

Let  $X$  = number of H show that  $X$  is a R. V.

$S = \{HH, HT, TH, TT\}$

~~$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0$~~

$Rx = \{x; x = 0, 1, 2\}$  countable



Note: We shall use  $X$  to denote of R. V.  $X$  and  $x$  to denote of value of R. V.  $X$ ,  $x \in X: 0, 1, 2$  (in ex. "1").

Ex2.: Choose a point from interval  $(0, 1)$ .

Let  $X$  be the chosen point, to show  $X$  is a R. V.

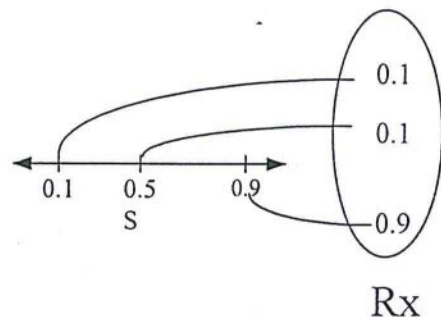
$S$  consist all point in  $(0, 1)$

$\therefore S$  has infinite number of points

The points in  $S$  are mapped to a real numbers.

then  $X$  is a R. V.

$Rx = \{x; 0 < x < 1\}$  uncountable.



Def.: A random variable  $X$  is said to be discrete r. v. if  $Rx$  is countable.

See ex. "1" above, denoted by d. r. v.

Def.: A random variable  $x$  is said to be continuous r. v. if  $Rx$  is uncountable denoted by c.r.v.

See ex, "2" above.

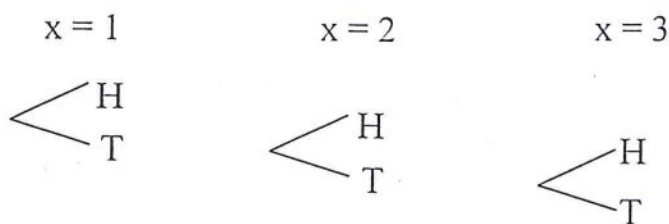
Ex.3: Toss a coin until first H appears.

Let  $x$ : number of tosses. show that  $x$  is d.r.v.





Sol.:



$x \in X: 1, 2, 3, 4, \dots$

$\therefore R_X = \{x; x \in \mathbb{N}\}$  countable

$\therefore x$  is d.r.v.

Def.: Probability Mass function (P. M. f).

Let  $X$  be a d.r.v.

A function  $f$  is a p.m.f. of  $X$  if  $f(x) = p(X = x)$ , and satisfy the following conditions: -

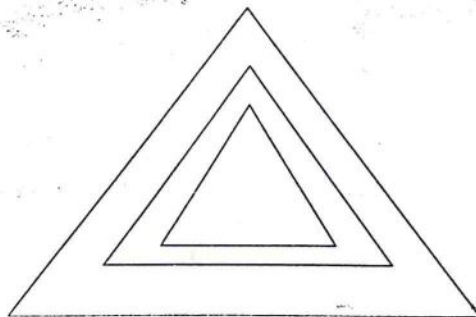
1.  $f(x) \geq 0, \forall x \in X$
2.  $\sum_{x \in X} f(x) = 1$

Note: 1. condition (1) shows the graph of  $f(x)$  above of the  $x$  - axis.

2. Also, if  $A \subset S$  then  $P(x \in A) = \sum_{x \in A} f(x) = 1$

Ex.: Given  $f(x) = \begin{cases} \frac{x}{10}, & \text{for } x = 0, 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$

Show that  $f(x)$  is a p.M.f.



cond. "1": T.p.  $f(x) \geq 0 \forall x \in X$

$$f(0)=0, f(1)=\frac{1}{10}, f(2)=\frac{2}{10}, f(3)=\frac{3}{10}, f(4)=\frac{4}{10}$$

$$\therefore f(x) \geq 0 \forall x \in X$$

$\therefore$  cond (1) satisfied.

$$\text{cond "2": T.p. } \sum_{x=0}^4 f(x) = 1$$

$$\sum_{x=0}^4 f(x) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = \frac{10}{10} = 1.$$

$\therefore f(x)$  is a p. M.f

$$* p(x=1) = f(1) = \frac{1}{10}$$

$$* p(x=8) = f(8) = 0$$

$$* p(x \geq 3) = p[(x=3) \cup (x=4)] = p(x=3) + p(x=4) \\ = f(3) + f(4) = \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$$

$$\text{or by note 2 } p(x \geq 3) = \sum_{x=3}^4 f(x) = f(3) + f(4) = \frac{7}{10}$$

$$* p(x \leq 2) = p[(x=2) \cup (x=1)] = p(x=2) + p(x=1) \\ = f(2) + f(1) = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$$

Ex.: Given a p. m. f.  $f(x) = \begin{cases} \frac{x}{k}, & \text{for } x = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$

find the value of k and sketch  $f(x)$ .

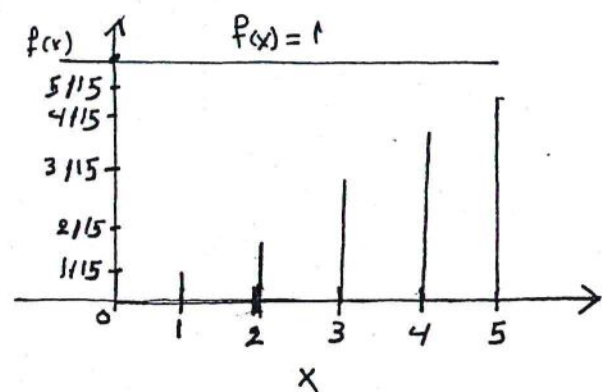
Sol.:  $\therefore f(x)$  is a p. m. f.

$$\therefore \text{ by cond "2" we get } \sum_{x=1}^5 f(x) = 1$$

$$f(1) + f(2) + f(3) + f(4) + f(5) = 1$$

$$\frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} = 1 \Rightarrow \therefore k = 15$$

$$\therefore f(x) = \begin{cases} \frac{x}{15}, & \text{for } x = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$



Ex.: Toss a coin 3-times. Let  $x$  = number of H. find the p. M.f. of  $x$  and sketch it's graph.

Sol.:  $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$\therefore S$  has (8) elements

$X$  = no. of H,  $x \in X$ ;  $x = 0, 1, 2, 3$

$R_X = \{x; x = 0, 1, 2, 3\}$ ,  $R_X$  is a countable

$x$  is d. r. v.

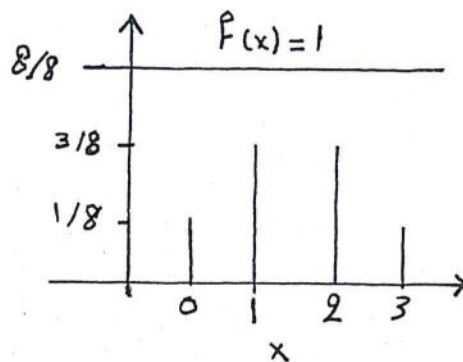
Event  $(X = x)$ : to get  $xH$ ,  $x = 0, 1, 2, 3$  when toss a coin 3- times

$\binom{3}{x}$  : number of samples in event  $(X = x)$  when toss a coin 3-times

$$f(x) = p(X = x) = \begin{cases} \frac{\binom{3}{x}}{8} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

| $x$ | $f(x) = \frac{\binom{3}{x}}{8}$ |
|-----|---------------------------------|
| 0   | 1/8                             |
| 1   | 3/8                             |
| 2   | 3/8                             |
| 3   | 1/8                             |

$\sum_{x=0}^3 f(x) = 1$



Def.: "probability distribution"

A probability dist. of a r.v.  $X$  is a set of all ordered pair of  $x$  and  $f(x)$ ,  $\forall x \in X$ .

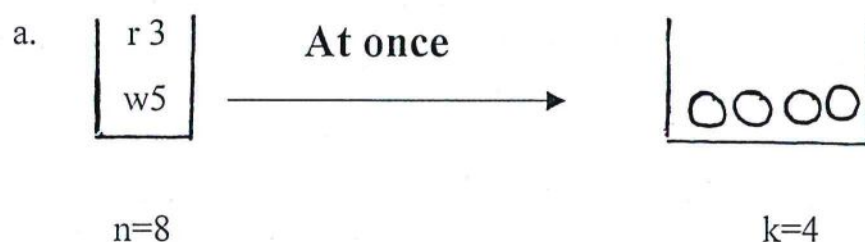
i. e.: pr. dist. of  $x = \{(x_i, f(x_i)); \forall x_i \in X\}$  of  $X$ .

In ex. above: pr. dist. of  $x = \left\{ \left(0, \frac{1}{8}\right), \left(1, \frac{3}{8}\right), \left(2, \frac{3}{8}\right), \left(3, \frac{1}{8}\right) \right\}$

Ex.: Given (3) red and (5) white balls choose a sample of any (4) balls, let  $x$  = number of white ball in a sample.

- Find the P. m. f of  $x$
- Find the pr. that a sample has (2) white balls.
- Find the pr. that a sample has at least (3) white balls.

Sol.:



$\binom{8}{4}$ : no. of sample in  $S$

$x$  = no. of w ball in a sample,  $x = 1, 2, 3, 4$

Event  $(X = x)$ : to get  $x$  w ball and  $(4-x)$  r ball in a sample.

$\therefore$  Number of samples  $\in (X = x)$  is  $\binom{5}{x} \binom{3}{4-x}$

$$f(x) = P(X = x) = \begin{cases} \frac{\binom{5}{x} \binom{3}{4-x}}{\binom{8}{4}}, & \text{for } x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{b. } p(x = 2) = f(2) = \frac{\binom{5}{2} \binom{3}{2}}{\binom{8}{4}}$$

$$\text{c. } p(x \geq 3) = \sum_{x=3}^4 f(x) = f(3) + f(4)$$



Uniform Distribution of discrete random variable:

Def.: Given  $k$  integers:  $1, 2, 3, \dots, k$  choose one integer

Let  $x =$  the chosen int.,  $S = \{1, 2, 3, \dots, k\}$

$$\therefore f(x) = p(X = x) = \begin{cases} \frac{1}{k}, & \text{for } x = 1, 2, 3, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

is called uniform dist. of  $k$  integers.

To show  $f(x)$  is a p. m. f. ?

cond. "1": T. p.  $f(x) \geq 0$

$$\therefore k > 0 \Rightarrow \frac{1}{k} > 0$$

$$\therefore f(x) = \frac{1}{k} > 0 \quad \text{for } x = 1, 2, 3, \dots, k$$

$$= 0 \quad \text{otherwise}$$

$$\therefore f(x) \geq 0$$

cond "2": T. p.  $\sum_{x=1}^k f(x) = 1$

$$\sum_{x=1}^k f(x) = \underbrace{\frac{1}{k} + \frac{1}{k} + \dots + \frac{1}{k}}_{k \text{ - times}} = \frac{k}{k} = 1$$

$\therefore f(x)$  is a p.m.f.

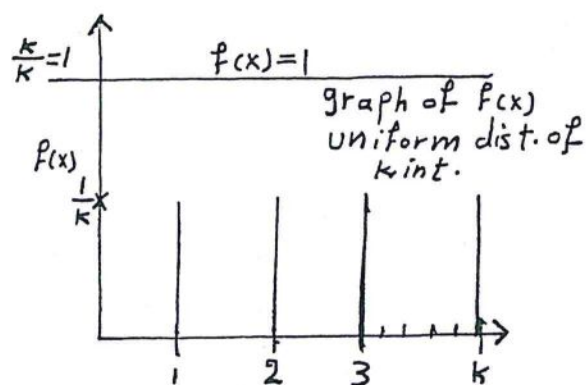
Exercise:

1. A r.v.  $x$  has a discrete dist. with p.m.f.,  $f(x) = \begin{cases} cx, & x = 1, 2, 3, 4, 5 \\ 0, & \text{o.w.} \end{cases}$   
Find the pr. dist. of  $x$ .

2.  $x$  has a uniform dist. on six integers:  $2, 3, 4, 5, 6, 7$  find the p.m.f. of  $x$ .

3. Given a set of integers  $2, 3, \dots, 15$  choose one integer which divisible by 3.

Let  $x =$  the chosen int. find the p.m.f. of  $x$ .





4. Given a set of integers  $\{1, 2, \dots, 10\}$  choose an integer and determine its divisors.

Let  $x$  = number of divisors. find the p.m.f. of  $x$ .

Probability density function (P.d.f):

Let  $x$  be a c.r.v.

A function  $f(x)$  is a p.d.f of  $x$  if for any interval  $A \subset \mathbb{R}_x$

$p(x \in A) = \int_A f(x)dx$ . and satisfy the following conditions:

$$1. f(x) \geq 0 \quad \forall x \in \mathbb{R}, \quad 2. \int_{-\infty}^{\infty} f(x)dx = 1$$

$X(s) = x \in \mathbb{R}_x$ ,  $\mathbb{R}_x$  is uncountable  $A \subset \mathbb{R}_x$

$A$  is a set of real no.

Suppose that  $A = \{x; a < x < b\}$ .

$$p(x \in A) = p(a < x < b) = \int_a^b f(x)dx$$

= Area under curve from  $a$  to  $b$ .

$f(x)$  is cont. over  $(a, b)$

let  $c \in (a, b)$

$f(x)$  is cont. at  $(c)$

$$p(x = c) = \text{no. Area} = 0 \Rightarrow \int_c^c f(x)dx = 0$$

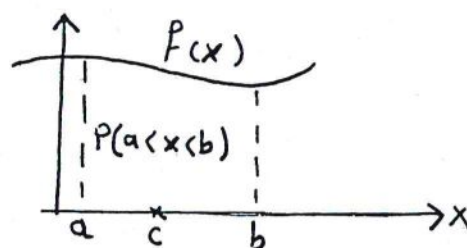
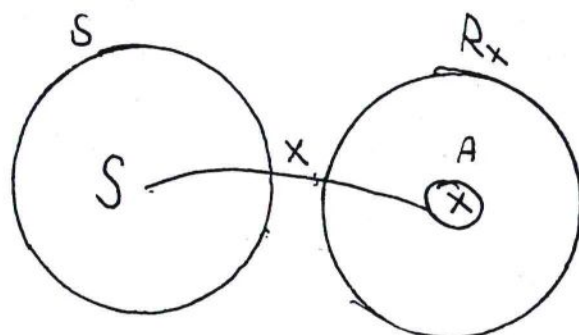
$$p(a \leq x \leq b) = p(x = a) + p(a < x < b) + p(x = b)$$

$$= p(a < x < b)$$

Note: When  $x$  is a d. r. v.

1.  $p(a \leq x \leq b)$  not necessary equal to  $p(a < x < b)$ .

2.  $p(x = c) = f(c)$  not necessary equal to zero.



Ex "1": Given a p.d.f,  $f(x) = \begin{cases} kx & \text{for } 0 < x < 2 \\ k & \text{for } 2 \leq x < 4 \\ 0 & \text{otherwise} \end{cases}$

a. find the value of  $k$  and sketch  $f(x)$ .

b. find  $p(x > 1)$ ,  $p(x < 3)$ ,  $p(\frac{3}{2} < x < \frac{5}{2})$ ,  $p(x < 1 | \frac{1}{2} < x < \frac{3}{2})$

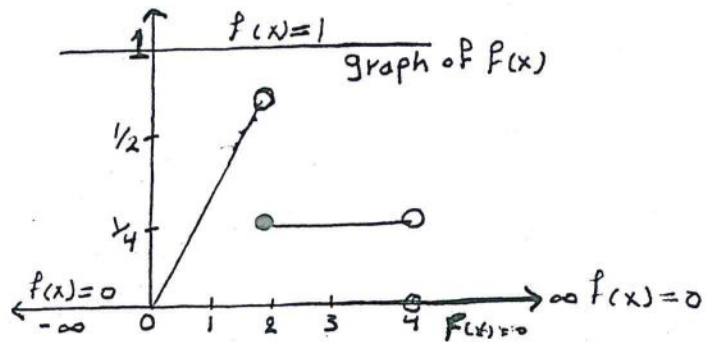
Sol.: a. by cond. "2" of p.d.f.  $\Rightarrow \int_{-\infty}^{\infty} f(x) dx = 1$

$$1 = \int_0^2 kx dx + \int_2^4 k dx \Rightarrow 1 = \frac{k}{2} x^2 \Big|_0^2 + kx \Big|_2^4$$

$$\Rightarrow 1 = \frac{k}{2} [4 - 0] + k[4 - 2] \Rightarrow 1 = 2k + 2k$$

$$\Rightarrow 1 = 4k \Rightarrow k = \frac{1}{4}$$

$$f(x) = \begin{cases} \frac{x}{4} & \text{for } 0 < x < 2 \\ \frac{1}{4} & \text{for } 2 \leq x < 4 \\ 0 & \text{o.w.} \end{cases}$$



# MATHEMATICS

ex.  $f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{o.w.} \end{cases}$

$$b. p(x > 1) = \int_1^2 \frac{x}{4} dx + \int_2^4 \frac{1}{4} dx$$

$$= \frac{1}{8} x^2 \Big|_1^2 + \frac{1}{4} x \Big|_2^4 = \frac{1}{8} (4 - 1) + (4 - 2) = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}$$

$$p(x < 3) = \int_0^2 \frac{x}{4} dx + \int_2^3 \frac{1}{4} dx$$

$$= \frac{1}{8} x^2 \Big|_0^2 + \frac{1}{4} x \Big|_2^3 = \frac{1}{8} (4 - 0) + \frac{1}{4} (3 - 2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$p\left(\frac{3}{2} < x < \frac{5}{2}\right) = \int_{\frac{3}{2}}^2 \frac{x}{4} dx + \int_2^{\frac{5}{2}} \frac{1}{4} dx$$

$$= \frac{1}{8} x^2 \Big|_{\frac{3}{2}}^2 + \frac{1}{4} x \Big|_2^{\frac{5}{2}} = \frac{1}{8} \left(4 - \frac{9}{4}\right) + \frac{1}{4} \left(\frac{5}{2} - 2\right)$$

$$= \frac{1}{8} \left(\frac{7}{4}\right) + \frac{1}{4} \left(\frac{1}{2}\right) = \frac{7}{32} + \frac{1}{8} = \frac{11}{32}$$

$$P(x < 1 | \frac{1}{2} < x < \frac{3}{2}) = P(A | B) = \frac{P(AB)}{P(B)}$$

$$A = \{x; x < 1\} = \{x; 0 < x < 1\}, \quad B = \{x; \frac{1}{2} < x < \frac{3}{2}\}$$

$$AB = \{x, \frac{1}{2} < x < 1\}$$

$$P(B) = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{x}{4} dx = \frac{1}{8} x^2 \Big|_{\frac{1}{2}}^{\frac{3}{2}} = \frac{1}{8} \left(\frac{9}{4} - \frac{1}{4}\right) = \frac{1}{4}$$

$$P(AB) = \int_{\frac{1}{2}}^1 \frac{x}{4} dx = \frac{1}{8} x^2 \Big|_{\frac{1}{2}}^1 = \frac{1}{8} \left(1 - \frac{1}{4}\right) = \frac{1}{8} \left(\frac{3}{4}\right) = \frac{3}{32}$$

$$P(x < 1 | \frac{1}{2} < x < \frac{3}{2}) = \frac{3/32}{1/4} = \frac{3}{8}$$

Ex "2": Given a p. d. f  $f(x) = \begin{cases} ke^{-x} & \text{for } x > 0 \\ 0 & \text{for o.w} \end{cases}$

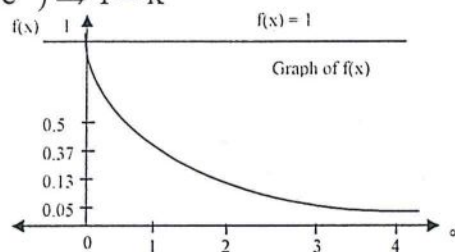
a. find the value of k and sketch  $f(x)$

b. find  $p(x < 2)$ .

Sol.: a. by cond. "2" of p.d.f  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$1 = k \int_{-\infty}^{\infty} e^{-x} dx \Rightarrow 1 = -ke^{-x} \Big|_0^{\infty} \Rightarrow 1 = -k(e^{-\infty} - e^0) \Rightarrow 1 = k$$

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{for o.w} \end{cases}$$



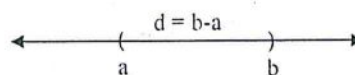
| x        | $f(x) = e^{-x}$                               |
|----------|-----------------------------------------------|
| 0        | $e^0 = 1$                                     |
| 1        | $e^{-1} = \frac{1}{e} = \frac{1}{2.7} = 0.37$ |
| 2        | $e^{-2} = 0.13$                               |
| 3        | $e^{-3} = 0.05$                               |
| .        |                                               |
| $\infty$ | $e^{\infty} = 0$                              |

$$\begin{aligned} b. p(x < 2) &= \int_{-\infty}^2 f(x) dx = \int_{-\infty}^0 0 dx + \int_0^2 e^{-x} dx = -e^{-x} \Big|_0^2 \\ &= -[e^{-2} - e^0] = 1 - e^{-2} = 1 - 0.13 = 0.87 \end{aligned}$$

Uniform distribution on interval (a, b):

Given an interval (a, b),  $b > a$

Choose a point x from (a, b) then  $a < x < b$



A function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{o.w} \end{cases}$$



Is called uniform dist. on (a, b)

To show that  $f(x)$  a p.d.f

Cond. "1": T.p  $f(x) \geq 0$

$$\because b - a > 0 \text{ since } a < b$$

$$\therefore \frac{1}{b-a} > 0 \Rightarrow f(x) = \frac{1}{b-a} > 0 \text{ for } a < x < b$$

$$f(x) \geq 0 \quad \quad \quad = 0 \quad \quad \quad \text{o.w}$$

Cond "2": T.p  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^a 0 dx + \int_a^b \frac{1}{b-a} dx + \int_b^{\infty} 0 dx \\ &= \int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} x \Big|_a^b \\ &= \frac{1}{b-a} (b-a) = 1 \end{aligned}$$

$\therefore f(x)$  is a p.d.f

Ex "1": If  $x$  has a uniform dist. on  $(-2, 3)$  [ $x \sim \text{unif. } (-2, 3)$ ]

a. find a p.d.f of  $x$  and sketch  $f(x)$ .

b. Find  $f(x > 0 | -\frac{1}{2} < x < 2)$

$$\text{Sol.: a. } f(x) = \begin{cases} \frac{1}{3 - (-2)} = \frac{1}{5} & \text{for } -2 < x < 3 \\ 0 & \text{o.w} \end{cases}$$

$$\text{b. } p(x > 0 | -\frac{1}{2} < x < 2) = \frac{p(AB)}{P(B)}$$

$$A = \{x; 0 < x < 3\}$$

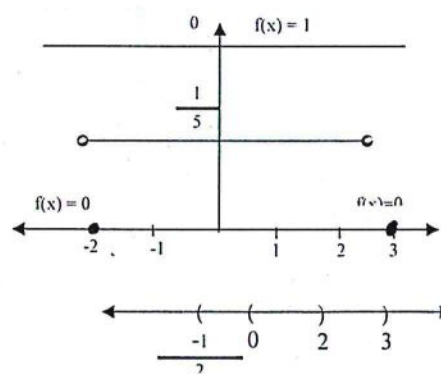
$$B = \left\{x; -\frac{1}{2} < x < 2\right\}$$

$$AB = \{x; 0 < x < 2\}$$

$$P(B) = \int_{-\frac{1}{2}}^2 \frac{1}{5} dx = \frac{1}{5} x \Big|_{-\frac{1}{2}}^2 = \frac{1}{5} \left(2 + \frac{1}{2}\right) = \frac{1}{5} \left(\frac{5}{2}\right) = \frac{1}{2}$$

$$P(AB) = \int_0^2 \frac{1}{5} dx = \frac{1}{5} x \Big|_0^2 = \frac{1}{5} (2 - 0) = \frac{2}{5}$$

$$\therefore p(x < 0 | -\frac{1}{2} < x < 2) = \frac{2/5}{1/2} = \frac{4}{5}$$





H.W.:

1.  $x \sim \text{unif.}(-1, 3)$  find p.d.f of  $x$  and find  $p(0 \leq x < 2)$ ,  $p(x < 1 | -1 < x < 2)$ .

2. Given  $f(x) = \frac{1}{\pi(1+x^2)}$  for  $-\infty < x < \infty$ . Show that  $f(x)$  is a p.d.f.

Note:  $f(x)$  above is called cauchy dist.

Cumulative Distribution Function (c. d. f)

(Distribution Function (d. f)) or (Cumulative Distribution Fun.)  
c. d. f

Def.: Let  $x$  be a r. v. either d. r.v. or c.r.v

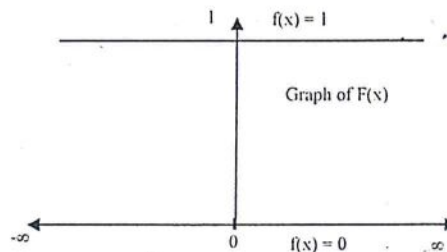
A function  $F(x)$  is a c.d.f of  $x$  iff

$$F(x) = p(X \leq x); -\infty < x < \infty$$

Event  $(X \leq x)$  = all point in  $(-\infty, x]$ ,  $0 \leq p(X \leq x) \leq 1 \Rightarrow 0 \leq F(x) \leq 1$

That is mean  $F(x)$  is bounded by zero and one.

i.e.: The graph of  $F(x)$  lies between the line  $F(x) = 0$  and the line  $F(x) = 1$



Note: We shall use  $F(x)$  to denote of a c.d.f of  $x$  and  $f(x)$  to denote of p.m.f or p.d.f.

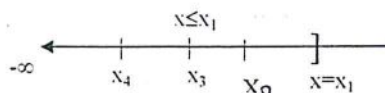
To find  $F(x)$  when  $x$  is d.r.v.

Let  $x$  be a d.r.v. with p.m.f  $f(x)$

$R_x = \{x_j \in I, j = 1, 2, 3, \dots\}$  countable

$F(x) = p(X \leq x)$  by def.

Choose  $x$  and let  $x = x_1$  ( $x_1 \in I$ )



Event  $(X \leq x) = \{x_j \leq x_i, i, j = 1, 2, \dots\}$

$(X \leq x) = \{(X=x_1) \cup (X=x_2) \cup (X=x_3) \cup \dots \cup (X=x_j) \cup \dots\}$

$$P(X \leq x) = P(X=x_1) + P(X=x_2) + P(X=x_3) + \dots + P(X=x_j) + \dots$$

$$= f(x_1) + f(x_2) + f(x_3) + \dots$$

$$p(X \leq x) = \sum_{x_j \leq x} f(x_j)$$

$$F(x) = \sum_{x_j \leq x} f(x_j) \quad \text{c.d.f of } x \text{ when it's d.r.v.}$$

Ex. "1": Given a p.m.f  $f(x) = \begin{cases} \frac{x}{10} & \text{for } x = 1, 2, 3, 4 \\ 0 & \text{o.w} \end{cases}$  ← d.r.v.

a. Find the c.d.f  $F(x)$  and sketch it's graph.

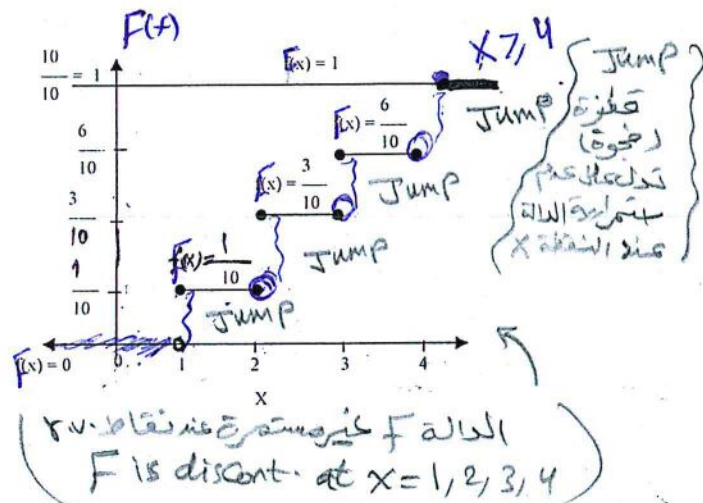
b. Find  $p(x \leq 2)$ ,  $p(x > 3)$ ,  $p(x < 2)$ ,  $p(1 < x \leq \frac{3}{2})$ ,  $p(1 \leq x < \frac{3}{2})$

Sol.: a.  $R_x = \{x, x = 1, 2, 3, 4\}$  count.  $\rightarrow x$  is d.r.v.

$$F(x) = \sum_{x_j \leq x} f(x_j)$$

| Interval of x  | $x_j \in I$ | $f(x) = \frac{x}{10}$ | $F(x) = \sum_{x_j \leq x} f(x_j)$                                                                                               |
|----------------|-------------|-----------------------|---------------------------------------------------------------------------------------------------------------------------------|
| $x < 1$        | 0           | 0                     | $F(x) = 0$                                                                                                                      |
| $1 \leq x < 2$ | 1           | $f(1) = \frac{1}{10}$ | $F(x) = 0 + \frac{1}{10} = \frac{1}{10}$                                                                                        |
| $2 \leq x < 3$ | 2           | $f(2) = \frac{2}{10}$ | $F(x) = 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$                                                                         |
| $3 \leq x < 4$ | 3           | $f(3) = \frac{3}{10}$ | $F(x) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = \frac{6}{10}$                                                          |
| $x \geq 4$     | 4           | $f(4) = \frac{4}{10}$ | $F(x) = 1 = \frac{10}{10} = F(4)$<br>$F(4) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = \frac{10}{10} = 1$ |

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{10} & \text{for } 1 \leq x < 2 \\ \frac{3}{10} & \text{for } 2 \leq x < 3 \\ \frac{6}{10} & \text{for } 3 \leq x < 4 \\ 1 & \text{for } x \geq 4 \end{cases}$$



NOTE: 1-  $F(X)$  is discontinuous at  $x=1,2,3,4$ .

2-  $F(X)$  is continuous to the right for each interval of continuity.

b.  $p(x \leq 2) = F(2) = \frac{3}{10}$

$(F(a) = p(x \leq a))$

or  $p(x \leq 2) = \sum_{x=1}^2 f(x) = f(1) + f(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$

or  $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2) = 1 - \frac{3}{10} = \frac{7}{10}$   
 $P(X > 2) = P(X=3 \cup X=4) = P(X=3) + P(X=4) = f(3) + f(4) = \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$   
 $P(X > 3) = 1 - P(X \leq 3) = 1 - F(3) = 1 - \frac{6}{10} = \frac{4}{10}$

Or  $p(x > 3) = p(x=4) = f(4) = \frac{4}{10}$

$P(x < 2) \neq F(2)$   $[p(x \leq x) = F(x)]$  or  $P(x \leq 1) = F(1) = \frac{1}{10}$

$P(x < 2) = p(x=1) = f(1) = \frac{1}{10}$  (by p.f.  $f(x)$ )

$P(1 < x \leq \frac{3}{2}) = p(\emptyset) = 0$

Since  $\frac{3}{2}$  is not an integer  $\in (1, \frac{3}{2}]$  since  $X$  is d.r.v.

$P(1 \leq x < \frac{3}{2}) = p(x=1) = f(1) = \frac{1}{10}$

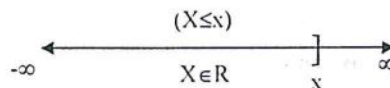
H.W.: Given an integers, 5,6,7,...,15 Choose one even integer. Let  $x$  be the chosen integer. Find  $F(x)$  and sketch its  $\rightarrow$  graph.

To find  $F(x)$  when  $x$  is c.r.v.

Let  $x$  be a c.r.v. with p.d.f  $f(x)$

$F(x) = p(X \leq x)$  by def.

To find  $p(X \leq x)$





$$\because x \text{ has a p.d.f } f(x) \Rightarrow p(X \leq x) = \int_{-\infty}^x f(t) dt$$

$$\therefore F(x) = \int_{-\infty}^x f(t) dt \quad [\text{c.d.f of } x \text{ when } x \text{ is c.r.v}]$$

$f(x)$  is cont.  $\forall x \in (-\infty, \infty)$

Ex. "1": Given a p.d.f  $f(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

a. Find  $F(x)$  and sketch it's graph.

b. Find  $p(x \leq \frac{1}{3})$ ,  $p(x < \frac{1}{3})$ ,  $p(x > 1)$ ,  $p(x \leq -\frac{1}{2})$ ,  $p(-\frac{1}{3} < x < \frac{1}{4})$

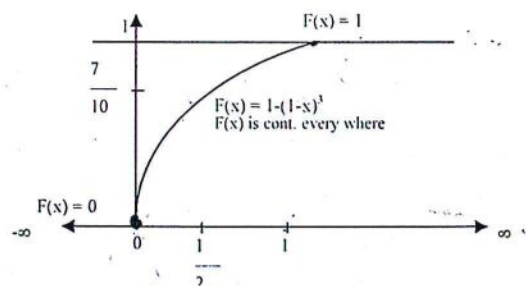
Sol.: a.  $F(x) = \int_{-\infty}^x f(t) dt$

$$= \int_{-\infty}^0 0 dt + \int_0^x 3(1-t)^2 dt = 3 \int_0^x (1-t)^2 dt$$

$$= -3 \frac{(1-t)^3}{3} \Big|_0^x = -[(1-x)^3 - 1] = 1 - (1-x)^3$$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - (1-x)^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

| x             | $F(x) = 1 - (1-x)^3$ |
|---------------|----------------------|
| 0             | 0                    |
| $\frac{1}{2}$ | $\frac{7}{8}$        |
| 1             | 1                    |



$$b. p(x \leq \frac{1}{3}) = F(\frac{1}{3}) = 1 - (1 - \frac{1}{3})^3 = 1 - \frac{8}{27} = \frac{19}{27}$$

$$p(x < \frac{1}{3}) \neq F(\frac{1}{3})$$

$$p(x < \frac{1}{3}) = 3 \int_0^{\frac{1}{3}} (1-x)^2 dx = -3 \frac{(1-x)^3}{3} \Big|_0^{\frac{1}{3}} = -(1-x)^3 \Big|_0^{\frac{1}{3}}$$

$$= -[(1 - \frac{1}{3})^3 - 1] = -[(\frac{2}{3})^3 - 1] = 1 - \frac{8}{27} = \frac{19}{27}$$

$$p(x > 1) = 1 - p(x \leq 1) = 1 - F(1) = 1 - 1 = 0$$

$$p(x \leq -\frac{1}{2}) = F(-\frac{1}{2}) = 0$$

$$p(-\frac{1}{3} < x < \frac{1}{4}) = \int_{-\frac{1}{3}}^{\frac{1}{4}} f(x) dx = \int_{-\frac{1}{3}}^0 0 dx + \int_0^{\frac{1}{4}} 3(1-x)^2 dx$$

$$= -(1-x)^3 \Big|_0^{\frac{1}{4}} = -[(1 - \frac{1}{4})^3 - 1] = 1 - (\frac{3}{4})^3$$

$$= 1 - \frac{27}{64} = \frac{37}{64}$$

H.W.:

$$1. \text{ Given a p.d.f } f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w} \end{cases}$$

a. Find c.d.f of x

b. Find  $p(x > 4.2)$  and  $p(x < 3.5)$  and  $p(x \leq -1)$

$$2. \text{ Given a p.d.f } f(x) = \begin{cases} Ce^{-x} & \text{for } x > 0 \\ 0 & \text{o.w} \end{cases}$$

a. Find the value of c and sketch F(x).

b. Find F(x) and sketch its graph

c.  $p(x \leq 3)$ ,  $p(x < 3)$ .



To find p.d.f  $f(x)$  <sup>from</sup> c.d.f  $F(x)$ :

Let  $x$  be a c.r.v. that has a c.d.f  $F(x)$  defined p.d.f  $f(x)$

$$F(x) = \int_{-\infty}^x f(t) dt$$

$$\frac{dF(x)}{dx} = \frac{d}{dx} \left[ \int_{-\infty}^x f(t) dt \right] = f(x)$$

$$\therefore \frac{dF(x)}{dx} = f(x) \quad \forall x, \text{ where } F(x) \text{ diff. \& } f(x) \text{ is cont.}$$

Ex. "1": Given a c.d.f  $F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{x}{1+x} & \text{for } x \geq 1 \end{cases}$   
Find p.d.f  $f(x)$ .

Sol.:  $F(x)$  is not cont. at  $x=1$ ,  $F(x)$  is not diff. At  $x=1$

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} \frac{d0}{dx} = 0 & \text{for } x \leq 1 \\ \frac{d}{dx} \left( \frac{x}{1+x} \right) = \frac{1}{(1+x)^2} & \text{for } x > 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{(1+x)^2} & \text{for } x > 1 \\ 0 & \text{o.w} \end{cases}$$

H.W.: Given a c.d.f  $F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 1 \\ 0 & \text{o.w} \end{cases}$

**Find the p.d.f  $f(x)$ .**

**Properties of c.d.f  $F(x)$ :**

Theorem "1":  $F(x)$  is non-decreasing function.

Proof: Let  $x_1 < x_2$  T.p  $F(x_1) \leq F(x_2)$ .

$$(x \leq x_1) \subseteq (x \leq x_2)$$

$$p(x \leq x_1) \leq p(x \leq x_2) \quad [\text{by th.}]$$

$$F(x_1) \leq F(x_2)$$

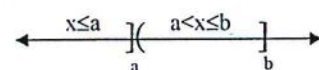
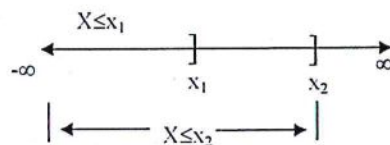
Theorem "2": If  $x$  is a r.v. that has a c.d.f  $F(x)$ , then

$$p(a < x \leq b) = F(b) - F(a), \quad a < b.$$

$$\text{Proof: } (x \leq b) = (x \leq a) \cup (a < x \leq b)$$

$$P(x \leq b) = p(x \leq a) + p(a < x \leq b) \quad (\text{by Ax. 3})$$

$$F(b) = F(a) + p(a < x \leq b) \quad [\text{by def. of } F(x)]$$



Find the p.m.f of  $X$  from C.d.f of  $X$  :

$$f(x_n) = F(x_n) - F(x_{n-1})$$

example:  $f(3) = F(3) - F(2)$

example

Given a C.d.f.  $F$  of  $X$  :

$$F(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ \frac{1}{10} & \text{for } 1 < x \leq 2 \\ \frac{3}{10} & \text{for } 2 < x \leq 3 \\ \frac{6}{10} & \text{for } 3 < x \leq 4 \\ 1 & \text{for } x \geq 4 \end{cases}$$

→ (Sketch  $F(x)$  first)

sol.

$x = 1, 2, 3, 4$  (d.r.v.)

$$f(x) = F(x) - F(x-1)$$

$$f(1) = F(1) - 0 = \frac{1}{10} = \boxed{\frac{1}{10}}$$

$$f(2) = F(2) - F(1) = \frac{3}{10} - \frac{1}{10} = \boxed{\frac{2}{10}}$$

$$f(3) = F(3) - F(2) = \frac{6}{10} - \frac{3}{10} = \boxed{\frac{3}{10}}$$

$$f(4) = F(4) - F(3) = 1 - \frac{6}{10} = \boxed{\frac{4}{10}}$$

$$f(x) = \begin{cases} \frac{x}{10} & \text{for } x = 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases}$$

(Check  $f$  is p.m.f)  
i.e. (Is Cond. 1 & 2 satisfied?)



$$P(a < x \leq b) = F(b) - F(a)$$

Theorem "3":  $F(-\infty) = 0$  &  $F(\infty) = 1$

Proof:  $F(-\infty) = \lim_{x \rightarrow -\infty} F(x)$

$$F(-\infty) = \lim_{x \rightarrow -\infty} p(X \leq x)$$

$$F(-\infty) = \lim_{x \rightarrow -\infty} p(X \leq x)$$

Choose a very small real no.  $x$  say at  $(-\infty)$

Event  $(X \leq x: x \rightarrow -\infty) = \phi$

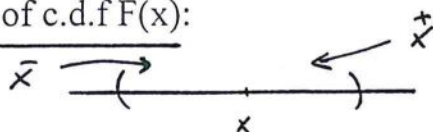
$$F(-\infty) = \lim_{x \rightarrow -\infty} p(X \leq x) = p(\phi) = 0$$

Choose a very large real no.  $x$  say at  $(\infty)$

Event  $(X \leq x: x \rightarrow \infty) = \{x: -\infty < x < \infty\} = S$

$$F(\infty) = \lim_{x \rightarrow \infty} p(X \leq x) = p(S) = 1$$

Continuity of c.d.f  $F(x)$ :



$F(x^-)$  and  $F(x^+)$

The left limit of  $F(x)$ :



Choose a point  $x$ , and

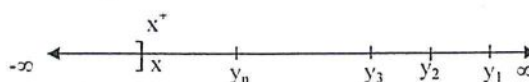
Consider a set of points  $y_n; n = 1, 2, 3, \dots$

Such that  $y_1 < y_2 < \dots < y_n < \dots < x$

$y_n \rightarrow x$  when  $n \rightarrow \infty$

$\therefore y_n = x^-$

$$F(x^-) = \lim_{\substack{n \rightarrow \infty \\ y_n < x}} F(y_n) \Rightarrow \therefore \left\{ F(x^-) = \lim_{\substack{n \rightarrow \infty \\ y_n < x}} p(x \leq y_n) \right\}$$





### The right limit of $F(x)$ :

Choose a point  $x$ , and consider

a set of points  $y_n, n = 1, 2, 3, \dots$

Such that  $y_1 > y_2 > \dots > y_n > \dots > x$

$y_n \rightarrow x$  when  $n \rightarrow \infty$

$\therefore y_n = x^+$

$$F(x^+) = \lim_{\substack{n \rightarrow \infty \\ y_n > x}} F(y_n) \Rightarrow \therefore \left\{ F(x^+) = \lim_{\substack{n \rightarrow \infty \\ y_n > x}} p(x \leq y_n) \right\}$$

Def.:  $F(x)$  is cont. at a point  $x$  iff  $F(x^+) = F(x^-)$

Theorem "4": For any point  $x$ ,  $F(x)$  is cont. to the right.

i.e.:  $F(x) = F(x^+)$

Theorem "5": At any point  $x$ , then  $p(X < x) = F(x^-)$

Theorem "6": At any point  $x$ , then  $p(X = x) = F(x^+) - F(x^-)$

Proof:  $(X \leq x) = (X = x) \cup (X < x)$

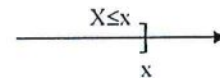
$$P(X \leq x) = p(X = x) + p(X < x)$$

$$F(x^+) = p(X = x) + F(x^-)$$

$$P(X = x) = F(x^+) - F(x^-)$$

Ex.: Given a c.d.f

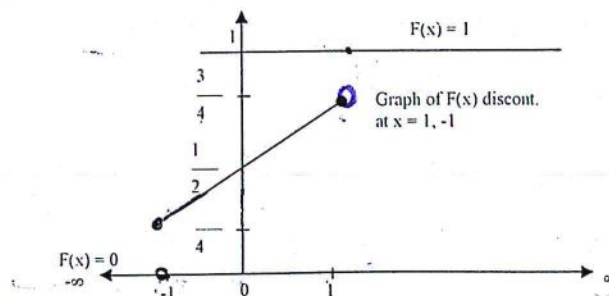
$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+2}{4} & \text{for } -1 \leq x \leq 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$



1. Sketch  $F(x)$ ,      2. Find  $p(x = -1)$ ,  $p(x = \frac{1}{2})$ ,  $p(x = 1)$ .

2. Find  $p(-\frac{1}{2} < x \leq \frac{1}{4})$ ,  $p(0 < x \leq 1)$ ,  $p(-\frac{1}{2} < x < \frac{1}{4})$ ,  $p(0 < x < 1)$ .

Sol.: 1.



$$2. p(x = -1) = F(-1^+) - F(-1^-) = \frac{1}{4} - 0 = \frac{1}{4}$$

$$p(x = 1) = F(1^+) - F(1^-) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$p(x = \frac{1}{2}) = F(\frac{1}{2}^+) - F(\frac{1}{2}^-) = (\frac{\frac{1}{2} + 2}{4}) - (\frac{\frac{1}{2} + 2}{4}) = 0$$

$F(x)$  is cont. at  $x = \frac{1}{2}$

$$3. p(-\frac{1}{2} < x \leq \frac{1}{4}) = F(\frac{1}{4}) - F(-\frac{1}{2}) = (\frac{\frac{1}{4} + 2}{4}) - (\frac{-\frac{1}{2} + 2}{4}) = \frac{3}{16}$$

$$p(0 < x \leq 1) = F(1) - F(0) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$p(-\frac{1}{2} < x < \frac{1}{4}) = p(-\frac{1}{2} < x \leq \frac{1}{4})$$

when  $F(x)$  is cont. at  $(\frac{1}{4})$

$$\therefore p(-\frac{1}{2} < x < \frac{1}{4}) = \frac{3}{16}$$

or

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} \frac{1}{4} & \text{for } -1 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

$$p(-\frac{1}{2} < x < \frac{1}{4}) = \int_{-\frac{1}{2}}^{\frac{1}{4}} \frac{1}{4} dx = \frac{1}{4} (\frac{1}{4} + \frac{1}{2}) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$$

$$p(0 < x < 1) \neq p(0 < x \leq 1)$$

Since  $F(x)$  discont at (1)

$$p(0 < x < 1) = \int_0^1 \frac{1}{4} dx = \frac{1}{4} (1 - 0) = \frac{1}{4}$$

$$\text{H.W.: Given a c.d.f } F(x) = \begin{cases} 1 - \frac{4}{x^2} & \text{for } x > 2 \\ 0 & \text{for } x \leq 2 \end{cases}$$

1. Find  $f(x)$  and sketch  $F(x)$  &  $f(x)$ .
2. Find  $p(x \leq 4)$ ,  $p(x > 8)$ ,  $p(2 < x \leq 8)$ ,  $p(x=2)$ .

### Transformation of random variables:

1. Transf. of d. r. v.  $x$ :

If  $x$  be a d. r. v. have a p.m.f  $f(x)$ , and let  $A$  be the space of  $x$  for  $f(x) > 0$ , and let  $y = u(x)$  define a (1-1) transformation that maps  $A$  onto  $B$  (The space of  $y$ ).

If we solve  $y = u(x)$  for  $x$  in term of  $y$ .

Say  $x = u^{-1}(y) \in A$  then for  $y = u(x) \in B$ , we have  $x = u^{-1}(y) \in A$

Know if we consider the r.v.  $Y = u(x)$  if  $y \in B$ , then  $x = u^{-1}(y) \in A$  and the events  $Y = y$  [or  $u(x) = y$ ] and  $x = u^{-1}(y)$  are equivalent.

Then the p.m.f  $g(y)$  of r.v.  $Y$  is define as follows:

$$g(y) = p[Y = y] = p[Y = u(x)] = p[x = u^{-1}(y)] = f[u^{-1}(y)]$$

$$\therefore g(y) = f[u^{-1}(y)]$$

$$\text{Ex. "1": Given a p.m.f } f(x) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w} \end{cases}$$

Find the p.m.f of  $Y = 2x$

$$\text{Sol.: } Y = 2x = u(x)$$

$$A = \{x; x = 0, 1, 2, \dots\}, \quad X = \frac{y}{2} = u^{-1}(y)$$

$$B = \{y; y = 0, 2, 4, 6, \dots\}, \quad g(y) = f[u^{-1}(y)] = f\left[\frac{y}{2}\right]$$

$$\therefore g(y) = \begin{cases} \frac{e^{-m} m^{y/2}}{(y/2)!} & \text{for } y = 0, 2, 4, \dots \\ 0 & \text{o.w} \end{cases}$$

1. Find  $f(x)$  and sketch  $F(x)$  &  $f(x)$ .

2. Find  $p(x \leq 4)$ ,  $p(x > 8)$ ,  $p(2 < x \leq 8)$ ,  $p(x = 2)$ .

Transformation of random variables:

1. Transf. of d. r. v.  $x$ :

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If we solve  $y = u(x)$  for  $x$  in term of  $y$ .

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Know if we consider the r.v.  $Y = u(x)$  if  $y \in B$ , then  $x = u^{-1}(y) \in A$  and the events  $Y = y$  [or  $u(x) = y$ ] and  $x = u^{-1}(y)$  are equivalent.

Then the p.m.f  $g(y)$  of r.v.  $Y$  is define as follows:

$$g(y) = p[Y = y] = p[Y = u(x)] = p[x = u^{-1}(y)] = f[u^{-1}(y)]$$

$$\therefore g(y) = f[u^{-1}(y)]$$

Ex. "1": Given a p.m.f  $f(x) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w} \end{cases}$

Find the p.m.f of  $Y = 2x$

Sol.:  $Y = 2x = u(x)$

$$A = \{x; x = 0, 1, 2, \dots\}, \quad x = \frac{y}{2} = u^{-1}(y)$$

$$B = \{y; y = 0, 2, 4, 6, \dots\}, \quad g(y) = f[u^{-1}(y)] = f\left[\frac{y}{2}\right]$$

$$\therefore g(y) = \begin{cases} \frac{e^{-m} m^{y/2}}{(y/2)!} & \text{for } y = 0, 2, 4, \dots \\ 0 & \text{o.w} \end{cases}$$

توزيع بواسون (Poisson)  
 $X \sim P_0(m)$   
 ملاحظة: توزيع بواسون  
 كمية ثابتة معلومة





Ex. "2": Let  $x$  have the following p.M.f

$$f(x) = \begin{cases} \binom{5}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{5-x} & x = 0, 1, \dots, 5 \\ 0 & \text{o.w} \end{cases}$$

توزيع ثنائي الحدين  
 $X \sim b(n, \frac{1}{4})$   
 $X \sim b(n, p)$

Find the p.M.f of  $y = x^2$

Sol.:  $y = x^2 = u(x)$

$$A = \{x; x = 0, 1, 2, 3, 4, 5\}, \quad x = u^{-1}(y) = \sqrt{y}$$

$$B = \{y; y = 0, 1, 4, 9, 16, 25\}$$

$$\therefore g(y) = f[u^{-1}(y)] = f(\sqrt{y})$$

$$= \begin{cases} \binom{5}{\sqrt{y}} \left(\frac{1}{4}\right)^{\sqrt{y}} \left(\frac{3}{4}\right)^{5-\sqrt{y}} & \text{for } y = 0, 1, 4, 9, 16, 25 \\ 0 & \text{o.w} \end{cases}$$

H. W ~~7~~ 1. If  $x$  has a uniform dist. on  $\{1, 2, 3\}$  then find the p.m.f of  $y = 4x + 2$ .

$$2. \text{ Given a p.m.f. } f(x) = \begin{cases} \left(\frac{1}{8}\right)^x, \frac{512}{73} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$$

Find the p.m.f of  $y = x^3$ .

$$= \begin{cases} \binom{3}{x} \left(\frac{1}{8}\right)^x \left(\frac{7}{8}\right)^{3-x} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$$

or  $f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^x & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$

دالة كثافة احتمالية  
G.S.

## 2. Transfor of c.r.v. $x$ :

Let  $x$  be a c.r.v. have a p.d.f  $f(x)$ ,  $f(x) > 0$  for each point of the space  $A$  of  $x$ , let  $y = u(x)$  define a (1-1) transf. from  $A$  to space  $B$  of  $y$ .

If we can write  $x = u^{-1}(y)$  as the inverse of the function  $u$ , then the d.p.f. of  $y$  is:

$$g(y) = f[u^{-1}(y)] \cdot [J] \quad \text{where } J = [u^{-1}(y)]' = x' = \frac{dx}{dy}$$

Jacobian

$$\text{Ex. "1": } f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w} \end{cases} \quad \text{Find the p.d.f of } y = 8x^3$$

Sol.:  $Y = 8x^3 = u(x)$



$$A = \{x; 0 < x < 1\}, \quad x = \frac{\sqrt[3]{y}}{2} = u^{-1}(y)$$

$$B = \{y; 0 < y < 8\}$$

$$P[u^{-1}(y)] = P\left[\frac{\sqrt[3]{y}}{2}\right] = 2 \cdot \frac{\sqrt[3]{y}}{2} = \sqrt[3]{y} \quad (\text{since } f(x) = 2x)$$

$$J = \frac{dx}{dy} = \frac{1}{6\sqrt[3]{y^2}}$$

$$\therefore g(y) = f[u^{-1}(y)] \cdot |J| = \begin{cases} \frac{\sqrt[3]{y}}{6\sqrt[3]{y^2}} & \text{for } 0 < y < 8 \\ 0 & \text{o.w} \end{cases}$$

Ex. "2": If  $x \sim \text{unif}(0, 1)$  then find the p.d.f of  $y = -2\ln x$ .

sol.:  $\because x \sim \text{unif}(0, 1)$

$$\therefore f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

$$y = u(x) = -2 \ln x, \quad x = u^{-1}(y) = e^{-y/2}$$

$$|J| = \left| \frac{dx}{dy} \right| = \left| -\frac{1}{2} e^{-y/2} \right| = \frac{1}{2} e^{-y/2}$$

$$g(y) = f(u^{-1}(y)) \cdot |J| = 1 \cdot \frac{1}{2} e^{-y/2} \quad (\text{since } f[u^{-1}(y)] = f[e^{-y/2}] = 1; \text{ since } f(x) = 1)$$

The space of  $y = \{y; 0 < y < \infty\}$ ;  $(y(0) = -2\ln(0) = \infty)$   
 $(y(1) = -2\ln(1) = 0)$

$$g(y) = \begin{cases} \frac{e^{-y/2}}{2} & \text{for } 0 < y < \infty \\ 0 & \text{o.w} \end{cases}$$

H. W 8 1.  $f(x) = \begin{cases} \frac{x^2}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w} \end{cases}$  Find p.d.f of  $y = x^3$

2. Given a p.d.f  $f(x) = \begin{cases} 2xe^{-x^2} & \text{for } 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$  Find p.d.f of  $y = x^2$



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ex Let  $X$  be a d.r.v. has the following p.m.f.

$$f(x) = \begin{cases} 1/5 & \text{for } x = -2 \\ 1/6 & \text{for } x = -1 \\ 1/5 & \text{for } x = 0 \\ 1/15 & \text{for } x = 1 \\ 11/30 & \text{for } x = 2 \end{cases} \Rightarrow A = \{-2, -1, 0, 1, 2\}$$

Let  $Y = X^2$ , then Find the p.m.f. of  $Y$

Sol.

|    |    |   |   |   |   |
|----|----|---|---|---|---|
| -2 | -1 | 0 | 1 | 2 | A |
| 4  | 1  | 0 | 1 | 4 | B |

Remark

$y = 0$  from  $x = 0$

$y = 1$  from  $x = -1$  &  $x = +1$

$y = 4$  from  $x = -2$  &  $x = +2$

$$g(y) = \begin{cases} 1/5 & y = 0 \\ 7/30 & y = 1 \\ 17/30 & y = 2 \end{cases}$$

$[x=0 \rightarrow f(x) = 1/5]$   
 $[x=\pm 1 \rightarrow f(-1) = 1/6, f(1) = 1/15 \Rightarrow \frac{7}{30}]$   
 $[x=\pm 2 \rightarrow f(-2) = 1/5, f(2) = 11/30]$

$$g(y) = \sum_{y=U(x_i)} P[U^{-1}(x)] = \sum_{\substack{y=x \\ y=U(x)}} P(x)$$

$$* P(Y < 2) = \sum_{y=0}^1 g(y) = \frac{1}{5} + \frac{7}{30} = \frac{13}{30} \in [0, 1]$$

$$* P(X < 0) = \sum_{x=-2}^{-1} f(x) = \frac{1}{5} + \frac{1}{6} = \frac{11}{30} \in [0, 1]$$

$$\Rightarrow \textcircled{1} g(y=1) = \sum_{\substack{x=-1, 1 \\ 1=U(x)=f(x)^2}} f(x) = f(-1) + f(1) = \frac{1}{6} + \frac{1}{15} = \frac{7}{30}$$

$$\textcircled{2} g(y=2) = \sum_{\substack{y=U(x)=f(x)^2 \\ 4=U(x)=f(x)^2}} f(x) = f(-2) + f(2) = \left(\frac{1}{5}\right) + \left(\frac{11}{30}\right) = \frac{17}{30}$$

التاريخ: ٢٠٠ / /

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ex.2 Given p.d.f.  $f(x)$  as below

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ for } -\infty < x < \infty$$

Let  $y = x^2 \rightarrow$  Find p.d.f. of  $y$

Sol. A:  $-\infty < x < \infty$

B:  $0 < x^2 = y < \infty$

We know,  $g(y) = f[u'(y)] \cdot |J|$

$$\therefore y = u(x) = x^2$$

$$x = u'(y) = \pm \sqrt{y} \rightarrow J = \frac{dx}{dy} = \pm \frac{1}{2\sqrt{y}}$$

Note  $y = x^2$  not (1-1) since  $x$  has positive and negative values.

$$\therefore g(y) = \sum_{x=\pm\sqrt{y}} f[u'(y)] \cdot |J| = \sum_{x=\pm\sqrt{y}} f(x) |J|$$

$$\rightarrow g(y) = [f(-\sqrt{y}) + f(\sqrt{y})] \cdot |J|$$

$$\begin{aligned} g(y) &= \sum_{y=u(x)=(\pm\sqrt{y})^2} f(x) \cdot |J| \\ &= [f(-\sqrt{y}) + f(\sqrt{y})] |J| \\ &= \left[ \frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \right] \cdot \frac{1}{2\sqrt{y}} \\ &= \frac{2}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}} \\ &= \frac{y^{1/2} e^{-y/2}}{\sqrt{\pi} \cdot 2^{1/2}} \quad \text{for } 0 < y < \infty \\ &\quad \text{o.w} \end{aligned}$$

Gamma  $(\frac{1}{2}, 2)$

chi square with  $r=1$



[illegible]



# Transformations

Q1: If  $X$  has a uniform dist. on  $\{1, 2, 3\}$  then find the p.m.f. of  $Y = 4X + 2$ .

Sol.  $X \sim \text{unif.}(3)$  ;  $X = 1, 2, 3$  (d.r.v.)

$$A = \{X; X = 1, 2, 3\}$$

$$B = \{Y; Y = 6, 10, 14\}$$

$$Y = 4X + 2 = u(X) \in B$$

$$X = \frac{Y-2}{4} = u^{-1}(Y) \in A$$

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

$$g(y) = f[u^{-1}(y)] = f\left(\frac{y-2}{4}\right)$$

$$g(y) = \begin{cases} \frac{1}{3} & \text{for } y = 6, 10, 14 \\ 0 & \text{o.w.} \end{cases} \Rightarrow Y \sim \text{unif.}(3)$$

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Q2: Given a p.m.f. of  $X$ :

$$f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases} \leftarrow f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$Y = X^3 = u(X) \in B$$

$$X = \sqrt[3]{Y} = u^{-1}(Y) \in A$$

$$A = \{X; X = 1, 2, 3, \dots\}$$

$$B = \{Y; Y = 1, 8, 27, \dots\}$$

$$g(y) = f[u^{-1}(y)] = f(\sqrt[3]{y})$$

$$\therefore g(y) = \begin{cases} 7\left(\frac{1}{8}\right)^{\sqrt[3]{y}} & \text{for } y = 1, 8, 27, \dots \\ 0 & \text{o.w.} \end{cases}$$

Check  $g(y)$  is p.m.f

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(Transformations)

$$Q_3: f(x) = \begin{cases} \frac{x^2}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

Find p.d.f. of  $y = x^3$ .

Sol.  $y = x^3 = u(x) \in B$   
 $x = \sqrt[3]{y} = u^{-1}(y) \in A$

$$A = \{x; 0 < x < 3\}$$

$$B = \{y; 0 < y < 27\} \quad (y(0) = 0 \text{ \& } y(3) = 27)$$

$$J = [u^{-1}(y)]' = |x'| = \left| \frac{dx}{dy} \right|$$

$$J = (\sqrt[3]{y})' = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{y^2}}$$

$$|J| = \left| \frac{1}{3\sqrt[3]{y^2}} \right| = \frac{1}{3\sqrt[3]{y^2}}$$

$$\begin{aligned} f[u^{-1}(y)] &= f[\sqrt[3]{y}] \text{ \& } f(x) = \frac{x^2}{9} \\ &= \frac{(\sqrt[3]{y})^2}{9} = \frac{\sqrt[3]{y^2}}{9} \end{aligned}$$

$$\begin{aligned} \therefore g(y) &= f[u^{-1}(y)] |J| \\ &= \frac{(\sqrt[3]{y^2})}{9} \cdot \frac{1}{3(\sqrt[3]{y^2})} = \frac{1}{27} \end{aligned}$$

$$\therefore g(y) = \begin{cases} \frac{1}{27} & \text{for } 0 < y < 27 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore y \sim \text{unif}(0, 27)$$

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(Transformations)

Q4: Given a.p.d.f. of  $X$ :

$$f(x) = \begin{cases} 2xe^{-x^2} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find p.d.f. of  $Y = X^2$ .

Sol:  $Y = X^2 = u(X) \in A$   
 $X = \sqrt{Y} = u^{-1}(Y) \in B$

$$A = \{X; 0 < X < \infty\}$$

$$B = \{Y; 0 < Y < \infty\} \longrightarrow \begin{cases} Y(0) = (0)^2 = 0 \\ Y(\infty) = (\infty)^2 = \infty \end{cases}$$

$$\begin{aligned} f[u^{-1}(y)] &= f[\sqrt{y}] ; f(x) = 2xe^{-x^2} \\ &= 2\sqrt{y} e^{-(\sqrt{y})^2} = 2\sqrt{y} e^{-y} \\ &= 2\sqrt{y} e^{-y} \end{aligned}$$

$$\begin{aligned} J &= [u^{-1}(y)]' = X' = \frac{dx}{dy} \\ &= (\sqrt{y})' = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}} \end{aligned}$$

$$|J| = \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2\sqrt{y}}$$

$$\begin{aligned} \therefore g(y) &= f[u^{-1}(y)] |J| \\ &= (2\sqrt{y} e^{-y}) \left( \frac{1}{2\sqrt{y}} \right) = e^{-y} \end{aligned}$$

$$\therefore g(y) = \begin{cases} e^{-y} & \text{for } 0 < y < \infty \\ 0 & \text{o.w.} \end{cases}$$

(check  $g(y)$  is p.d.f.)

$$\therefore Y \sim \text{EXP}(\lambda=1) ; \lambda=1$$

التوزيع الأسي  
توزيع أسي



## (Transformations)

### Exercises (H.W.)

Q<sub>1</sub>:  $X \sim \text{unif}(0,1)$  &  $Y = -2 \ln X$ ; then show that:  
 $Y \sim \chi^2(2)$  or  $Y \sim G(1,2)$ .

Q<sub>2</sub>: If  $X \sim \text{unif}(0,1)$  &  $Y = -4 \ln X$ ; then find p.d.f. of  $Y$ .

Q<sub>3</sub>: If  $X \sim P(3)$ ;  $Y = 2X$ ; find p.d.f. of  $Y$ .

Q<sub>4</sub>: If  $X \sim b(3, \frac{1}{2})$ ;  $Y = X^2$ ; find p.f. of  $Y$ .

Q<sub>5</sub>: If  $X \sim f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$   
&  $Y = 8X^3$ , then find the p.d.f. of  $Y$ .

Q: Given a p.d.f. of  $X$ :

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

① Let  $Y = X(4-X)$  be a transformation.

Find a p.d.f. of  $Y$  ( $f_Y(y)$ ) & C.d.f.  $G(y)$ .

② Let  $Y = X(2-X)$  be a transformation.

Find a p.d.f. of  $Y$  ( $f_Y(y)$ ) & C.d.f.  $G(y)$ .

Sol. ①  $Y = X(4-X) = 4X - X^2 = u(x) \in B$

$$A = \{x; 0 < x < 2\}$$





## Chapter Title

### Random Variables and Probability Distribution

#### Exercise/P.6

Q<sub>1</sub>: A r.v.  $X$  has a discrete distribution with p.m.f.

$$f(x) = \begin{cases} cx & \text{for } x=1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Find the pr. dist. of  $X$ .

Sol.  $\because f(x)$  is a p.m.f.

$\because$  Condition (2) is satisfied  $\sum_{x=1}^5 f(x) = 1$

$$\sum_{x=1}^5 f(x) = f(1) + f(2) + f(3) + f(4) + f(5) = 1$$

$$= c[1+2+3+4+5] = 1$$

$$= c \left[ \frac{5(5+1)}{2} \right] = 1$$

$$= 15c = 1 \Rightarrow c = \frac{1}{15}$$

$$\therefore f(x) = \begin{cases} \frac{x}{15} & \text{for } x=1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{pr. dist. of } X = \{ (x_i, f(x_i)) , \forall x_i \in X \}$$

$$= \left\{ \left(1, \frac{1}{15}\right), \left(2, \frac{2}{15}\right), \left(3, \frac{3}{15}\right), \left(4, \frac{4}{15}\right), \left(5, \frac{5}{15}\right) \right\}$$

Q<sub>2</sub>:  $X$  has a uniform dist. on six integers: 2, 3, 4, 5, 6, 7. Find the p.m.f. of  $X$ .

Sol.  $X = 2, 3, 4, 5, 6, 7 \Rightarrow 6\text{-integers}$

$X \sim \text{uniform}(6)$

$$\therefore f(x) = \begin{cases} \frac{1}{K} & \text{for } x=1, 2, \dots, K \\ 0 & \text{o.w.} \end{cases} = \begin{cases} \frac{1}{6} & \text{for } x=2, 3, 4, 5, 6, 7 \\ 0 & \text{o.w.} \end{cases}$$





Q3: Given a set of integers 2, 3, ..., 15 Choose one integer which divisible by 3.

Let  $X =$  the chosen int. find the p.m.f. of  $X$ .

Sol.  $X = \{3, 6, 9, 12, 15\} = \text{d.r.v.}$

$R_X = \{3, 6, 9, 12, 15\}$  is Countable.

$X \sim \text{uniform}(5)$

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 3, 6, 9, 12, 15 \\ 0 & \text{o.w.} \end{cases}$$

p.m.f.

Q4: Given a set of integers  $\{1, 2, \dots, 10\}$ , choose an integer and determine its divisors.

Let  $X =$  no. of divisors.

Find the p.m.f. of  $X$ .

Sol.  $X =$  no. of divisors

| integer | divisor     | $X = \text{no. of divisor}$ | $f(x) = \text{p.m.f.} = \frac{X}{10}$ |
|---------|-------------|-----------------------------|---------------------------------------|
| 1       | 1           | 1                           | $\frac{1}{10}$                        |
| 2       | 1, 2        | 2                           | $\frac{2}{10}$                        |
| 3       | 1, 3        | 2                           | $\frac{2}{10}$                        |
| 4       | 1, 2, 4     | 3                           | $\frac{3}{10}$                        |
| 5       | 1, 5        | 2                           | $\frac{2}{10}$                        |
| 6       | 1, 2, 3, 6  | 4                           | $\frac{4}{10}$                        |
| 7       | 1, 7        | 2                           | $\frac{2}{10}$                        |
| 8       | 1, 2, 4, 8  | 4                           | $\frac{4}{10}$                        |
| 9       | 1, 3, 9     | 3                           | $\frac{3}{10}$                        |
| 10      | 1, 2, 5, 10 | 4                           | $\frac{4}{10}$                        |

$$\therefore f(x) = \begin{cases} \frac{1}{10} & \text{for } x=1 \\ \frac{2}{10} & \text{for } x=2 \\ \frac{3}{10} & \text{for } x=3 \\ \frac{4}{10} & \text{for } x=4 \\ 0 & \text{o.w.} \end{cases}$$

p.m.f.



## Exercise / P. 12

Q1:  $X \sim \text{unif.}(-1, 3)$ . Find p.d.f. of  $X$  and find  $P(0 \leq X \leq 2)$ ,  $P(X < 1 / -1 \leq X \leq 2)$

Sol. <sup>given</sup>  $X \sim \text{unif.}(-1, 3)$

$$\therefore f(x) = \begin{cases} \frac{1}{3-(-1)} & \text{for } -1 \leq x \leq 3 \\ 0 & \text{o.w.} \end{cases}$$
$$= \begin{cases} \frac{1}{4} & \text{for } -1 \leq x \leq 3 \\ 0 & \text{o.w.} \end{cases}$$

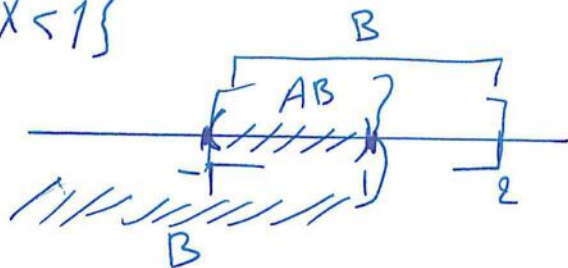
$$P(0 \leq X \leq 2) = \int_0^2 f(x) dx = \int_0^2 \frac{1}{4} dx = \frac{1}{4} x \Big|_0^2 = \frac{1}{4} [2-0] = \frac{1}{2}$$

$$P(\underbrace{X < 1}_A / \underbrace{-1 \leq X \leq 2}_B) = P(A|B) = \frac{P(AB)}{P(B)}$$

$$A = \{X : X < 1\} = \{X : -1 \leq X < 1\}$$

$$B = \{X : -1 \leq X \leq 2\}$$

$$AB = \{X : -1 \leq X < 1\}$$



$$P(B) = \int_{-1}^2 \frac{1}{4} dx = \frac{1}{4} x \Big|_{-1}^2 = \frac{1}{4} [2+1] = \frac{3}{4} \in [0, 1]$$

$$P(AB) = \int_{-1}^1 \frac{1}{4} dx = \frac{1}{4} x \Big|_{-1}^1 = \frac{1}{4} (1+1) = \frac{1}{2} \in [0, 1]$$

$$P(X < 1 / -1 \leq X \leq 2) = \frac{1/2}{3/4} = \frac{2}{3} \in [0, 1]$$

Q2: Given  $f(x) = \frac{1}{\pi(1+x^2)}$  for  $-\infty < x < \infty$ . Show that  $f(x)$  is a p.d.f.

Note:  $f(x)$  is called Cauchy dist.

Sol. I.P. Cond. ①  $f(x) \geq 0 \quad \forall x \in \mathbb{R}_x = (-\infty, \infty)$





$$x^2 \geq 0 \Rightarrow \pi(1+x^2) \geq 0 = \frac{1}{\pi(1+x^2)} \geq 0$$

$$= f(x) \geq 0$$

$$x \in (-\infty, \infty)$$

$$\text{if } x < \infty \rightarrow f(x) \geq 0$$

$$\text{if } x > -\infty \rightarrow f(x) \geq 0$$

$$\therefore f(x) \geq 0 \quad \forall x \in (-\infty, \infty)$$

T.P. Cond. ② is satisfied

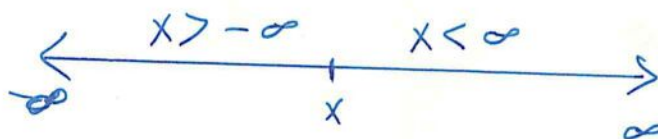
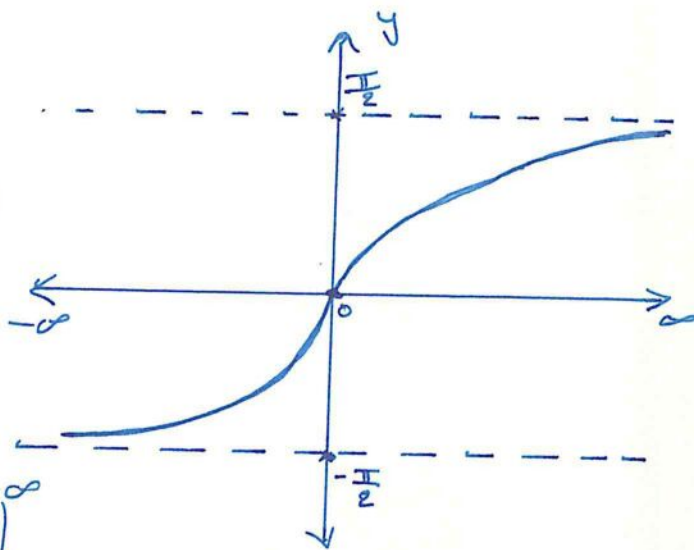
i.e. T.P.  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{1}{\pi} \tan^{-1} x \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi} [\tan^{-1}(\infty) - \tan^{-1}(-\infty)] = \frac{1}{\pi} \left[ \frac{\pi}{2} - \left[ -\frac{\pi}{2} \right] \right] = \frac{1}{\pi} (\pi) = 1$$

$\therefore$  Cond. ② is satisfied.

$\therefore f(x)$  is a p.d.f.



H.w. P.14 Given an integer 5, 6, 7, ..., 15. Choose one even integer. Let  $X$  be the chosen integer. Find  $F(x)$  and sketch its graph.

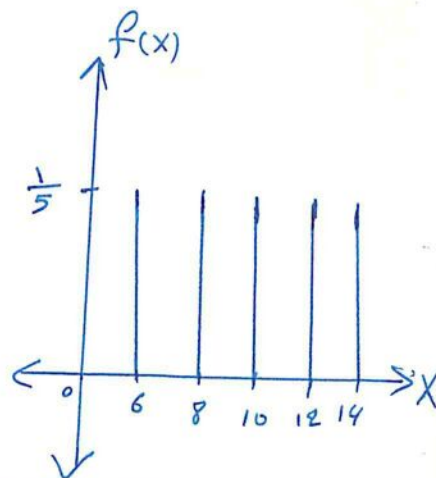
Sol.  $X$  = The even integer

$$\therefore X = 6, 8, 10, 12, 14$$

$$R_X = \{x : x = 6, 8, 10, 12, 14\} = \text{d.r.v.}$$

$$X \sim \text{unif.}(5)$$

$$\therefore f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 6, 8, 10, 12, 14 \\ 0 & \text{o.w.} \end{cases}$$

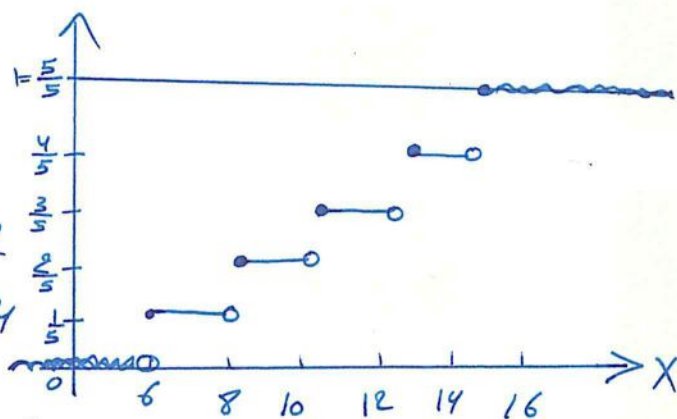


جواب  $\rightarrow$



| interval of $X$  | $X$<br>integer | P.m.f<br>$f(x) = \frac{1}{5}$ | $F(x) = \text{c.d.f.}$                                                                                        |
|------------------|----------------|-------------------------------|---------------------------------------------------------------------------------------------------------------|
| $X < 6$          | no. int.       | 0                             | 0 ← مبدأ الصفر                                                                                                |
| $6 \leq X < 8$   | 6              | $\frac{1}{5}$                 | $0 + \frac{1}{5} = \frac{1}{5}$                                                                               |
| $8 \leq X < 10$  | 8              | $\frac{1}{5}$                 | $0 + \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$                                                                 |
| $10 \leq X < 12$ | 10             | $\frac{1}{5}$                 | $0 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$                                                   |
| $12 \leq X < 14$ | 12             | $\frac{1}{5}$                 | $0 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5}$                                     |
| $X \geq 14$      | 14             | $\frac{1}{5}$                 | $0 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{5}{5} = 1$<br>→ تتصل بالواحد |

$$F(x) = \begin{cases} 0 & \text{for } x < 6 \\ \frac{1}{5} & \text{for } 6 \leq x < 8 \\ \frac{2}{5} & \text{for } 8 \leq x < 10 \\ \frac{3}{5} & \text{for } 10 \leq x < 12 \\ \frac{4}{5} & \text{for } 12 \leq x < 14 \\ 1 & \text{for } x \geq 14 \end{cases}$$



$$0 \leq F(x) \leq 1$$

### Exercise / P.16

Q<sub>1</sub>: Given a p.d.f.

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$$

a. Find c.d.f. of  $X$

b. Find  $P(X > 4.2)$  and  $P(X < 3.5)$  and  $P(X \leq -1)$

Sol.  $F(x) = \int_{-\infty}^x f(t) dt = \int_1^x \frac{1}{t^2} dt = \frac{-1}{t} \Big|_1^x = -\left[\frac{1}{x} - 1\right] = 1 - \frac{1}{x}$





$$\therefore F(x) = \begin{cases} 1 - \frac{1}{x} & \text{for } x \geq 1 \\ 0 & \text{o.w.} \end{cases}$$

Note:  $F(x)$  is conts. at  $x=1$

Since  $\lim_{x \rightarrow 1^+} F(x) = 1 - \frac{1}{1} = 0 = F(1^+)$

$\lim_{x \rightarrow 1^-} F(x) = 0 = F(1^-)$

$$\begin{aligned} \text{Now, } P(X > 4.2) &= 1 - P(X \leq 4.2) \\ &= 1 - F(4.2) \\ &= 1 - \left(1 - \frac{1}{4.2}\right) = 0.238 \end{aligned}$$

$$P(X < 3.5) = \int_1^{3.5} \frac{1}{x^2} dx = \left. -\frac{1}{x} \right|_1^{3.5} = -\left[\frac{1}{3.5} - 1\right] = 0.714$$

$$P(X \leq -1) = F(-1) = 0$$

Q2: Given a p.d.f.

$$f(x) = \begin{cases} ce^{-x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

a. Find the value of  $c$  and sketch  $F(x)$ .

b. Find  $F(x)$  and sketch its graph.

c.  $P(X \leq 3)$ ,  $P(X < 3)$ .

Sol.  $\therefore f(x)$  is p.d.f.

a.  $\therefore$  cond. ② is satisfied :  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} ce^{-x} dx = -ce^{-x} \Big|_0^{\infty} = -c \left[ \lim_{x \rightarrow \infty} e^{-x} - e^0 \right] = -c(-1) = c = 1$$

$$\therefore \boxed{c=1}$$





$$\therefore f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \text{b. } F(x) &= \int_{-\infty}^x f(t) dt \\ &= \int_0^x e^{-t} dt = -e^{-t} \Big|_0^x = -[e^{-x} - \underline{e^0}] = 1 - e^{-x} \end{aligned}$$

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{o.w.} \end{cases} \leftarrow F(x) \text{ is conts at } x=0 \text{ since } \begin{cases} F(0^+) = F(0^-) \\ F(0^+) = 1 - e^0 = 1 - 1 = 0 \\ F(0^-) = 0 \end{cases}$$

$$\text{c. } P(X \leq 3) = F(3) = 1 - e^{-3}$$

$$P(X \leq 3) = \int_0^3 e^{-x} dx = -e^{-x} \Big|_0^3 = -[e^{-3} - \underline{e^0}]$$

$$= 0.9502$$

$$\text{or } P(X < 3) = P(X \leq 3) = F(3) \text{ since (C.V.)}$$

H.W / P.17 Given a c.d.f.  $F(x)$ :

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the p.d.f.  $f(x)$ .

Sol.  $F(x)$  is conts. function.  $\Rightarrow F(x)$  is diff.

$$f(x) = \frac{dF(x)}{dx} = F'(x)$$

$$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$$

But is  $F(x)$  conts. at  $x=1$ ?

$$F(1^+) = \lim_{x \rightarrow 1^+} F(x) = 1 - e^{-1} \neq 0$$

Z



$$F(1) = \lim_{x \rightarrow 1^-} F(x) = 0$$

$\therefore F(1) \neq F(1) \Rightarrow F(x)$  is not conts. at  $x=1$

H.W / P. 20 Given a c.d.f.

$$F(x) = \begin{cases} 1 - \frac{4}{x^2} & \text{for } x > 2 \\ 0 & \text{for } x \leq 2 \end{cases}$$

1. Find  $f(x)$  and sketch  $F(x)$  &  $f(x)$  (H.W.)
2. Find  $P(X \leq 4)$ ,  $P(X > 8)$ ,  $P(2 < X \leq 8)$ ,  $P(X=2)$ .

Sol<sup>n</sup>  $F(x)$  is conts at  $x=2$

Since  $F(2^+) = F(2^-)$

$$F(2^+) = \lim_{x \rightarrow 2^+} F(x) = 1 - \frac{4}{4} = 0$$

$$F(2^-) = \lim_{x \rightarrow 2^-} F(x) = 0$$

$$f(x) = F'(x) = \begin{cases} \frac{8}{x^3} & \text{for } x > 2 \\ 0 & \text{for } x \leq 2 \end{cases}$$

$$P(X \leq 4) = F(4) = 1 - \frac{4}{16} = 1 - \frac{1}{4} = \frac{3}{4} \quad (\text{By def. of } F(x))$$

$$\begin{aligned} P(X > 8) &= 1 - P(X \leq 8) \\ &= 1 - F(8) \\ &= 1 - (1 - \frac{1}{16}) = 1 - \frac{15}{16} = \frac{1}{16} \end{aligned}$$

$$\begin{aligned} P(2 < X \leq 8) &= F(8) - F(2) \\ &= (1 - \frac{4}{64}) - 0 \\ &= 1 - \frac{1}{16} = \frac{15}{16} \end{aligned}$$

By Th.  $P(a < X \leq b) = F(b) - F(a)$

$$\begin{aligned} P(X=2) &= F(2^+) - F(2^-) \\ &= 0 \end{aligned} \quad \text{By Th. } P(X=x) = F(x^+) - F(x^-)$$

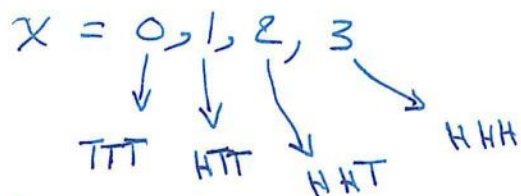
Since  $F(x)$  is a conts. at  $x=2$





Q<sub>1</sub>: Toss a coin 3-times. If  $X$  be the number of head (H). Find the p.f. of  $X$ .

Sol.  $X \equiv$  number of H.



$R_X = \{X : X = 0, 1, 2, 3\} = \text{Countable}$   
 $\therefore X$  is d.r.v.

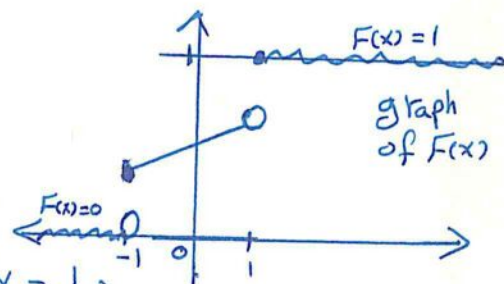
$\binom{3}{x} \equiv$  number of sample in event  $(X=x)$  when toss a coin 3-times.

$$P(X=x) = f(x) = \begin{cases} \frac{\binom{3}{x}}{8} & \text{for } x=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

p.f. o.w.

Q<sub>2</sub>: Given a d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+2}{4} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$



① Sketch  $F(x)$  and find  $P(X=-1)$  and  $P(X=\frac{1}{2})$ .

② If  $F(x)$  conts. at  $x=\frac{1}{2}$ ? why?

Sol.  $P(X=-1) = F(-1^+) - F(-1^-)$

$$= \frac{-1+2}{4} - 0 = \frac{1}{4}$$

By Th.  $P(X=x) = F(x^+) - F(x^-)$

$$P(X=\frac{1}{2}) = \frac{\frac{1}{2}+2}{4} - \frac{\frac{1}{2}+2}{4} = 0$$

$F(x)$  is conts. at  $x=\frac{1}{2}$  since  $P(X=\frac{1}{2}) = 0$



Q3: Given a p.d.f.  $f(x)$ :

$$f(x) = \begin{cases} 1-|x| & \text{for } |x| < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the d.f.  $F(x)$  and find  $P(X > 0)$  &  $P(|X| > \frac{1}{2})$ .  
Sol.

$$|x| < 1 \Rightarrow -1 < x < 1$$

$$|x| = -x \Rightarrow -1 < x < 0$$

$$|x| = x \Rightarrow 0 \leq x < 1$$

$$f(x) = \begin{cases} 1+x & \text{for } -1 < x < 0 \\ 1-x & \text{for } 0 \leq x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$F(x) = \int_{-\infty}^x f(t) dt$$

$$F(x) = \begin{cases} \int_{-1}^x (1+t) dt & \text{for } -1 \leq x < 0 \\ \int_0^x (1-t) dt & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x < -1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

$$\begin{aligned} \text{Find } P(X > 0) &= 1 - P(X \leq 0) \\ &= 1 - F(0) = 1 - \int_0^x (1-t) dt = \dots \end{aligned}$$

$$\begin{aligned} P(|X| > \frac{1}{2}) &= 1 - P(|X| \leq \frac{1}{2}) \\ &= 1 - P(-\frac{1}{2} \leq x \leq \frac{1}{2}) \\ &= 1 - \left[ \int_{-\frac{1}{2}}^0 f(x) dx + \int_0^{\frac{1}{2}} f(x) dx \right] \\ &= 1 - \left[ \int_{-\frac{1}{2}}^0 (1+x) dx + \int_0^{\frac{1}{2}} (1-x) dx \right] \\ &= \dots \end{aligned}$$





Q 4: Two dice are thrown once, let  $X$  be the absolute difference between the two numbers shown by the dice. Find the p.m.f. and c.d.f. of  $X$ .

Sol.  $S = \{ (d_1, d_2) : 1 \leq d_i \leq 6 ; i=1,2 \} = 6^2 = 36$

$$X = \left\{ \begin{array}{lll} |1-1|=0 & , & |2-1|=1 & , \dots , & |6-1|=5 \\ |1-2|=1 & & |2-2|=0 & & |6-2|=4 \\ |1-3|=2 & & |2-3|=1 & & |6-3|=3 \\ \vdots & & \vdots & & \vdots \\ |1-6|=5 & & |2-6|=4 & & |6-6|=0 \end{array} \right\}$$

$\therefore X = 0, 1, 2, 3, 4, 5$

Event  $(X=0) = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} = (6) \text{ elements}$

$f(x) = P(X=x)$

$f(0) = P(X=0) = \frac{6}{36}$

Event  $(X=1) = \{(2,1), (1,2), (2,3), (3,2), (4,3), (3,4), (4,5), (5,4), (5,6), (6,5)\} = (10) \text{ elements}$

$f(1) = P(X=1) = \frac{10}{36}$

Event  $(X=2) = \{(1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4)\} = (8) \text{ elements}$

$f(2) = P(X=2) = \frac{8}{36}$

Event  $(X=3) = \{(1,4), (4,1), (2,5), (5,2), (3,6), (6,3)\} = (6) \text{ elements}$

$f(3) = P(X=3) = \frac{6}{36}$

Event  $(X=4) = \{(1,5), (5,1), (2,6), (6,2)\} = (4) \text{ elements}$

$f(4) = P(X=4) = \frac{4}{36}$

Event  $(X=5) = \{(1,6), (6,1)\} = (2) \text{ elements}$

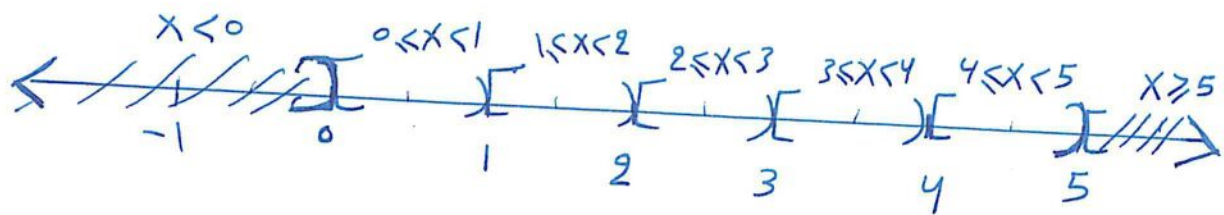
$f(5) = P(X=5) = \frac{2}{36}$





$$f(x) = \begin{cases} \frac{6}{36} & \text{for } x=0 \\ \frac{10}{36} & \text{for } x=1 \\ \frac{8}{36} & \text{for } x=2 \\ \frac{6}{36} & \text{for } x=3 \\ \frac{4}{36} & \text{for } x=4 \\ \frac{2}{36} & \text{for } x=5 \\ 0 & \text{o.w.} \end{cases}$$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{6}{36} & \text{for } 0 \leq x < 1 \\ \frac{16}{36} & \text{for } 1 \leq x < 2 \\ \frac{24}{36} & \text{for } 2 \leq x < 3 \\ \frac{30}{36} & \text{for } 3 \leq x < 4 \\ \frac{34}{36} & \text{for } 4 \leq x < 5 \\ 1 & \text{for } x \geq 5 \end{cases}$$





Q5: Given a p.f.

$$f(x) = \begin{cases} K \left(\frac{2}{3}\right)^x & \text{for } x=1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Find the value of  $K$ .

Sol: Since  $f(x)$  is p.m.f. then cond. ②  $\sum_{x \in R_x} f(x) = 1$  is satisfied.

By cond. ② we get:  $\sum_{x=1}^{\infty} K \left(\frac{2}{3}\right)^x = 1$

$$1 = K \sum_{x=1}^{\infty} \left(\frac{2}{3}\right)^x$$

$$1 = K \left[ \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^n + \dots \right]$$

$$1 = \left(\frac{2}{3}\right) K \left[ 1 + \left(\frac{2}{3}\right) + \dots + \left(\frac{2}{3}\right)^{n-1} + \dots \right]$$

Geometric Series

→  $r = \left(\frac{2}{3}\right) < 1 \Rightarrow$  the series is convergent.

$$1 = \left(\frac{2}{3}\right) K \left[ \frac{1}{1 - \left(\frac{2}{3}\right)} \right] = \left(\frac{2}{3}\right) K \left[ \frac{1}{\frac{1}{3}} \right] = \left(\frac{2}{3}\right) K (3) = 2K$$

→ مجموع متوالية هندسية  
حدها الأول واحد ونسبتها  $\left(\frac{2}{3}\right)$

$$\therefore 2K = 1 \Rightarrow \boxed{K = \frac{1}{2}}$$

$$\therefore f(x) = \begin{cases} \frac{1}{2} \left(\frac{2}{3}\right)^x & \text{for } x=1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Q6: Given a p.f.:

$$f(x) = \begin{cases} \frac{e^{-1}}{x!} & \text{for } x=0, 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Show that  $f(x)$  is a p.m.f.

→ يتبع





Sol. T.P. Cond. ①:  $f(x) \geq 0 \quad \forall x \in \mathbb{R}_x$

$$\frac{1}{e} > 0, \frac{1}{x!} > 0 \quad \forall x = 0, 1, 2, \dots$$

$$\text{Since if } x=0 \Rightarrow \frac{1}{e} > 0, \frac{1}{0!} = 1 > 0$$

$$\& \quad x=1 \Rightarrow \frac{1}{e} > 0, \frac{1}{1!} = 1 > 0$$

$\vdots$

$$\Rightarrow \frac{1}{e x!} > 0 \Rightarrow \text{Cond. ① is satisfied.}$$

$$\text{T.P. Cond. ②: } \sum_{x=0}^{\infty} \frac{1}{e x!} = 1$$

$$\sum_{x=0}^{\infty} \frac{1}{e x!} = \frac{1}{e} \sum_{x=0}^{\infty} \frac{1}{x!} = \frac{1}{e} \cdot e = 1$$

$\therefore$  Cond. ② is satisfied.  $\Rightarrow \therefore f(x)$  is p.m.f.

Q2: Given a p.d.f.

$$f(x) = \begin{cases} \frac{1}{8} x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the Value (t) s.t.  $P(X \leq t) = \frac{1}{4}$

(b) Find the Value (t) s.t.  $P(X \leq t) = \frac{1}{2}$

Sol.

$$(a) \quad P(X \leq t) = \frac{1}{4} \quad \dots \text{①}$$

$$\frac{1}{8} \int_0^t x dx = \frac{1}{16} x^2 \Big|_0^t = \frac{1}{16} t^2 \Rightarrow P(X \leq t) = \frac{1}{16} t^2 = \frac{1}{4} \quad (\text{by ①})$$

$$\frac{1}{4} t^2 = 1 \Rightarrow t^2 = 4 \Rightarrow t = \sqrt{4} = \pm 2 \Rightarrow \boxed{t=2} \text{ only}$$

since  $\{0 \leq x \leq 4\} = \mathbb{R}_x$  is positive.

$$(b) \quad P(X \leq t) = \frac{1}{2}$$

$$P(X \leq t) = \frac{1}{8} \int_0^t x dx = \frac{1}{16} x^2 \Big|_0^t = \frac{1}{16} t^2 \Rightarrow \frac{1}{16} t^2 = \frac{1}{2} \rightarrow \text{سيع}$$



$\frac{1}{8} t^2 = 1 \Rightarrow t^2 = 8 \Rightarrow t = \pm 2\sqrt{2} \Rightarrow t = +2\sqrt{2}$  (only)  
 since  $R_x = \{x: 0 \leq x \leq 4\}$  is positive.

Now; Find  $P(X \leq 2\sqrt{2})$ ?

$$P(X \leq 2\sqrt{2}) = \frac{1}{8} \int_0^{2\sqrt{2}} x \, dx = \frac{1}{16} x^2 \Big|_0^{2\sqrt{2}} = \frac{1}{16} (2\sqrt{2})^2 = \frac{1}{16} (8) = \underline{\underline{\frac{1}{2}}}$$

(c) Find the c.d.f. of  $X$ .

$$F(x) = \int_0^x f(t) \, dt = \int_0^x \frac{1}{8} t \, dt = \frac{x^2}{16}$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{16} & \text{for } 0 \leq x < 4 \\ 1 & \text{for } x \geq 4 \end{cases}$$

Note  $F(x)$  is conts. at  $x=0$  since  $F(0^+) = \frac{0^2}{16} = 0 = F(0^-)$ .

Q8: Suppose that the c.d.f. of a.r.v.  $X$  is as follows;

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{x^2}{9} & \text{for } 0 < x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}$$

Find and sketch (H.W.) the p.d.f of  $X$ .

Sol.  $f(x) = \frac{dF(x)}{dx} = F'(x)$  and  $F(x)$  is conts. and diff.

$$f(x) = F'(x) = \frac{2}{9} x$$

$$\therefore f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Note  $F(3^+) = 1$   
 $F(3^-) = \frac{3^2}{9} = 1$

$\therefore F(3^+) = F(3^-)$

$\therefore F(x)$  is conts. at  $x=3$





Q9: Answer by (true) or (False) and given the reasons:

① If  $X$  is c.r.v., then  $P(a \leq X \leq b) \neq P(a < X < b)$ .

(False) since  $P(a \leq X \leq b) = P(a < X < b)$  if  $X$  is c.r.v., because  $P(X=a)=0$  &  $P(X=b)=0$  when  $X$  is c.r.v.

② If  $X$  is d.r.v. with c.d.f.  $F(x)$ , then  $f(x) = \frac{d}{dx} F(x)$ .

(False), only when  $X$  is c.r.v.,  $f(x) = \frac{d}{dx} F(x)$ .

③ If  $X$  is a.r.v. with c.d.f.  $F(x)$ , then  $P(a < X < b) = F(b) - F(a)$ .

(False), since by Th.  $P(a < X \leq b) = F(b) - F(a)$

④ If a c.d.f.  $F(x)$  conts. at  $x=a$ , then  $P(X=a)=0$ .

(True), since if  $F(x)$  conts. at  $x=a$ , then  $F(a^+) = F(a^-)$  and  $P(X=a) = F(a^+) - F(a^-)$  (by Theorem).

⑤  $F(\infty) = 0$  &  $F(-\infty) = 1$

(False),  $F(\infty) = 1$  and  $F(-\infty) = 0$  (By Theorem).

⑥ state (Cauchy dist.):

$$f(x) = \begin{cases} \frac{1}{\pi(1+x^2)} & \text{for } -\infty < x < \infty \\ 0 & \text{o.w.} \end{cases} \quad \text{and p.m.f.}$$

(False), Cauchy dist. is p.d.f. since  $\mathbb{R}_X = \{x; -\infty < x < \infty\}$  and  $f(x) = \frac{1}{\pi(1+x^2)}$  for  $-\infty < x < \infty$  (only).

---

H.w.: Given a c.d.f.  $F(x) = \begin{cases} x^{-3} & \text{for } x \leq 3 \\ 0 & \text{for } x > 3 \end{cases}$

① Find and sketch the p.d.f.  $f(x)$ .

② Is  $F(x)$  conts. at  $x=3$ ? why?





Q10: Show that there does not exist any number (c) s.t. the following function  $f(x)$  would be a p.d.f.

$$f(x) = \begin{cases} \frac{c}{1+x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

Sol. If cond. ② is satisfied then  $f(x)$  is p.d.f.

I.e. T.P.  $\int_{-\infty}^{\infty} f(x) dx \stackrel{?}{=} 1$  (if we can that) لا يوجد عدد (c) يجعل هذه  $f(x)$  دالة احتمالية

$$\int_0^{\infty} \frac{c}{1+x} dx = c \ln(1+x) \Big|_0^{\infty} = c [\ln(\infty) - \ln(1)] = c [\infty - 0] = \infty$$

$\therefore$  c is not exist.

Then does not exist any no. (c) s.t.  $f(x)$  be a p.f.

H.w. Given a p.d.f.  $f(x)$ :

$$f(x) = \begin{cases} ce^{-2x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value c and sketch its graph.

② Find the c.d.f. of X ( $F(x)$ ) and sketch its graph.

Q11: Show that the following function be a p.f.

Sol. T.P. cond. ①

$$f(x) = \begin{cases} 2 \sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

note:  $2 \sin x \cos x = \sin 2x$

Sol. T.P. cond. ①  $f(x) \geq 0 \quad \forall x \in (0, \frac{\pi}{2})$

$$2 \sin x \cos x \geq 0 \quad \forall x \in (0, \frac{\pi}{2})$$

T.P. cond. ②

$$\int_0^{\frac{\pi}{2}} 2 \sin x \cos x dx = \int_0^{\frac{\pi}{2}} \sin 2x dx = -\frac{1}{2} \cos 2x \Big|_0^{\frac{\pi}{2}} = -\frac{1}{2} [\cos \pi - \cos 0] = -\frac{1}{2} [-1 - 1] = 1$$

$\therefore$  By cond. ① and ② we get:  $f(x)$  is p.d.f.



Q12: Show that if the following function be p.r.f.:

$$f(x) = \begin{cases} \frac{2}{\pi} \sin^2\left(\frac{x}{2}\right) & \text{for } 0 < x < \pi \\ 0 & \text{o.w.} \end{cases}$$

Sol. T.P. Cond. ①

i.e.  $f(x) \geq 0 \quad \forall x \in (0, \pi)$

$$\sin^2 \frac{x}{2} \geq 0 \Rightarrow \frac{2}{\pi} \sin^2 \frac{x}{2} \geq 0 \quad \forall x \in (0, \pi)$$

T.P. Cond. ②

Note:  $\sin^2 x = 1 - \cos x$

$$\begin{aligned} \frac{1}{\pi} \int_0^{\pi} 2 \sin^2 \frac{x}{2} dx &= \frac{1}{\pi} \int_0^{\pi} (1 - \cos x) dx \\ &= \frac{1}{\pi} (x - \sin x) \Big|_0^{\pi} = \frac{1}{\pi} [(\pi - 0) - (0 - 0)] \\ &= \frac{\pi}{\pi} = 1 \end{aligned}$$

$\therefore f(x)$  is a p.r.d.f.

Q13: A box has (4) red and (3) black balls. Choose a sample of (3) balls one by one without replacement. Let  $X$  be a number of black in a sample. Find the p.m.f. of  $X$  and sketch its graph. Also find the pr. that a sample has at least (2) red balls.

Sol.

|     |                                           |     |
|-----|-------------------------------------------|-----|
| 4 r | $\xrightarrow[\text{without rep}]{(1-1)}$ | K=3 |
| 3 b |                                           |     |

$X \equiv$  no. of black balls in a sample.

$\therefore X = 0, 1, 2, 3 = \text{d.r.v.}$

$f(x) = P(X=x) = \text{p.m.f.}$

Event  $(X=x)$ : a sample have (x) black and  $\overset{x}{(3-x)}$  red balls.





Event  $(X=x)$  has  $P_x^3 P_{3-x}^4$  and  $n(s) = P_3^7$   
 no. of black balls      no. of red balls

$$f(x) = P(X=x) = \begin{cases} \frac{P_x^3 P_{3-x}^4}{P_3^7} & \text{for } x=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

H.w. Find  $F(x)$  of this problem.

Q14: Consider the following c.d.f

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2}(x^3+1) & \text{for } -1 \leq x \leq 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

- Find ①  $P(-1 < x < 1)$  ,  $P(-1 \leq x \leq 1)$   
 ② p.d.f.  $f(x)$   
 ③  $P(-2 < x < 2)$

Sol.

$$\textcircled{1} P(-1 < x < 1) = P(-1 \leq x \leq 1) = F(1) - F(-1) \text{ (by Th.)}$$

$$= 1 - \frac{1}{2}((-1)^3 + 1)$$

$$= 1$$

$\uparrow$   
x is c.v.

$$\textcircled{2} f(x) = F'(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$\uparrow$   
p.d.f. of x

Note:  $F(x)$  is conts. at  $x = -1$

$$\textcircled{3} P(-2 < x < 2) = P(-2 \leq x \leq 2) = F(2) - F(-2) \text{ (by Th.)}$$

$$= 1 - 0$$

$$= 1$$

Note  $F(-2) = F(-2^+) - F(-2^-)$

$$= 0 - 0 = 0 \Rightarrow F \text{ is conts. at } x = -2$$



Q15: Given a p.d.f.

$$f(x) = \begin{cases} b \sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

Find the value of  $b$ .

Sol.  $\because f(x)$  is a p.d.f.  $\Rightarrow$  Cond. (2) is satisfied.

$$\because b \int_0^{\frac{\pi}{2}} \sin x \cos x dx = 1$$

$$1 = b \left[ \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2x dx \right] \quad (\text{since } \sin 2x = 2 \sin x \cos x)$$

$$1 = -\frac{b}{2} \cos 2x \Big|_0^{\frac{\pi}{2}}$$

$$1 = -\frac{b}{2} [\cos \pi - \cos 0]$$

$$1 = -\frac{b}{2} [-1 - (1)] \Rightarrow b = 1$$

$$\therefore f(x) = \begin{cases} \sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

---

Q16: Given a d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < -2 \\ \frac{x+3}{5} & \text{for } -2 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

(1) Find  $P(X = -2)$ ,  $P(-3 < X < -2)$

(2) Find  $f(x)$

(3) Is  $F(x)$  conts. at  $x = 2$ ? why?

Sol.

$$(1) P(X = -2) = F(-2^+) - F(-2^-) \\ = \left( \frac{-2+3}{5} \right) - 0 = \frac{1}{5}$$

$$\text{by Th. } P(X=x) = F(x^+) - F(x^-)$$

$$P(-3 < X < -2) \neq P(-3 < X \leq -2)$$

since  $(F(x) \text{ is disconts. at } x = -2)$

→



$$(2) f(x) = \frac{dF(x)}{dx} = F'(x) = \begin{cases} \frac{1}{5} & \text{for } -2 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

$$P(-3 < x < -2) = \int_{-3}^{-2} f(x) dx = 0$$

$$(3) F(x) \text{ is conts. at } x=2 \text{ since } F(2^-) = F(2^+), \\ F(2^+) = 1, F(2^-) = \frac{2+3}{5} = \frac{5}{5} = 1$$

Q17: Given the following p.m.f.

$$f(x) = \begin{cases} K \left(\frac{2}{3}\right)^{x-2} & \text{for } x=2, 3, 4, \dots \\ 0 & \text{o.w.} \end{cases}$$

Find the value of  $K$ .

sol.  $\because f(x)$  is p.m.f. then cond. (2) is satisfied.

$$\therefore K \sum_{x=2}^{\infty} \left(\frac{2}{3}\right)^{x-2} = 1$$

$$\underline{\text{Note}} \quad K \sum_{x=2}^{\infty} \left(\frac{2}{3}\right)^{x-2} = K \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^x$$

$$\therefore 1 = K \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^x = K \left[ \left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \dots + \left(\frac{2}{3}\right)^n + \dots \right]$$

$$1 = K \left[ 1 + \left(\frac{2}{3}\right) + \dots + \left(\frac{2}{3}\right)^n + \dots \right]$$

Geometric series

$\therefore |r| = \left|\frac{2}{3}\right| < 1 \Leftrightarrow$  the series is convergent

$$1 = K \left[ \frac{1}{1 - \left(\frac{2}{3}\right)} \right] = K \cdot \frac{1}{\left(\frac{1}{3}\right)} = 3K$$

$$\Rightarrow 3K = 1 \Rightarrow \boxed{K = \frac{1}{3}}$$

$$\therefore f(x) = \begin{cases} \frac{1}{3} \left(\frac{2}{3}\right)^x & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$





Q18: Given a set of 3 boys and 4 girls. Choose a sample of four students. If  $y$  be a random variable represents the number of boys in a sample:

- 1- Find the pr. distribution of  $y$ .
- 2- Find the c.d.f. of  $y$  and sketch its graph.
- 3- Find the pr. of the following by two methods:
  - (i) If a sample has at least (2) girls.
  - (ii) If a sample has at most (1) girl.

Sol.



$Y \equiv$  no. of boys in a sample.

$$Y = \{0, 1, 2, 3\} = \text{d.r.v.}$$

$$n(s) = {}_7^7$$

$$\therefore f(y) = \begin{cases} \frac{{}_3^y {}_4^{4-y}}{{}_7^7} & \text{for } y = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

o.w.

$$y = 0 \rightarrow f(0) = P(Y=0) = \frac{{}_0^3 {}_4^4}{{}_7^7} = \frac{1}{35}$$

$$y = 1 \rightarrow f(1) = P(Y=1) = \frac{{}_1^3 {}_3^4}{{}_7^7} = \frac{12}{35}$$

$$y = 2 \rightarrow f(2) = P(Y=2) = \frac{{}_2^3 {}_2^4}{{}_7^7} = \frac{18}{35}$$

$$y = 3 \rightarrow f(3) = P(Y=3) = \frac{{}_3^3 {}_1^4}{{}_7^7} = \frac{4}{35}$$

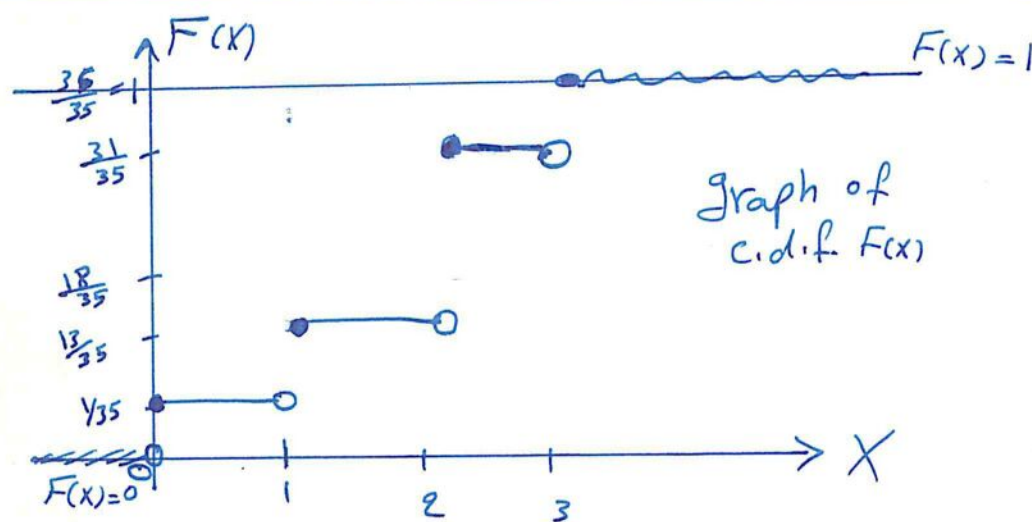
pr. dist. of  $X = \{ (x_i, f(x_i)) ; \forall x_i \in R_X \}$

$$\therefore \text{pr. dist. of } X = \left\{ \left(0, \frac{1}{35}\right), \left(1, \frac{12}{35}\right), \left(2, \frac{18}{35}\right), \left(3, \frac{4}{35}\right) \right\}$$



(ii) To find c.d.f.  $F(y)$

| interval       | integer    | $f(y)$  | $F(y)$  |
|----------------|------------|---------|---------|
| $y < 0$        | no integer | 0       | 0       |
| $0 \leq y < 1$ | 0          | $1/35$  | $1/35$  |
| $1 \leq y < 2$ | 1          | $12/35$ | $13/35$ |
| $2 \leq y < 3$ | 2          | $18/35$ | $31/35$ |
| $y \geq 3$     | 3          | $4/35$  | 1       |



$$(iii) P(y \leq 2) = \sum_{y=0}^2 f(y) = f(0) + f(1) + f(2) = \frac{1}{35} + \frac{12}{35} + \frac{18}{35} = \frac{31}{35}$$

$$\text{or } P(y \leq 2) = F(2) = \frac{31}{35}$$

$$P(y=3) = f(3) = \frac{4}{35}$$

$$\text{or } P(y=3) = F(3^+) - F(3^-) = 1 - \frac{31}{35} = \frac{4}{35}$$

$$\begin{array}{c|c} g & b \\ \hline 2 & 2 \\ 3 & 1 \\ 4 & 0 \end{array} \left. \vphantom{\begin{array}{c|c} g & b \\ \hline 2 & 2 \\ 3 & 1 \\ 4 & 0 \end{array}} \right\} \rightarrow y \leq 2$$

$K=4$

$$\begin{array}{c|c} g & b \\ \hline 0 & 4 \\ 1 & 3 \end{array} \rightarrow y=3$$

$K=4$





Q.19: A box has (3) bad and (4) good tubes. Choose (3) tubes at once from this box.

Let  $X$  be a r.v. represents the number of bad tubes in a sample.

1- Find the pr. function of  $X$  and sketch its graph.

2- Find the c.d.f. of  $X$  and sketch its graph.

3- (i) Find  $P(X < 1 \mid 1 < X < 3)$ .

(ii) Find the pr. that a sample has at least (1) good tube.

Sol.

$$\begin{array}{|c|c|} \hline 3 \text{ b} \\ 4 \text{ g} \\ \hline \end{array} \xrightarrow{\text{at once}} \boxed{\phantom{000}} \quad , \quad n(s) = \binom{7}{3}$$

$n=7 \qquad k=3$

$X \equiv$  no. of bad tube in a sample.

$$\therefore R_X = \{0, 1, 2, 3\}$$

$X = 0, 1, 2, 3 = \text{d.r.v.}$

Event  $(X=x)$ : a sample has  $x$ -bad and  $(k-x)$  good tubes.

(1)  $f(x) = P(X=x) = \text{p.m.f.}$

$$\therefore f(x) = \begin{cases} \frac{\binom{3}{x} \binom{4}{3-x}}{\binom{7}{3}} & \text{for } x=0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

(2) C.d.f.  $F(x) = \sum_{x_j \leq x} f(x_j)$

| interval       | integer    | $f(x) = \text{p.m.f.}$ | C.d.f. $F(x)$ |
|----------------|------------|------------------------|---------------|
| $X < 0$        | no integer | 0                      | 0             |
| $0 \leq X < 1$ | 0          | $4/35$                 | $4/35$        |
| $1 \leq X < 2$ | 1          | $18/35$                | $22/35$       |
| $2 \leq X < 3$ | 2          | $12/35$                | $34/35$       |
| $X \geq 3$     | 3          | $1/35$                 | 1             |

\* Sketch its graph.



$$(i) \frac{P(X < 1)}{A} / \frac{P(1 < X < 3)}{B} = P(A|B) = \frac{P(AB)}{P(B)}$$

$$A = \{X : 0 < X < 1\}$$

$$B = \{X : 1 < X < 3\}$$

$$AB = \emptyset$$



$$\therefore P(X < 1 / 1 < X < 3) = \frac{P(\emptyset)}{P(X=2)} = \frac{0}{f(2)} = \frac{0}{18/35} = 0$$

$$(ii) P(X \leq 2) = F(2) \\ = \frac{34}{35}$$

$$\begin{aligned} \text{or } P(X \leq 2) &= f(0) + f(1) + f(2) \\ &= \frac{4}{35} + \frac{18}{35} + \frac{12}{35} \\ &= \frac{34}{35} \end{aligned}$$

| g   |  | b |                        |
|-----|--|---|------------------------|
| 1   |  | 2 | } \rightarrow X \leq 2 |
| 2   |  | 1 |                        |
| 3   |  | 0 |                        |
| K=3 |  |   |                        |

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Group (1)

Q1: Given a.p.d.f. of  $X$ :

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the c.d.f. of  $X$  ( $F(x)$ ).

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t) dt \\ &= 2 \int_0^x t dt = \frac{2t^2}{2} \Big|_0^x = x^2 \end{aligned}$$

$$\therefore F(x) = \begin{cases} x^2 & \text{for } 0 \leq x < 1 \\ 0 & \text{o.w.} \end{cases}$$

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Q2: If a.r.v.  $X \sim \text{unif}(0,10)$ , Calculate the pr. that:

①  $P(X < 3)$  ②  $P(X > 6)$

Sol.  $\therefore X \sim \text{uniform}(0,10)$

$$\therefore f(x) = \begin{cases} \frac{1}{10} & \text{for } 0 < x < 10 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X < 3) = \int_0^3 \frac{1}{10} dx = \frac{x}{10} \Big|_0^3 = \frac{1}{10} (3-0) = \frac{3}{10}$$

$$P(X > 6) = \int_6^{10} \frac{1}{10} dx = \frac{x}{10} \Big|_6^{10} = \frac{1}{10} (10-6) = \frac{4}{10}$$

Q3: Given a c.d.f. of  $X$ :

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{16} & \text{for } 0 \leq x < 4 \\ 1 & \text{for } x \geq 4 \end{cases}$$



Find the pr. of: ①  $P(X=4)$ , ②  $P(X>0)$

Sol.  $P(X=4) = F(4^+) - F(4^-)$   
 $= 1 - \frac{4^2}{16} = 1 - \frac{16}{16} = 1 - 1 = \text{Zero}$

$P(X>0) = 1 - P(X \leq 0)$   
 $= 1 - F(0)$  by def of  $F(x)$   
 $= 1 - F(0^+)$  by theorem  $F(x) = F(x^+)$   
 $= 1 - \frac{0^2}{16} = 1 - 0 = 1$

Q<sub>4</sub>: Given a.c.d.f. of  $X(F(x))$ , if  $F$  is continuous at  $x=2$ ,  
 Is  $P(0 < X < 2) = F(2) - F(0)$ ? Why?

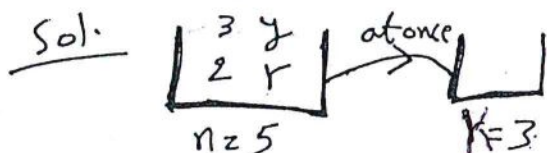
Yes, since  $X$  is c.r.v. &  $P(X=2)=0$

then  $P(0 < X < 2) = P(0 < X \leq 2) = F(2) - F(0)$  by theorem.

### Group (2)

Q<sub>1</sub>: A box has (3) yellow and (2) red cards. Choose a sample of (3) cards. Let  $X$  be a number of yellow cards in a sample, then find:

- ① the pr. f. of  $X$  ( $f(x)$ ),
- ② the c.d.f. of  $X$  ( $F(x)$ ),
- ③ the pr. that a sample has (2) yellow cards.



$X \equiv$  no. of yellow cards in a sample.

$X = 1, 2, 3$  (d.r.v.)

$f(x) = \begin{cases} \frac{C_x^3 C_{3-x}^2}{C_5^5} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$   
 p.m.f.



| interval       | no. of integer | p.f.<br>$f(x)$                           | c.d.f.<br>$F(x) = \sum_{x_j \leq x} f(x_j)$ |
|----------------|----------------|------------------------------------------|---------------------------------------------|
| $X < 1$        | no integer     | 0                                        | 0                                           |
| $1 \leq X < 2$ | 1              | $f(1) = \frac{{}^3C_1 {}^2C_2}{{}^5C_3}$ | $F(1) = f(1) + 0$                           |
| $2 \leq X < 3$ | 2              | $f(2) = \frac{{}^3C_2 {}^1C_1}{{}^5C_3}$ | $F(2) = 0 + f(1) + f(2)$                    |
| $X \geq 3$     | 3              | $f(3) = \frac{{}^3C_3 {}^0C_0}{{}^5C_3}$ | $F(3) = 0 + f(1) + f(2) + f(3) = 1$         |

$$\therefore F(x) = \begin{cases} 0 & \text{for } X < 1 \\ F(1) & \text{for } 1 \leq X < 2 \\ F(2) & \text{for } 2 \leq X < 3 \\ F(3) = 1 & \text{for } X \geq 3 \end{cases}$$

$$\textcircled{3} P(X=2) = P(\text{2-yellow cards in a sample}) = f(2) = \frac{{}^3C_2 {}^1C_1}{{}^5C_3}$$

Q<sub>2</sub> Show that there does not exist any number (c) such that the following function  $f(x)$  would be a p.f. of  $X$ :

$$f(x) = \begin{cases} ce^x & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

Sol. by Condition (2): (If  $f(x)$  p.f. then it satisfy this condition).

$$c \int_0^{\infty} e^x dx = 1 \Rightarrow c \int_0^{\infty} e^x dx = c e^x \Big|_0^{\infty} = c [e^{\infty} - e^0] = c [\infty - 1] = c(\infty) \neq 1$$

$\therefore$  there does not exist any number (c) such that the following function  $f(x)$  would be a p.f. of  $X$ .

Q<sub>3</sub>: Given a p.d.f. of  $X$ :

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{o.w.} \end{cases}$$



Find the value ( $t$ ) such that  $P(X \leq t) = \frac{1}{16}$ .

Sol.

$$P(X \leq t) = \frac{1}{8} \int_0^t x dx = \frac{1}{8} \cdot \frac{x^2}{2} \Big|_0^t = \frac{1}{16} (t^2 - 0^2) = \frac{t^2}{16} = \frac{1}{16}$$

$$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1 \Rightarrow t = 1 \text{ only since } t \in [0, 4]$$

### Group (3)

Q1: Given the d.f.  $F(x)$  of  $X$ :

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+1}{2} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find: ①  $P(X=1)$ , ②  $P(X < -1)$ , ③  $P(-\frac{1}{2} < X < \frac{1}{2})$

Sol.

$$\begin{aligned} ① P(X=1) &= F(1^+) - F(1^-) \quad (\text{by theorem}) \\ &= 1 - \left(\frac{1+1}{2}\right) \\ &= 1 - \frac{2}{2} = 1 - 1 = \text{Zero} \end{aligned}$$

$$\begin{aligned} ② P(X < -1) &= F(-1^-) \quad (\text{by theorem}) \\ &= 0 \end{aligned}$$

$$\begin{aligned} ③ P(-\frac{1}{2} < X < \frac{1}{2}) &= P(-\frac{1}{2} < X \leq \frac{1}{2}) \quad \text{Since } F \text{ is Contr. at } x = \frac{1}{2} \\ &= F(\frac{1}{2}) - F(-\frac{1}{2}) \quad \text{by theorem} \\ &= \frac{(\frac{1}{2}+1)}{2} - \frac{(-\frac{1}{2}+1)}{2} \\ &= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2} \end{aligned}$$

Q2: Find the value of ( $c$ ) from the following Pr. function:

$$f(x) = \begin{cases} c \left(\frac{1}{4}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Sol. by Cond. (2), we get:  $\sum_{x=1}^{\infty} f(x) = 1$



$$\therefore \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = 1$$

$$C \left[ \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots \right] = \frac{1}{4} C \left[ 1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \dots \right]$$

Geometric series

where  $r = \frac{1}{4} < 1 \Rightarrow$  this series is convergent (by theorem)

$$\therefore \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = \frac{1}{1 - \frac{1}{4}} = \frac{1}{3/4} = \frac{4}{3} \quad (\text{by theorem})$$

$$\Rightarrow C \left( \frac{1}{4} \right) \left( \frac{4}{3} \right) = 1 \Rightarrow \frac{C}{3} = 1 \Rightarrow C = 3$$

$$\therefore f(x) = \begin{cases} 3 \left(\frac{1}{4}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Q<sub>3</sub>: Is the following function is pr. function? Why?

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ①  $\sum_{x=1}^5 \frac{1}{5} = \frac{6}{5} > 1$

$\therefore$  Cond. ① doesn't satisfied.

$\therefore f(x)$  is not pr. function.

Q<sub>4</sub>: Find the c.d.f.  $F(x)$  from the following pr. f of  $x$ :

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \leq x < 2 \\ 0 & \text{o.w.} \end{cases}$$

Sol.  $F(x) = \int_{-\infty}^x f(t) dt = \int_0^x t dt + \int_1^x (2-t) dt$

$$= \left. \frac{t^2}{2} \right|_0^x + \left( 2t - \frac{t^2}{2} \right) \Big|_1^x$$

$$= \frac{1}{2} x^2 + \left( 2x - \frac{1}{2} x^2 \right) - \left( 2 - \frac{1}{2} \right)$$

$$\therefore F(x) = \begin{cases} \left( 2x - \frac{3}{2} - \frac{1}{2} x^2 \right) & x \in [1, 2] \\ \left( \frac{1}{2} x^2 \right) & x \in [0, 1] \end{cases}$$



$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} x^2 & \text{for } 0 \leq x < 1 \\ 2x - \frac{1}{2} x^2 - \frac{3}{2} & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

Group (4)

Q<sub>1</sub>: Find the pr. distribution of X from the following p.f. of

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

$$\begin{aligned} \text{Pr. dist. of } X &= \{ (x_i, f(x_i)) ; i = 1, 2, 3 \} \\ &= \left\{ \left(1, \frac{1}{3}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{3}\right) \right\} \end{aligned}$$

Q<sub>2</sub>: Show that there does not exist any number (c) such that the following function f(x) would be a p.f.:

$$f(x) = \begin{cases} ce^{-x} & \text{for } x < 0 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

If f(x) is p.f. then Cond. (2) is satisfied :

$$\begin{aligned} c \int_{-\infty}^0 e^{-x} dx &= -ce^{-x} \Big|_{-\infty}^0 = -c [e^0 - e^{\infty}] = -c [1 - \infty] = 1 \\ &= -c + \infty = 1 \end{aligned}$$

$\Rightarrow \therefore c \neq 1$   
 ∴ There does not exist any number (c) such that the following function f(x) would be a p.f.

Q<sub>3</sub>: Given a set of 3-boys and 3-girls. A sample of (8) students are choosing. If Z be a r.v. represents the number of boys in a sample, then find:



- ① the pr. distribution of  $Z$ .
- ② the c.d.f. of  $Z$ .
- ③ the pr. that a sample has 3-boys.

Sol.  $\begin{array}{|c|c|} \hline 3 & b \\ \hline 3 & g \\ \hline \end{array}$  at once  $r=2$   
 $n=6$

$Z \equiv$  no. of boys in a sample.

$Z = 0, 1, 2$

$$\textcircled{1} f(z) = \begin{cases} \frac{\binom{3}{z} \binom{3}{2-z}}{\binom{3}{2}} & \text{for } z = 0, 1, 2 \\ 0 & \text{o.w.} \end{cases}$$

pr. distribution of  $Z = \{ (z_i, f(z_i)); i = 0, 1, 2 \}$

$$= \{ (0, f(0)), (1, f(1)), (2, f(2)) \}$$

$$= \left\{ \left( 0, \frac{\binom{3}{0} \binom{3}{2}}{\binom{3}{2}} \right), \left( 1, \frac{\binom{3}{1} \binom{3}{1}}{\binom{3}{2}} \right), \left( 2, \frac{\binom{3}{2} \binom{3}{0}}{\binom{3}{2}} \right) \right\}$$

$$= \left\{ \left( 0, \frac{3(1)}{3} \right), \left( 1, \frac{(3)3}{3} \right), \left( 2, \frac{3(1)}{3} \right) \right\}$$

$$= \{ (0, 1), (1, 3), (2, 1) \}$$

$$\textcircled{2} F(x) = \begin{cases} 0 & \text{for } x < 0 \\ f(0) + 0 = F(0) & \text{for } 0 \leq x < 1 \\ f(1) + f(0) + 0 = F(1) & \text{for } 1 \leq x < 2 \\ f(2) + f(1) + f(0) + 0 = F(2) = 1 & \text{for } x \geq 2 \end{cases}$$

$$\textcircled{3} P(\text{A sample has 3-boys}) = P(Z=3) = f(3) = 0$$



### Group(5)

Q<sub>1</sub>: Given a p.d.f. of X:

$$f(x) = \begin{cases} ce^x & \text{for } x < 0 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value of c.

② Find F(x)

Sol. ① by cond. (2), we get:  $c \int_{-\infty}^0 e^x dx = 1$

$$\Rightarrow ce^x \Big|_{-\infty}^0 = 1 \Rightarrow c[e^0 - e^{-\infty}] = 1 \Rightarrow c[1 - 0] = 1$$

$$\therefore c = 1$$

$$\therefore f(x) = \begin{cases} e^x & \text{for } x < 0 \\ 0 & \text{o.w.} \end{cases} \quad \left( \Rightarrow \begin{matrix} x < 0 \\ \text{or } x < 0 \end{matrix} \right)$$

$$\textcircled{2} F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^x e^t dt = e^t \Big|_{-\infty}^x = e^x - e^{-\infty} = e^x$$

$$\therefore F(x) = \begin{cases} e^x & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

Q<sub>2</sub>: Show that the following function is P.f.:

$$f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ①:  $7\left(\frac{1}{8}\right)^1 = f(1) \geq 0$   
 $7\left(\frac{1}{8}\right)^2 = f(2) \geq 0$

$\therefore f(x) \geq 0$  & Cond. (1) is satisfied.

$$\text{Cond. ② } 7 \sum_{x=1}^{\infty} \left(\frac{1}{8}\right)^x = 7 \left[ \left(\frac{1}{8}\right)^1 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \dots \right]$$
$$= \left(\frac{7}{8}\right) \left[ 1 + \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \dots \right]$$

Geometric Series



by theorem &  $r = |\frac{1}{8}| < 1 \Rightarrow$  this series is convergent  
 by theorem, we get:  $\sum_{x=0}^{\infty} (\frac{1}{8})^x = \frac{1}{1 - \frac{1}{8}} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$

$$\therefore \sum_{x=1}^{\infty} (\frac{1}{8})^x = (\frac{8}{8}) - (\frac{1}{8}) = 1$$

$\therefore$  Cond. (2) is satisfied.

Q<sub>3</sub>: If  $X \sim \text{uniform}(0,1)$ , find p.f.  $f(x)$  of  $X$ .

Sol.

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Q<sub>4</sub>: Given a c.d.f. of  $X$

$$F(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{1}{2}(x^2+1) & \text{for } -1 < x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find the p.f. of  $X$  ( $f(x)$ ).

Sol.

$$f(x) = F'(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{3}{2}x^2 & \text{for } -1 < x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } -1 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

← since  $F$  is continuous at  $x = -1$

Q<sub>1</sub>: Is the following Group (6) function is pr. function? Why?

$$f(x) = \begin{cases} \frac{x-2}{5} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ①:  $f(1) = \frac{-1}{5} < 0$ ,  $f(2) = 0$ ,  $f(x) \geq 0$   
 $x = 3, 4, 5$   
 $\therefore f$  is not pr. function.



Q2: Given a c.d.f.  $F$  of  $X$  as follows:

$$F(x) = \begin{cases} 1 - (1+x)e^{-x} & \text{for } x=1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

① Find  $P(X \leq -2)$ ,  $P(X > 4)$

② Is  $F$  continuous at  $x=0$ ? Why?

Sol.

①  $P(X \leq -2) = F(-2)$  by def. of  $F$   
 $= 0$

$$\begin{aligned} P(X > 4) &= 1 - P(X \leq 4) \\ &= 1 - F(4) \\ &= 1 - [(1+4)e^{-4}] \end{aligned}$$

② Yes, since  $F(0^+) = F(0^-) = \text{Zero}$ .

Q3: State Cauchy distribution.

$$f(x) = \frac{1}{\pi(1+x^2)} \quad \text{for } -\infty < x < \infty$$

Q4: Given a c.d.f. of  $X$ :

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find the pr. f. of  $X$  ( $f(x)$ ).

Sol.

$$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

→ since  $F(0^-) = F(0^+)$   
&  $F$  is contrs.  
at  $x=0$

Group (7)

Consider the c.d.f. of  $X$ :

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$



- ① Find the p.f. of  $X$  ( $f(x)$ ).
- ② Is  $F$  conts. at  $x=1$ ? Why?
- ③ Find  $P(X=0)$ .

Sol.

$$① f(x) = F'(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x \geq 1 \end{cases} = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for o.w.} \end{cases}$$

② Yes, since  $F(1^+) = F(1^-) = 1$ .

③  $P(X=0) = F(0^+) - F(0^-)$   
 $= 0 - 0 = \text{Zero}$

Q2: Show that the following function is Pr. f. of  $X$ :

$$f(x) = \begin{cases} (\frac{1}{2})^x & \text{for } x=1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Sol.

Cond. ①  $(\frac{1}{2})^1 > 0, (\frac{1}{2})^2 > 0, \dots$   
 $\therefore f(x) \geq 0 \quad \forall x=1, 2, 3, \dots$

Cond. ②  $\sum_{x=1}^{\infty} (\frac{1}{2})^x = [(\frac{1}{2})^1 + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \dots]$   
 $= (\frac{1}{2}) [1 + (\frac{1}{2}) + (\frac{1}{2})^2 + \dots]$   
 $= \frac{1}{2} \left( \frac{1}{1 - \frac{1}{2}} \right) = \frac{1}{2} \cdot 2 = 1$   
Geometric Series

Q3: Toss a coin fourth times ( $4^{\text{th}}$  times). Find the C.d.f. of a r.v.  $X$ , when  $X$  represents the numbers of tail (T) in a sample.

Sol.

$$f(x) = \begin{cases} \frac{C_x}{16} & \text{for } x=0, 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases} \quad | \quad n(s) = 2^4 = 16$$

$$F(x) = \sum_{x_j \leq x} f(x_j)$$



$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ F(0) = 0 + f(0) & 0 \leq x < 1 \\ F(1) = 0 + f(0) + f(1) & 1 \leq x < 2 \\ F(2) = 0 + f(0) + f(1) + f(2) & 2 \leq x < 3 \\ F(3) = 0 + f(0) + f(1) + f(2) + f(3) & 3 \leq x < 4 \\ F(4) = 1 = 0 + f(0) + f(1) + f(2) + f(3) + f(4) & x \geq 4 \end{cases}$$

### Group (8)

Q<sub>1</sub>: Show that the following function is pr. function:

$$f(x) = \begin{cases} \frac{x+2}{25} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. (9)  $\frac{1}{25} \sum_{x=1}^5 (x+2) = \frac{1}{25} [3+4+5+6+7] = \frac{25}{25} = 1$

Cond. (1)  $f(1) = \frac{3}{25} > 0, f(2) = \frac{4}{25} > 0, \dots, f(5) = \frac{7}{25} > 0$

$\therefore f(x) \geq 0 \quad \forall x = 1, 2, 3, 4, 5$

$\therefore f(x)$  is pr. f.

Q<sub>2</sub>: Given a c.d.f. of  $X$ :

$$F(x) = \begin{cases} 1 - (1+x)e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find: (1)  $P(X \leq 2) = F(2)$  by def. of  $F$   
 $= 1 - (1+2)e^{-2}$

(2)  $P(1 < X < 3) = P(1 \leq X \leq 3)$  since  $F$  is cont. at  $x=3$   
 $= F(3) - F(1)$   
 $= (1 - (1+3)e^{-3}) - (1 - (1+1)e^{-1})$

(3)  $P(X > 4) = 1 - P(X \leq 4) = 1 - F(4) = 1 - (1 - (1+4)e^{-4})$

(4)  $P(X=0) = F(0^+) - F(0^-) = 1 - (1+0)e^0 - 0$



by theorem &  $r = |\frac{1}{8}| < 1 \Rightarrow$  this series is convergent  
 by theorem, we get:  $\sum_{x=0}^{\infty} (\frac{1}{8})^x = \frac{1}{1 - \frac{1}{8}} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$   
 $\therefore \sum_{x=1}^{\infty} (\frac{1}{8})^x = (\frac{7}{8}) \cdot (\frac{8}{7}) = 1$   
 $\therefore$  Cond. (2) is satisfied.

Q<sub>3</sub>: If  $X \sim \text{uniform}(0,1)$ , find p.f.  $f(x)$  of  $X$ .  
Sol.  

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Q<sub>4</sub>: Given a c.d.f. of  $X$   

$$F(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{1}{2}(x^3 + 1) & \text{for } -1 < x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find the p.f. of  $X$  ( $f(x)$ ).

Sol.  

$$f(x) = F'(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{3}{2}x^2 & \text{for } -1 < x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$
  

$$f(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } -1 < x < 1 \\ 0 & \text{o.w.} \end{cases} \leftarrow \text{since } F \text{ is continuous at } x = -1$$

Q<sub>1</sub>: Is the following Group (6) function is pr. function? Why?  

$$f(x) = \begin{cases} \frac{x-2}{5} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ①:  $f(1) = \frac{-1}{5} < 0$ ,  $f(2) = 0$ ,  $f(x) \geq 0$   
 $\therefore f$  is not pr. function.  
 $x = 3, 4, 5$



Q2: Given a c.d.f.  $F$  of  $X$  as follows:

$$F(x) = \begin{cases} 1 - (1+x)e^{-x} & \text{for } x=1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

① Find  $P(X \leq -2)$ ,  $P(X > 4)$

② Is  $F$  continuous at  $x=0$ ? Why?

Sol.

①  $P(X \leq -2) = F(-2)$  by def. of  $F$   
 $= 0$

$$\begin{aligned} P(X > 4) &= 1 - P(X \leq 4) \\ &= 1 - F(4) \\ &= 1 - [(1+4)e^{-4}] \end{aligned}$$

② Yes, since  $F(0^+) = F(0^-) = \text{Zero}$

Q3: State Cauchy distribution.

$$f(x) = \frac{1}{\pi(1+x^2)} \quad \text{for } -\infty < x < \infty$$

Q4: Given a c.d.f. of  $X$ :

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find the p.r.f. of  $X$  ( $f(x)$ ).

Sol.

$$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

→ since  $F(0^-) = F(0^+) = 0$   
&  $F$  is conts. at  $x=0$

Group (7)

Consider the c.d.f. of  $X$ :

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$



① Find the p.f. of  $X$  ( $f(x)$ ).

② Is  $F$  conts. at  $x=1$ ? Why?

③ Find  $P(X=0)$ .

Sol.

$$① f(x) = F'(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x \geq 1 \end{cases} = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for o.w.} \end{cases}$$

② Yes, since  $F(1^+) = F(1^-) = 1$ .

$$③ P(X=0) = F(0^+) - F(0^-) \\ = 0 - 0 = \text{Zero}$$

Q2: Show that the following function is Pr. f. of  $X$ :

$$f(x) = \begin{cases} (\frac{1}{2})^x & \text{for } x=1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Sol.

Cond. ①  $(\frac{1}{2})^1 > 0, (\frac{1}{2})^2 > 0, \dots$

$\therefore f(x) \geq 0 \quad \forall x=1, 2, 3, \dots$

Cond. ②

$$\sum_{x=1}^{\infty} (\frac{1}{2})^x = [(\frac{1}{2})^1 + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \dots] \\ = (\frac{1}{2}) [1 + (\frac{1}{2}) + (\frac{1}{2})^2 + \dots] \quad ; r = \frac{1}{2} < 1 \\ = \frac{1}{2} \left( \frac{1}{1 - \frac{1}{2}} \right) = \frac{1}{2} \cdot 2 = 1$$

Geometric series

Q3: Toss a coin fourth times ( $4^{\text{th}}$  times). Find the C.d.f. of a r.v.  $X$ , when  $X$  represents the numbers of tail (T) in a sample.

Sol.

$$f(x) = \begin{cases} \frac{C_x^4}{16} & \text{for } x=0, 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases} \quad | \quad n(s) = 2^4 = 16$$

$$F(x) = \sum_{x_j \leq x} f(x_j)$$



$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ F(0) = 0 + f(0) & 0 \leq x < 1 \\ F(1) = 0 + f(0) + f(1) & 1 \leq x < 2 \\ F(2) = 0 + f(0) + f(1) + f(2) & 2 \leq x < 3 \\ F(3) = 0 + f(0) + f(1) + f(2) + f(3) & 3 \leq x < 4 \\ F(4) = 1 = 0 + f(0) + f(1) + f(2) + f(3) + f(4) & x \geq 4 \end{cases}$$

### Group (8)

Q<sub>1</sub>: Show that the following function is pr. function:

$$f(x) = \begin{cases} \frac{x+2}{25} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ②:  $\frac{1}{25} \sum_{x=1}^5 (x+2) = \frac{1}{25} [3 + 4 + 5 + 6 + 7] = \frac{25}{25} = 1$

Cond. ①  $f(1) = \frac{3}{25} > 0, f(2) = \frac{4}{25} > 0, \dots, f(5) = \frac{7}{25} > 0$

$\therefore f(x) \geq 0 \quad \forall x = 1, 2, 3, 4, 5$

$\therefore f(x)$  is pr. f.

Q<sub>2</sub>: Given a c.d.f. of  $X$ :

$$F(x) = \begin{cases} 1 - (1+x)e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find: ①  $P(X \leq 2) = F(2)$  by def. of  $F$   
 $= 1 - (1+2)e^{-2}$

②  $P(1 < X < 3) = P(1 \leq X \leq 3)$  since  $F$  is cont. at  $x=3$   
 $= F(3) - F(1)$   
 $= (1 - (1+3)e^{-3}) - (1 - (1+1)e^{-1})$

③  $P(X > 4) = 1 - P(X \leq 4) = 1 - F(4) = 1 - (1 - (1+4)e^{-4})$

④  $P(X=0) = F(0^+) - F(0^-) = 1 - (1+0)e^0 - 0$

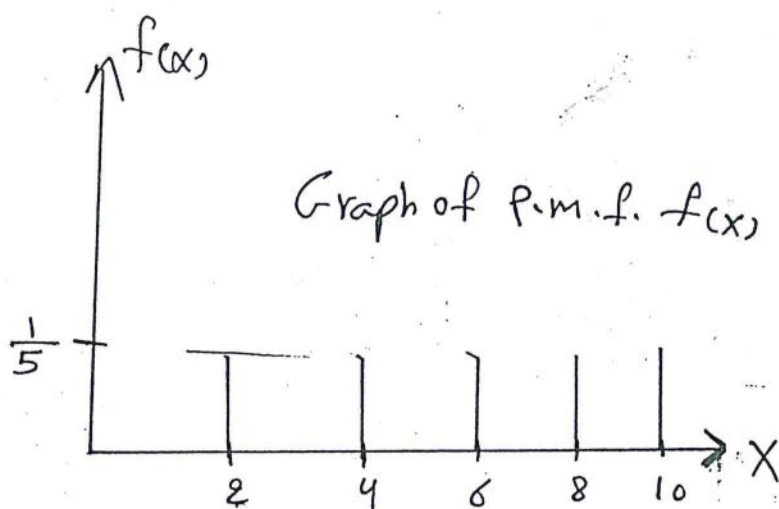


Q<sub>3</sub>: Answer by (true) or (False) and given the reason:  
If a c.d.f.  $F(x)$  continuous at  $x=2$ , then  $P(X=2) \neq 0$ .

Sol. No, since if  $F$  conts. at  $x=2$ , then  $F(2^+) = F(2^-) = F(2)$   
and  $P(X=2) = F(2^+) - F(2^-)$   
 $= \text{Zero}$

Q<sub>4</sub>: Given a set of integers  $\{2, 4, 6, 8, 10\}$ . Find the p.m.f.  $f(x)$  of  $X$  and sketch its graph.

Sol.  $X \sim \text{unif.}(5)$   
$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 2, 4, 6, 8, 10 \\ 0 & \text{o.w.} \end{cases}$$







ex: 1  $f(x) = \begin{cases} kx & 0 < x < 2 \\ k & 2 \leq x < 4 \\ 0 & \text{o.w.} \end{cases}$

p.d.f

Find 5 5 4

①  $k = ?$ , sketch  $f(x)$   
 ②  $P(X > 1)$ ,  $P(X < 3)$   
 $P(\frac{3}{2} < X < 5/2)$

③  $P(X < 1 | \frac{1}{2} < X < \frac{3}{2})$



$P(A|B) = \frac{P(AB)}{P(B)}$

X  
C

ex: 2

$f(x) = \begin{cases} k e^{-x} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$

$\begin{cases} x > 0 \\ 0 < x < \infty \end{cases}$

①  $k = ?$  sketch  
 ②  $P(X < 2)$ ,  $P(X > 1)$

uniform  
dist 2

ex: 3 uniform dist  $(a,b)$   $[-2,3]$

$f(x) = \begin{cases} \frac{1}{5} & \text{for } -2 < x < 3 \\ 0 & \text{o.w.} \end{cases}$

$P(X > 0 | -\frac{1}{2} < X < 2) = P(A|B)$

$P(AB) = P(0 < X < 2) = 2/5$   
 $P(B) = \frac{1}{2}$

$b-a > 0$   
 $\frac{1}{b-a} > 0$

Def

$f(x) = \frac{1}{\pi(1+x^2)}$  for  $-\infty < x < \infty$

Show that  $f(x)$  is p.d.f

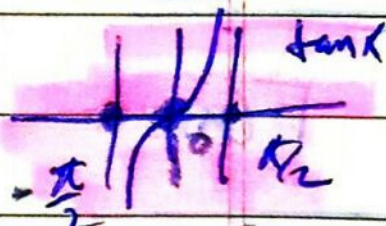
Cauchy dist

①  $f(x) \geq 0$

②  $\int_{-\infty}^{\infty} f(x) dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} d(\tan^{-1} x)$

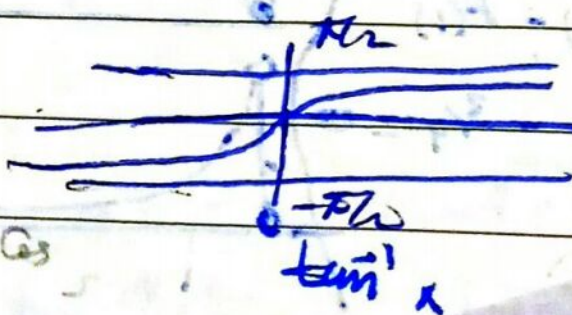
$\tan \frac{\pi}{2} = \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}} = \frac{1}{0} = \infty$

$= \frac{1}{\pi} \tan^{-1}(x) \Big|_{-\infty}^{\infty} = \frac{1}{\pi} [\tan^{-1}(\infty) - \tan^{-1}(-\infty)]$



$\tan \frac{\pi}{2} = \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}}$

$= \frac{1}{\pi} [\frac{\pi}{2} - (-\frac{\pi}{2})]$



$\tan^{-1}(\frac{\pi}{2}) = \frac{\pi}{2}$   
 $\tan^{-1}(-\infty) = -\frac{\pi}{2}$   
 $\tan^{-1}(\infty) = \frac{\pi}{2}$



# 2 Cumulative dist. fun (cdf)

5

Dist. fun (d.f)  $F(x)$

Def  $X$  is a r.v.  $\leftarrow$  d.r.v with p.m.f  $f(x)$  or p.d.f  $f(x)$

$$F(x) = P(X \leq x)$$

$$0 \leq F(x) \leq 1$$

(Sketch)  $F(x)$

1.  $P(X \leq 1)$ , 2.  $P(X \leq 2)$ , 3.  $P(X \leq 3)$

Cont.

d.r.v.

r.v.  $X$

$f(x)$  is p.m.f

$$F(x) = P(X \leq x)$$

$$0 \leq F(x) \leq 1$$

$$0 \leq F(x) \leq 1$$

$$F(x) = P(X \leq x)$$

$$= \sum_{x_j \leq x} f(x_j)$$

$$= \int_{-\infty}^x f(t) dt$$

ex Toss a coin twice  
 $X$  = no. of heads

- 1 - Find p.m.f  $f(x)$
- 2 -  $\hookrightarrow$  d.f.  $F(x)$
- 3 - sketch  $f(x)$  &  $F(x)$

4 -  $P(X \leq 2)$

$P(X \leq 2)$ ,  
 $P(X = 2)$ ,  
 $P(X > 2)$ ,  
 $P(X \geq 2)$

$P(1 < X \leq 3/2) = P(X=1)$

$$P(1 \leq X < 3/2) = P(X=1)$$

$$= f(1)$$

$$P(X \leq 3/2) = F(3/2) = F(1)$$

$$P(X < 3/2) = P(X=0 \cup X=1) = f(0) + f(1)$$

$$P(1 < X \leq 3/2) = F(3/2) - F(1)$$

$$= 3/4 - 1/4 = 1/2$$

Find & Sketch  $R(x) = 1 - F(x)$

$$P(X > 3.5) = 1 - P(X \leq 3.5) = 1 - P(X \leq 3) = 1 - 5/10 = 1/2 \rightarrow f(4) = 1/10$$

$$F(x) = 1 - e^{-x}$$

$$0 \quad 0$$

$$+1 \quad 1 - e^{-1} = 1 - 0.37 = 0.63$$

$$2 \quad 1 - e^{-2} = 1 - 0.13 = 0.87$$

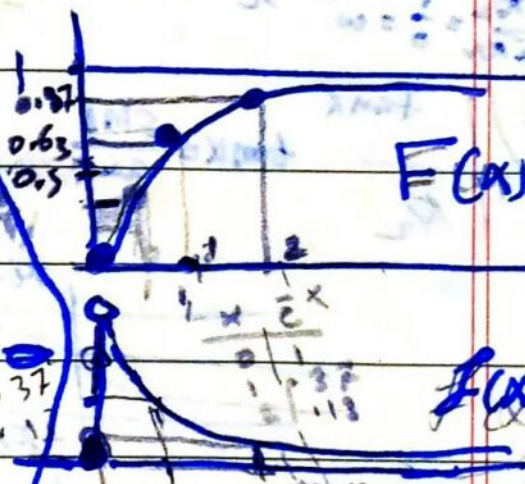
$$\infty \quad 1 - e^{-\infty} = 1$$

$$F(x) = -e^{-x}$$

$$= -[e^{-x}]_0^{\infty}$$

$$= [1 - e^{-x}]_{x=0}^{\infty}$$

$$0 - 0 = 0$$



$F(x)$

$f(x)$

$$e^{-x}$$



Ex.1 Given a p.m.f  $f(x) = \begin{cases} \frac{x}{6} & \text{for } x=1,2,3 \\ 0 & \text{otherwise} \end{cases}$

① - Find and sketch c.d.f  $[F(x)]$  and sketch its graph.

② - Find  $P(X \leq 2)$ ,  $P(X < 2)$ ,  $P(X > 2)$ ,  $P(X = 2)$

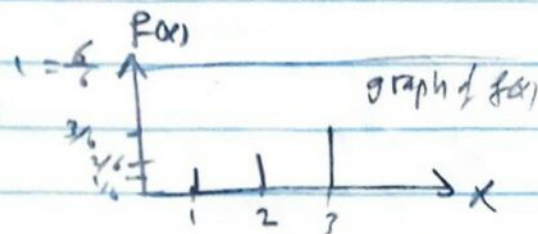
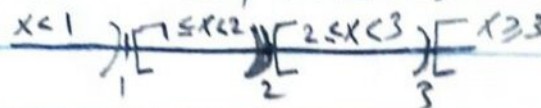
③ - Find  $P(X \leq 5/2)$ ,  $P(X \leq 3/2)$

$P(X < 5/2)$ ,  $P(X < 3/2)$

Sol.

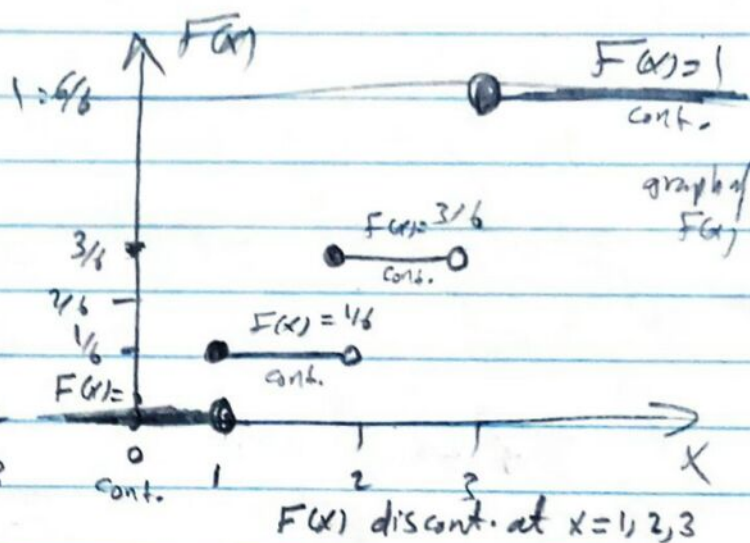
$X$  is d.r.v.,  $R_X = \{1, 2, 3\}$

(u.s.)  $F(x) = P(X \leq x) = \sum_{x_j \leq x} f(x_j)$



| interval       | int. | $f(x)$ | $F(x)$ |
|----------------|------|--------|--------|
| $x < 1$        | no.  | 0      | 0      |
| $1 \leq x < 2$ | 1    | $1/6$  | $1/6$  |
| $2 \leq x < 3$ | 2    | $2/6$  | $3/6$  |
| $x \geq 3$     | 3    | $3/6$  | 1      |

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1/6 & \text{for } 1 \leq x < 2 \\ 3/6 & \text{for } 2 \leq x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$



Note If  $F(x)$  given, we can find  $f(x) \forall x \in R_X$   
for above example,

$$\begin{aligned} F(3) &= F(2) + f(3) \\ f(3) &= F(3) - F(2) \\ &= 1 - 3/6 = 3/6 \end{aligned}$$

$$f(1) = F(1) - f(0)$$

$$= \frac{1}{6} - 0 = 1/6$$

$$\Rightarrow f(x_i) = F(x_i) - F(x_{i-1})$$

②  $P(X \leq 2) = \sum_{x=1}^2 f(x) = f(1) + f(2) = \frac{1}{6} + \frac{2}{6} = 3/6$   $f$  is d.r.v.

or  $P(X \leq 2) = F(2) = 3/6$   $F$  is c.d.f.

$P(X < 2) = P(X = 1) = f(1) = 1/6$

or  $P(X < 2) = P(X \leq 1) = F(1) = 1/6$



$$P(X > 2) = P(X=3) = f(3) = 3/6$$

$$\text{or } P(X > 2) = 1 - P(X \leq 2) = 1 - F(2) = 1 - \frac{3}{6} = \frac{3}{6}$$

$$P(X \geq 2) = \sum_{x=2}^3 f(x) = f(2) + f(3) = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$

$$\text{or } P(X \geq 2) = 1 - P(X < 2) = 1 - \frac{1}{6} = \frac{5}{6}$$

①

Suppose

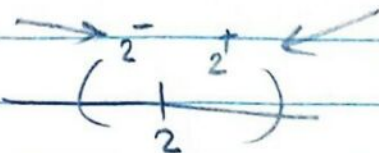
66

$$\text{or } \begin{cases} = 1 - P(X \leq 1) \\ = 1 - F(1) \end{cases} = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(X=2) = f(2) = \frac{2}{6}$$

$$\Rightarrow f(2) = F(2) - F(1)$$

$$= \frac{3}{6} - \frac{1}{6} = \frac{2}{6}$$



Note  
X is discrete

$$P(X=2) \neq 0$$

$\rightarrow F(x)$  discontinuous at  $x=2$

$x=2$  is a jump point of  $F$  with a positive probability mass

③

$$P(X \leq \frac{5}{2}) = P(X \leq 2.5) = \sum_{x=1}^2 f(x) = f(1) + f(2) = \frac{1}{6} + \frac{2}{6} = \frac{3}{6}$$

$$\text{or } P(X \leq \frac{5}{2}) = F(\frac{5}{2}) = F(2.5) = \frac{3}{6}$$

$$P(X < \frac{5}{2}) = P(X < 2.5) = \sum_{x=1}^2 f(x) = f(1) + f(2) = \frac{3}{6}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{6} & 1 \leq x < 2 \\ \frac{3}{6} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

$$\therefore P(X \leq \frac{5}{2}) = P(X < \frac{5}{2}) = \frac{3}{6} \Rightarrow F(x) \text{ continuous at } x = \frac{5}{2}$$

\* by the same method of  $P(X \leq 3/2)$ ,  $P(X < 3/2)$

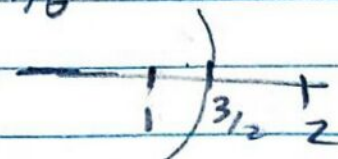
$$\begin{cases} P(X \leq 3/2) = P(X \leq 1.5) = f(1) = 1/6 \\ P(X < 3/2) = P(X < 1.5) = F(1.5) = 1/6 \end{cases}$$

by  $f$

by  $F$

$$P(X < 3/2) = P(X < 1.5) = P(X=1) = f(1) = 1/6$$

$$P(X \leq 3/2) = P(X \leq 1) = F(1) = 1/6$$



$$P(X \leq 3/2) = P(X < 3/2) = 1/6 \Rightarrow F(x) \text{ is continuous at } x = \frac{3}{2}$$



(6c)

$$f(x_j) = F(x_j) - F(x_{j-1})$$

$$f(1) = F(1) - F(0) = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} \quad ||$$

$$f(2) = F(2) - F(1) = 1 - \frac{3}{4} = \frac{1}{4} \quad ||$$

(3)  $p(X \leq 1) = \sum_{x=0}^1 f(x) = f(0) + f(1) = \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$   
 $p(X \leq 1) = F(1) = \frac{3}{4}$

$p(X < 1) = p(X=0) = f(0) = \frac{1}{4}$   
 $p(X < 1) = p(X \leq 0) = F(0) = \frac{1}{4}$   
 $p(X > 1) = p(X=2) = f(2) = \frac{1}{4}$   
 $p(X > 1) = 1 - p(X \leq 1) = 1 - F(1) = 1 - \frac{3}{4} = \frac{1}{4}$

$p(X=1) = f(1) = \frac{2}{4}$   
 $p(X=1) = f(1) = F(1) - F(0) = \frac{3}{4} - \frac{1}{4} = \frac{2}{4}$

(4)  $p(X \leq 3/2) = \sum_{x=0}^1 f(x) = f(0) + f(1) = \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$   
 $p(X \leq 3/2) = F(3/2) = \frac{3}{4}$

$p(X < 3/2) = \sum_{x=0}^1 f(x) = \frac{3}{4}$   
 $p(X < 3/2) = p(X \leq 1) = F(1) = \frac{3}{4}$

(5)  $p(1 < X \leq 3/2) = p(\emptyset) = 0$   
 $p(1 < X \leq 3/2) = F(3/2) - F(1) = \frac{3}{4} - \frac{3}{4} = 0$

$p(1 < X < 3/2) = p(\emptyset) = 0$   
 $p(1 < X < 3/2) = F(3/2) - F(1) = \frac{3}{4} - \frac{3}{4} = 0$

$\Rightarrow p(1 < X \leq 3/2) = p(1 < X < 3/2)$   
 since  $F$  is cont. at  $x=3/2$

(6)  $p(1 \leq X \leq 3/2) = f(1) = \frac{2}{4}$   
 $p(1 \leq X \leq 3/2) = p[X=1 \cup 1 < X \leq 3/2]$   
 $= p(X=1) + p(1 < X \leq 3/2)$   
 $= [F(1) - F(0)] + 0$   
 $= \frac{3}{4} - \frac{1}{4} = \frac{2}{4}$

ex. 2 Toss a coin twice  
 let  $x$  = no. of heads

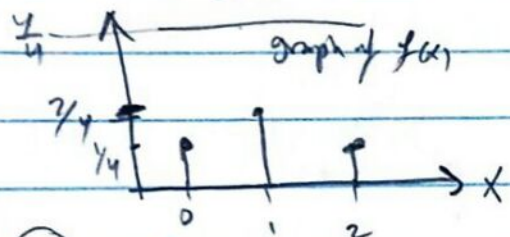
- Find and sketch  $f(x)$  (p.m.f) and pr. dist. of  $X$
- Find  $F(x)$  (c.d.f) and find  $f(1), f(2)$  using  $F(x)$
- Find  $p(X \leq 1), p(X < 1), p(X > 1), p(X=1)$  using  $f(x)$  &  $F(x)$
- Find  $p(X \leq 3/2), p(X < 3/2)$  using  $f$  &  $F$
- Find  $p(1 < X \leq 3/2), p(1 < X < 3/2)$
- Find  $p(1 \leq X \leq 3/2), p(1 \leq X < 3/2)$

Sol: ①  $n(s) = 2^2 = 4 \rightarrow s = \{HH, HT, TH, TT\}$

$$f(x) = p(X=x) = \frac{\binom{2}{x}}{4} \quad x=0,1,2$$

pr. dist. of  $X = \{(0, \frac{1}{4}), (1, \frac{2}{4}), (2, \frac{1}{4})\}$

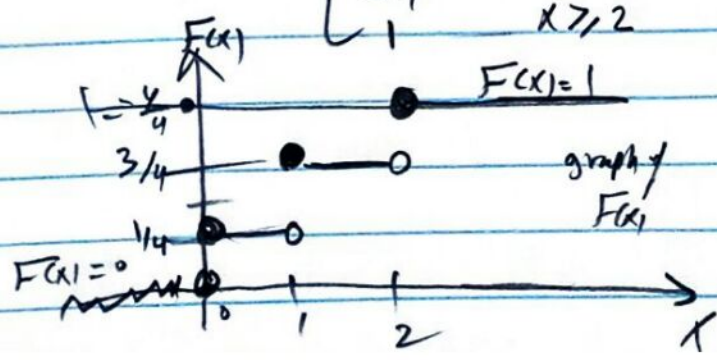
$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x=0,2 \\ \frac{2}{4} & \text{for } x=1 \\ 0 & \text{o.w} \end{cases}$$



②  $R_X = \{0, 1, 2\}$

| interval       | int. | $f(x)$ | $F(x) = \sum_{x \leq x_j} f(x_j)$ |
|----------------|------|--------|-----------------------------------|
| $x < 0$        | no.  | 0      | 0                                 |
| $0 \leq x < 1$ | 0    | $1/4$  | $1/4$                             |
| $1 \leq x < 2$ | 1    | $2/4$  | $3/4$                             |
| $x \geq 2$     | 2    | $1/4$  | 1                                 |

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1/4 & 0 \leq x < 1 \\ 3/4 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$



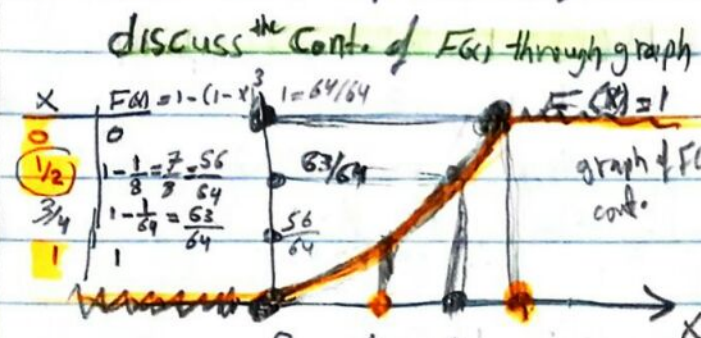


$$\begin{aligned}
 P(1 \leq X < \frac{3}{2}) &= P(X=1 \cup 1 < X < \frac{3}{2}) \\
 &= P(X=1) + P(1 < X < \frac{3}{2}) \quad \text{addition rule} \\
 &= [F(1) - F(0)] + [F(\frac{3}{2}) - F(1)] \\
 &= [\frac{3}{4} - \frac{1}{4}] + [\frac{1}{4} - \frac{1}{4}] = \frac{2}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{or } &= F(1) - F(0) + F(\frac{3}{2}) - F(1) \\
 &= F(\frac{3}{2}) - F(0) \\
 &= \frac{3}{4} - \frac{1}{4} = \frac{2}{4}
 \end{aligned}$$



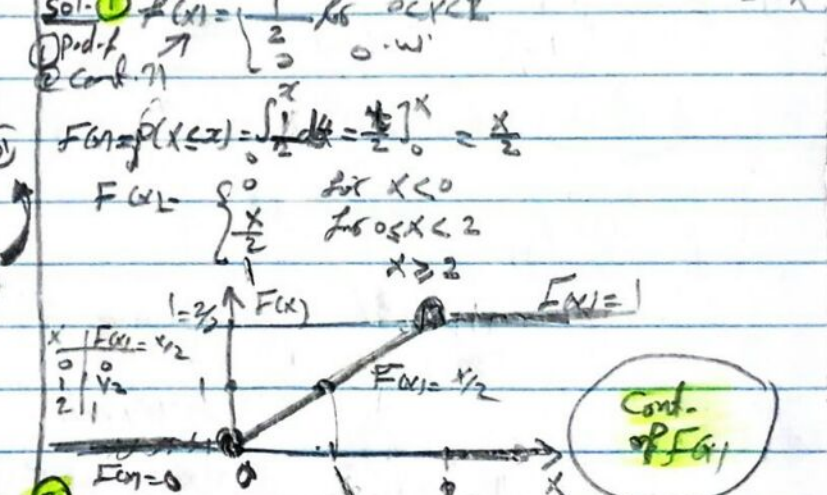
$f(x) = \begin{cases} 3(1-x)^2 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$   
 1) Find and sketch  $F(x)$  [c.d.f.]  
 2)  $P(X \leq 0)$ ,  $P(X \leq 1)$ ,  $P(X < 1/2)$   
 $P(X < 0)$ ,  $P(X < 1)$ ,  $P(X < 1/2)$   
 3)  $P(X=0)$ ,  $P(X=1)$ ,  $P(X=1/2)$   
 Sol:  $F(x) = P(X \leq x) = \int_0^x 3(1-t)^2 dt$   
 $= -\frac{3}{3}(1-t)^3 \Big|_0^x = -[(1-x)^3 - (1-0)^3]$   
 $= 1 - (1-x)^3$   
 $\therefore F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - (1-x)^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$



2)  $F(x)$  is cont. everywhere  
 $\left[ \begin{aligned} P(X \leq 0) &= F(0) = 0 \\ P(X < 0) &= \int_{-\infty}^0 0 dx = 0 \end{aligned} \right]$   $F(x)$  cont. at  $x=0$   
 $\left[ \begin{aligned} P(X \leq 1) &= F(1) = 1 \\ P(X < 1) &= \int_0^1 3(1-x)^2 dx = [1 - (1-x)^3]_0^1 = 1 - 0 = 1 \end{aligned} \right]$   
 $\left[ \begin{aligned} P(X \leq 1/2) &= F(1/2) = \frac{7}{8} = \frac{56}{64} \\ P(X < 1/2) &= \int_0^{1/2} 3(1-x)^2 dx = [1 - (1-x)^3]_0^{1/2} \\ &= \frac{1}{2} [1 - (1/2)^3] - [1 - (1-0)^3] \\ &= \frac{1}{2} [1 - 1/8] - [1 - 1] \\ &= \frac{7}{8} \end{aligned} \right]$   
 $\rightarrow P(X \leq 1/2) = P(X < 1/2) = 7/8$   
 since  $F(x)$  is cont. at  $x=1/2$   
 $\rightarrow P(X \leq 1) = P(X < 1) = 1$   
 since  $F(x)$  cont. at  $x=1$

3)  $P(X=0) = F(0^+) - F(0^-)$   
 $= [1 - (1-0)^3] - 0 = 0$   $F$  cont. at  $x=0$   
 $P(X=1) = F(1^+) - F(1^-)$   
 $= 1 - [1 - (1-1)^3] = 0$   $F$  cont. at  $x=1$   
 $P(X=1/2) = F(1/2^+) - F(1/2^-)$   
 $= [1 - (1-1/2)^3] - [1 - (1-1/2)^3] = 0$   $F$  cont. at  $x=1/2$

Ex. 2  $X \sim \text{unit } (0,2)$   
 1) Find and sketch  $F(x)$ ,  $F(x)$   
 2) Find  $P(X \leq 1)$ ,  $P(X < 1)$ ,  $P(X=1)$   
 3)  $P(X < 0)$ ,  $P(X < 2)$ ,  $P(X=0)$ ,  $P(X=2)$



2)  $P(X \leq 1) = F(1) = 1/2$   
 $P(X < 1) = \int_0^1 \frac{1}{2} dx = \frac{x}{2} \Big|_0^1 = \frac{1}{2}$   $F$  cont. at  $x=1$   
 $P(X=1) = F(1^+) - F(1^-) = (\frac{1}{2}) - (\frac{1}{2}) = 0$   $F(x)$  cont. at  $x=1$   
 $P(X=2) = F(2^+) - F(2^-) = 1 - \frac{2}{2} = 0$   
 $P(X=0) = F(0^+) - F(0^-) = \frac{0}{2} - 0 = 0$   $F$  cont. at  $x=0/2$   
 3)  $P(X < 0) = \int_{-\infty}^0 0 dx = 0$   
 $P(X \leq 0) = F(0) = \frac{0}{2} = 0$   
 $P(X < 2) = \int_0^2 \frac{1}{2} dx = 1$   
 $P(X \leq 2) = F(2) = \frac{2}{2} = 1$   $F$  cont. at  $x=2$

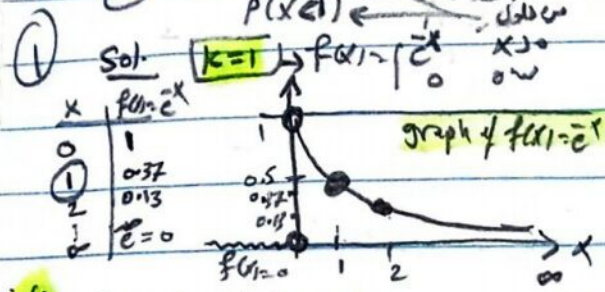


**ex3** Given a p.d.f  $f(x) = \begin{cases} k e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

① Find  $k$ ?, sketch  $f(x)$ , study cont./f.c.

② Find and sketch  $F(x)$

③ Find  $P(X \leq 1)$   $P(X=1)$



④ discuss continuity of  $F$  at  $x=1$  through  $P(X > 1)$  &  $P(X \geq 1)$  i.e.  $P(X=1)$

$$P(X=1) = F(1^+) - F(1^-) = [1 - e^{-1}] - [1 - e^{-1}] = 0$$

so  $F(x)$  is cont. at  $x=1$

$$P(X=0) = F(0^+) - F(0^-) = [1 - e^0] - 0 = 0$$

$F(x)$  is cont. at  $x=0$

● Think

$$F(x) = P(X \leq x)$$

$$R(x) = 1 - F(x) = P(X > x)$$

Discuss  $R(x)$

$$f(x) = e^{-x} \quad x \geq 0$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

②  $P(X \leq 0) = \int_0^0 e^{-x} dx = 0$

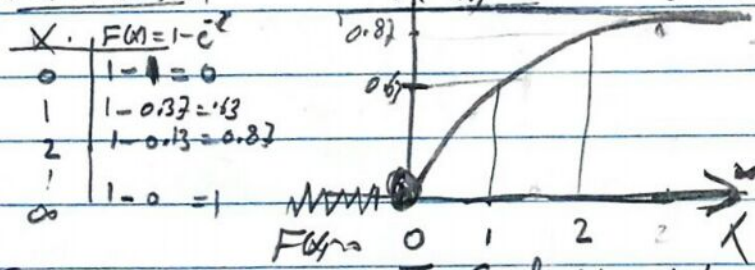
②  $P(X < 1) = \int_0^1 e^{-x} dx = -e^{-x} \Big|_0^1 = -[e^{-1} - e^0] = 1 - e^{-1} = 1 - 0.37 = 0.63$

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt = \int_0^x e^{-t} dt = -e^{-t} \Big|_0^x = -[e^{-x} - e^0] = 1 - e^{-x}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

by f.c.  
 $P(X=0) = 0$   
 $P(X \leq 0) = F(0) = 0$   
 $F$  is cont. at 0  
 $\rightarrow P(X=0) = 0$   
 $= F(0^+) - F(0^-) = 0 - 0 = 0$

$$F(1) = F(\infty) = 1$$



③  $P(X \leq 1) = F(1) = 0.63$

$P(X < 1) = \int_0^1 e^{-x} dx = 0.63$   $\rightarrow P(X \leq 1) = P(X < 1)$

$$P(X \leq 0) = F(0) = 0$$

$P(X < 0) = 0$

$P(X=1) = F(1^+) - F(1^-) = [1 - e^{-1}] - [1 - e^{-1}] = 0$   
 cont.

④  $P(X > 1) = 1 - P(X \leq 1) = 1 - F(1)$

$$= 1 - 0.63 = 0.37$$

$$P(X \geq 1) = 1 - P(X < 1) = 1 - F(1)$$

$$= 1 - 0.63 = 0.37$$

**ex4** Given a p.d.f.

$$f(x) = \begin{cases} \frac{k}{x^2} & x > 1 \\ 0 & x \leq 1 \end{cases}$$

① Find the value of  $k$  and sketch  $f(x)$  graph

② Find  $F(x)$  [c.d.f.] and sketch its graph

③ Find  $P(X \leq 2)$ ,  $P(X < 2)$ ,  $P(X > 2)$ ,  $P(X \geq 2)$  and  $P(X=2)$

Sol.



ex. 4 Given a p.d.f.  $f(x) = \begin{cases} \frac{k}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$

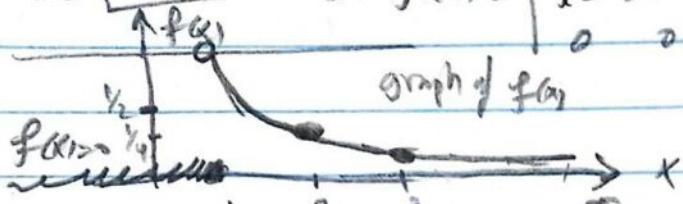
(1) Find the value of  $k$  and sketch the graph of  $f(x)$ .

(2) //  $F(x)$  [c.d.f.] and sketch its graph.

(3) Find  $P(X \leq 2)$ ,  $P(X < 2)$ ,  $P(X > 2)$ ,  $P(X \geq 2)$  and  $P(X=2)$ ,  $P(X=1)$  using  $f(x)$  &  $F(x)$

Sol. (1) by cond. (2) of p.d.f.  $\rightarrow \int_{-\infty}^{\infty} f(x) dx = 1 \Rightarrow \int_1^{\infty} \frac{k}{x^2} dx = 1$   
 $\rightarrow 1 = k \left[ -\frac{1}{x} \right]_1^{\infty} \rightarrow 1 = -k \left[ \frac{1}{x} \right]_1^{\infty} \rightarrow 1 = -k \left[ \frac{1}{\infty} - \frac{1}{1} \right]$

ob  $k=1 \rightarrow f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$



| x | f(x) = 1/x^2 |
|---|--------------|
| 1 | 1            |
| 2 | 1/4          |
| 3 | 1/9          |
| ∞ | 0            |

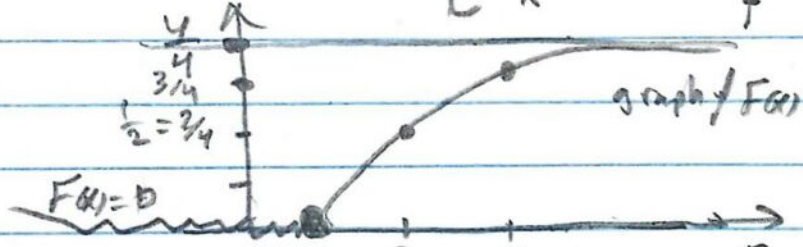
ob  $f(x)$  is discontin. at  $x=1 \rightarrow f(1^-) + f(1^+)$

(2) To find and sketch  $F(x)$

$F(x) = \int_{-\infty}^x f(t) dt = \int_1^x \frac{1}{t^2} dt = \left[ -\frac{1}{t} \right]_1^x = -\left[ \frac{1}{x} - 1 \right] = 1 - \frac{1}{x}$

ob  $F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1 - \frac{1}{x} & \text{for } x \geq 1 \end{cases}$

| x | F(x) = 1 - 1/x |
|---|----------------|
| 1 | 0              |
| 2 | 1/2 = 2/4      |
| 4 | 3/4            |
| ∞ | 1              |



ob  $F(x)$  is cont. every where.

(3) (i)  $P(X \leq 2) = \int_1^2 \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_1^2 = \left[ \frac{1}{2} - 1 \right] = \frac{1}{2}$

$P(X \leq 2) = F(2) = 1 - \frac{1}{2} = \frac{1}{2}$

(ii)  $P(X < 2) = P(X \leq 2) = \frac{1}{2}$   
 f.d.f.  $P(X < 2) = \int_1^2 f(x) dx = \frac{1}{2}$

(iii)  $P(X > 2) = 1 - P(X \leq 2) = 1 - \frac{1}{2} = \frac{1}{2}$   
 $P(X > 2) = 1 - P(X \leq 2)$   
 $= 1 - F(2) = \frac{1}{2}$

(iv)  $P(X \geq 2)$   
 by the same method

(v)  $P(X=2) = 0$  f.d.f.  $P(X=2) = 0$   
 $P(X=2) = F(2^+) - F(2^-)$

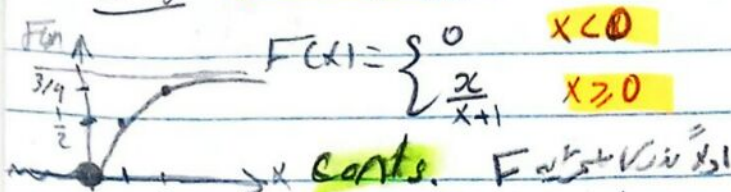
$F(2^+) = (1 - \frac{1}{2}) - (1 - \frac{1}{2}) = 0$

$P(X=1) = 0$  f.d.f.  $P(X=1) = 0$   
 $P(X=1) = F(1^+) - F(1^-)$

$F(1^+) = (1 - \frac{1}{1}) - 0 = 0$   
 $F(1^-) = 0 - 0 = 0$



Def  $f(x) = F'(x)$  when  $x$  is c.r.v.



$$f(x) = \frac{dF(x)}{dx} = \frac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$p(x=0)=0$   
 $F(0^-) = 0, F(0^+) = 0$   
 $0 = 0 = 0$

(7d)

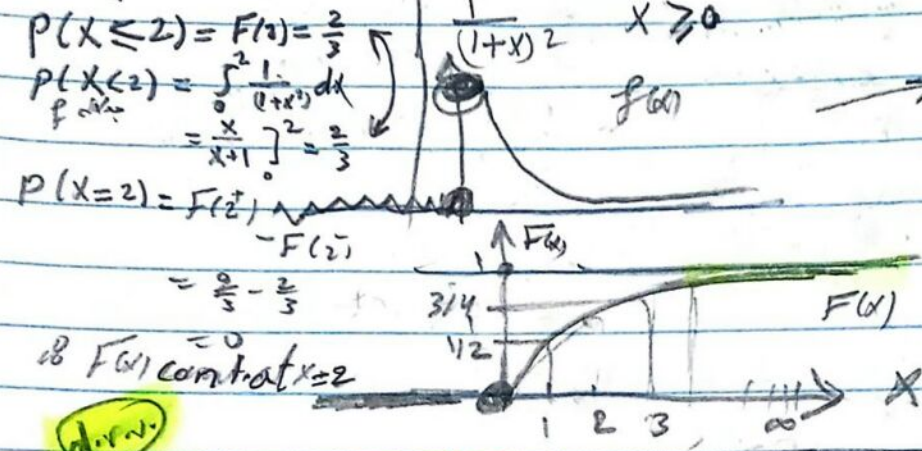
Note

H.W.

$F(x) = \begin{cases} 0 & x < 1 \\ \frac{x}{x+1} & x \geq 1 \end{cases}$   
 $F$  not c.r.v. at  $x=1$   
 $F(1^-) = 0, F(1^+) = 1/2$   
 $f(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{(x+1)^2} & x \geq 1 \end{cases}$   
 $P(X=1) = 1/2 \neq 0$   
 $\therefore$  d.r.v.

$P(X \leq 2) = F(2) = \frac{2}{3}$   
 $P(X < 2) = \int_0^2 \frac{1}{(x+1)^2} dx = \left[ -\frac{1}{x+1} \right]_0^2 = -\frac{1}{3} - (-1) = \frac{2}{3}$   
 $P(X=2) = F(2) - F(2^-) = \frac{2}{3} - \frac{2}{3} = 0$

| x | f(x)    |
|---|---------|
| 0 | 1       |
| 1 | 1/4     |
| 2 | 1/9     |
| 3 | 1/16    |
| ∞ | 1/∞ = 0 |



| x | F(x) = x/(x+1) |
|---|----------------|
| 0 | 0              |
| 1 | 1/2            |
| 2 | 2/3            |
| 3 | 3/4            |
| ∞ | 1              |

$\frac{10,000}{10,001} = 0.99990001$

ex.  $f(x) = \int_0^x \frac{1}{t^2} dt$   $x = 1, 2, 3$   
 $\therefore$  d.r.v.

$F(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \leq x < 2 \\ 3/6 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$

$f(x) = \begin{cases} \frac{1}{6} & x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$

to find  $f(x)$  from  $F(x)$   
 $X$  is d.r.v.

$f(3) = F(3) - F(2) = 1 - \frac{3}{6} = \frac{3}{6}$   
 $f(2) = F(2) - F(1) = \frac{3}{6} - \frac{1}{6} = \frac{2}{6}$   
 $f(1) = F(1) - F(0) = \frac{1}{6} - 0 = \frac{1}{6}$

| int            | int | F(x) | F(x)       |
|----------------|-----|------|------------|
| $x < 1$        | no  | 0    | 0 $F(0)$   |
| $1 \leq x < 2$ | 1   | 1/6  | 1/6 $F(1)$ |
| $2 \leq x < 3$ | 2   | 3/6  | 3/6 $F(2)$ |
| $x \geq 3$     | 3   | 3/6  | 1 $F(3)$   |

ex.  $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$

Find  $f(x)$  if exist. [ $F$  c.r.v.]

$f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$

$F(x) = \begin{cases} 0 & \text{o.w.} \\ 1/6 & x=1 \\ 2/6 & x=2 \\ 3/6 & x=3 \end{cases}$   
 $F(2) = F(1) + P(2)$   
 $f(2) = F(2) - F(1)$   
 $f(3) = F(3) - F(2)$   
 $f(1) = F(1) - F(0)$   
 $= 1/6 - 0 = 1/6$   
 $= \frac{1}{6} \times 1, 2, 3$   
 $= \frac{1}{6} \times 0 \text{ o.w.}$



# properties of $F(x)$

$F_{d_2}$

Th.1

$F(x)$  is a non-decreasing function

proof for  $x_1 < x_2$  T.T.  $F(x_1) \leq F(x_2)$



$$(-\infty, x_1] \subseteq (-\infty, x_2]$$

by th.4 incl.  $A \subseteq B$   
 $P(A) \leq P(B)$

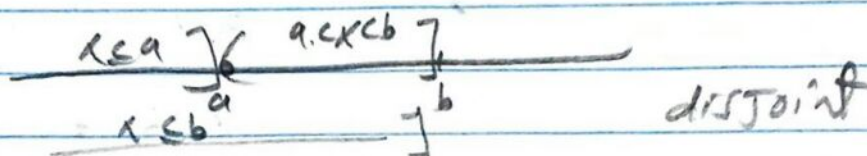
$$P(X \leq x_1) \leq P(X \leq x_2)$$

$$F(x_1) \leq F(x_2)$$

Th.2 If  $X$  is a r.v. has a d.f.  $F(x)$ , then

$$P(a < X \leq b) = F(b) - F(a) \text{ for } a < b \in \mathbb{R}$$

proof



$$X \leq b = X \leq a \cup (a < X \leq b)$$

$$P(X \leq b) = P(X \leq a) + P(a < X \leq b) \quad \text{by AX.3}$$

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) \\ = F(b) - F(a)$$

Th.3

$$F(-\infty) = 0$$

$$F(\infty) = 1$$

i.e.

$$\lim_{x \rightarrow -\infty} F(x) = 0 \\ \lim_{x \rightarrow \infty} F(x) = 1$$

proof

$$F(-\infty) = \lim_{x \rightarrow -\infty} P(X \leq x)$$

$$= \lim_{x \rightarrow -\infty} P(X \leq x; x \rightarrow -\infty)$$

$$F(\infty) = \lim_{x \rightarrow \infty} P(X \leq x)$$

$$= \lim_{x \rightarrow \infty} P(X \leq x; x \rightarrow \infty)$$

if we choose very small no.  $x$  say at  $-\infty$

$$\text{then Event } (X \leq x; x \rightarrow -\infty) = \emptyset$$

$$\therefore F(-\infty) = \lim_{x \rightarrow -\infty} P(X \leq x) = P(X \leq x; x \rightarrow -\infty) = P(\emptyset) = 0$$

if we choose very large no.  $x$  say at  $\infty$

$$\text{then Event } (X \leq x; x \rightarrow \infty) = \{X; -\infty < X < \infty\} = S$$

$$F(\infty) = \lim_{x \rightarrow \infty} P(X \leq x) = P(X \leq x; x \rightarrow \infty) = P(S) = 1$$

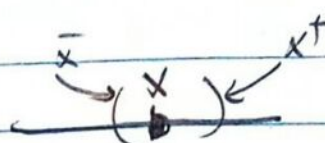


(7e)

Continuity of  $F(x)$ ① Limiting of  $F(x)$ ① Left limit of  $F(x)$ 

$$F(\bar{x}) = \lim_{\substack{n \rightarrow \infty \\ y_n < x}} F(y_n)$$

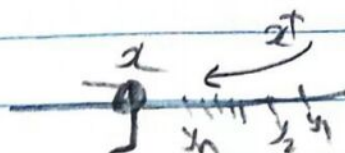
$$= \lim_{\substack{n \rightarrow \infty \\ y_n < x}} P(X \leq y_n)$$



$$F(x) = P(X \leq x)$$

② Right limit of  $F(x)$ 

$$F(x^+) = \lim_{\substack{n \rightarrow \infty \\ y_n > x}} F(y_n)$$

② Continuity of  $F(x)$  $F(x)$  is cont. at  $x$  iff  $F(\bar{x}) = F(x^+)$ (Th-4)  $F(x)$  is cont. to the right for any point  $x$ 

$$F(x) = F(x^+)$$

$$\text{Hint } F(x^+) = \lim_{\substack{n \rightarrow \infty \\ y_n > x}} P(X \leq y_n) = P(X \leq x) = F(x) \quad [x \in (X \leq y_n)]$$

(Th-5)  $P(X < x) = F(\bar{x})$  at any point  $x$ 

$$\text{Hint } F(\bar{x}) = \lim_{\substack{n \rightarrow \infty \\ y_n < x}} P(X \leq y_n) = P(X < x) \quad [y_n < x]$$

(Th-6)  $P(X=x) = F(x^+) - F(\bar{x})$   
proof

ex.  $F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+2}{4} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+2}{4} & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Sketch  $F(x)$ , find  $P(X=-1)$ ,  $P(X=1)$ ,  $P(X=0)$ ,  $P(X=1/2)$   
 $P(0 < X < 1)$  ;  $P(0 < X \leq 1)$  ,  $P(0 < X \leq 1/2)$

$$f(x) = \begin{cases} 0 & \text{or w.} \\ 1/4 & -1 < x < 1 \end{cases}$$

$$P(0 < X < 1/2)$$



## Ch.2 - Part 2

### Transformation of Random Variables

التحويلات العشوائية  
للمتغيرات العشوائية

[1]

#### [I] Transformation of discrete Random Variables

If  $X$  be a d.r.v. have a p.m.f.  $f(x)$

Let  $A$  be the space of  $X$  for  $f(x) > 0$

and let  $Y = U(X)$  define a (1-1) transformation that maps  $A$  on to  $B$ , where  $B$  is the space of  $Y$

If we solve  $y = U(x)$  for  $x$  in terms of  $y$

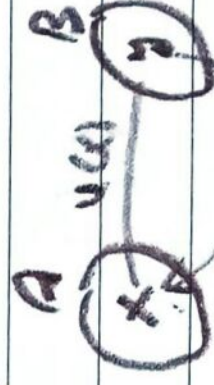
Say  $x = \bar{u}'(y) \in A$

then for  $y = U(x) \in B \rightarrow x = \bar{u}'(y) \in A$

then the p.m.f.  $g(y)$  of a r.v.  $Y$  is defined  $\bar{u}'(y)$

$$g(y) = P(Y=y) = P(Y=U(x)) = P(x=\bar{u}'(y)) = f[\bar{u}'(y)]$$

$$\therefore g(y) = f[\bar{u}'(y)]$$



A

2

ex.1 Given a p.m.f. of  $X$   $f(x) = \frac{e^{-m} m^x}{x!}$  for  $x = 0, 1, 2, \dots$

- Find the p.m.f. of  $Y = 2X$  o.w.

- Find  $P(Y < 4)$ ,  $P(Y = 2)$ ,  $P(Y = 1)$

Sol:  $Y = 2X = u(x)$  و  $A = \{0, 1, 2, \dots\}$

$\therefore B = \{0, 2, 4, \dots\}$

$$\therefore x = \frac{y}{2} = u(y)$$

$\frac{y}{2}$

حسب المتغير

$$\therefore g(y) = P[u^{-1}(y)] = f\left(\frac{y}{2}\right) = \begin{cases} \frac{e^{-m} m^{y/2}}{(\frac{y}{2})!} & \text{for } y = 0, 2, 4, 6, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$- P(Y < 4) = \sum_{y=0}^3 g(y) = g(0) + g(2)$$

$$= \frac{e^{-m} m^{0/2}}{0!} + \frac{e^{-m} m^{2/2}}{(\frac{2}{2})!}$$

$$P(Y=1) = 0 \quad \times \quad P(Y=2) = g(2) = \frac{e^{-m} m^{2/2}}{(\frac{2}{2})!} = \frac{e^{-m} m}{1!}$$



## II) Transformation of Continuous Random Variables

Let  $X$  be a C.R.V. have a p.d.f.  $f(x)$ ,  $f(x) > 0$  at each point of the space  $A$  of  $X$ .

Let  $y = u(x)$  define a (1-1) transformation that maps  $\text{Space}(A)$  on to  $\text{Space}(B)$  of  $Y$ .

If we can write  $x = u^{-1}(y)$  as the inverse of the function  $u$ , then the p.d.f. of  $Y$  is

$$g(y) = f[u^{-1}(y)] \cdot |J| = f[x] \cdot |J|$$

$$\text{where } J = [u^{-1}(y)]' = [x]' = \frac{dx}{dy}$$

Jacobian



4

ex. 1 Given a p.d.f. of  $X$   $f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

— Find the p.d.f. of  $Y = 8X^3$  [using transformation]

— Find  $P(Y < 1)$

Sol.  $Y = 8X^3$  ;  $A = \{x; 0 < x < 1\}$  space of  $X$

$\Rightarrow B = \{y; 0 < y < 8\}$

$$x = \frac{y}{8} \Rightarrow x = \frac{\sqrt[3]{y}}{2} = u(y)$$

$$\therefore J = \frac{dx}{dy} = \left[ \frac{\sqrt[3]{y}}{2} \right]' = \frac{1}{2} \left[ \frac{1}{3} y^{-2/3} \right] = \frac{\sqrt[3]{y}}{6}$$

$$\therefore g(y) = f[u(y)] \cdot |J| = 2 \left( \frac{\sqrt[3]{y}}{2} \right) \cdot \left| \frac{\sqrt[3]{y}}{6} \right| = \frac{1}{6\sqrt[3]{y}}$$

$$\therefore g(y) = \begin{cases} \frac{1}{6\sqrt[3]{y}} & \text{for } 0 < y < 8 \\ 0 & \text{o.w.} \end{cases}$$

$$P(Y < 1) = \int_0^1 \frac{1}{6\sqrt[3]{y}} dy = \frac{1}{6} \int_0^1 y^{-2/3} dy = \frac{1}{6} \left[ 3y^{1/3} \right]_0^1 = \frac{1}{2}$$

## Remarks

ملاحظات من خلال الأمثلة التالية

ex. 1

Given a p.m.f. of  $X$

$$f(x) = \begin{cases} 1/5 & \text{for } x=2 \\ 1/6 & \text{for } x=1 \\ 1/5 & \text{for } x=0 \\ 1/5 & \text{for } x=-1 \\ 1/30 & \text{for } x=-2 \end{cases}$$

not  
(1-1)

Find p.m.f. of  $Y = X^2$

Sol.  $Y = X^2 \rightarrow X = \pm\sqrt{Y}$

$$A = -2, -1, 0, 1, 2$$

$$B = 0, 1, 4$$

for  $y=0$

$$g(y) =$$

$$\left\{ \frac{1}{5} \right.$$

for  $y=1$

$$\left[ \frac{7}{30} \right.$$

for  $y=4$

$$\left[ \frac{17}{30} \right.$$

$$\rightarrow$$

$$[X=0 \rightarrow f(0)=1/5]$$

$$[X=\pm 1 \rightarrow f(\pm 1)=1/5]$$

$$[X=\pm 2 \rightarrow f(\pm 2)=1/30]$$

$$g(y) = f[\sqrt{y}] + f[-\sqrt{y}]$$

$$X = \pm 2 \rightarrow f(2) = \frac{1}{30} = \frac{1}{30}$$

$$f(2) = 1/30$$

$$\Sigma f(x) = 17/30$$

$$\Sigma f(x) = \frac{1}{5} + \frac{1}{5} + \frac{1}{30} = \frac{17}{30}$$



6

ex. 2  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$  for  $-\infty < x < \infty$  p.d.f. of  $x$

Let  $y = x^2 \rightarrow$  Find p.d.f. of  $y$

Sol.

$$y = x^2 \rightarrow x = \pm \sqrt{y} \rightarrow \frac{dy}{dx} = \pm \frac{2x}{2\sqrt{y}}$$

$$\begin{aligned} \infty > y > 0 \\ 0 < x^2 < \infty \\ -\infty < x < \infty \end{aligned}$$

$$|f(x)| \cdot \left[ f(\sqrt{y}) + f(-\sqrt{y}) \right] = f(y)$$

for  $0 < y < \infty$

$$= \left[ \frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \right]$$

$$= \frac{2}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-y/2}}{\sqrt{y}} = \frac{1}{\sqrt{2\pi y}} e^{-y/2}$$

∴



# Transformation of random variables

## I Transformation of d.r.v.

Suppose  $X$  is d.r.v. has p.m.f.  $f(x)$   
and Let  $A$  be the space of  $x$  for  $f(x) \geq 0$

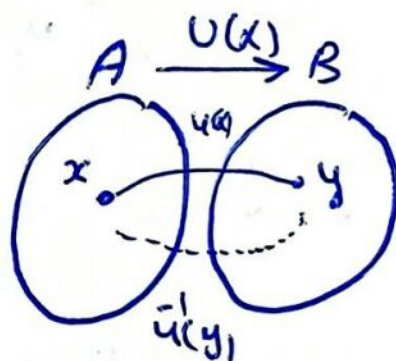
Let  $Y = U(X)$

Let  $B$  be the space of  $y$

To define a (1-1) transformation that  
 maps  $A$  onto  $B$

$$Y = U(X)$$

$$X = U^{-1}(Y) \in A$$



then the p.m.f of  $Y$  can be define  
 as below

$$g(y) = P(Y=y) = P(Y=U(X))$$

$$= P(X = U^{-1}(y))$$

$$= f(U^{-1}(y))$$

$$\therefore g(y) = f[U^{-1}(y)]$$

• In case when the transformation  
 is not (1-1)

((  $\tilde{f} \rightarrow \rightarrow$  ))

ex. Given a p.m.f. of  $X$  as below  

$$f(x) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

Find the p.m.f. of  $Y=2X$ .

Sol.

$$Y=2X=U(X)$$

$$A = \{X, X=0, 1, 2, \dots\}$$

$$\therefore X = \frac{Y}{2} = U^{-1}(Y)$$

$$B = \{Y; Y=0, 2, 4, \dots\}$$

$$g(Y) = f[U^{-1}(Y)] = f(Y/2)$$

$$= \begin{cases} \frac{e^{-m} m^{Y/2}}{(Y/2)!} & \text{for } Y=0, 2, 4, \dots \\ 0 & \text{o.w.} \end{cases}$$

اضافة  
الحل \* If  $m=2$ , Find  $P(X < 2)$   
 $P(Y < 2)$

$$\begin{aligned} P(X < 2) &= \sum_{x=0}^1 f(x) = f(0) + f(1) \\ &= \frac{e^{-2} \binom{2}{0}}{0!} + \frac{e^{-2} \binom{2}{1}}{1!} \\ &= 3e^{-2} = 3(0.13) = 0.39 \in [0, 1] \end{aligned}$$

$$\begin{aligned} P(Y < 2) &= P(Y=0) \\ &= \frac{e^{-2} \binom{2}{0}}{0!} = 0.13 \in [0, 1] \end{aligned}$$



## II] Transformation of c.r.v.

Suppose  $X$  is c.r.v. has p.d.f  $f(x)$   
 $f(x) > 0 \quad \forall x \in A$  (space  $X$ )

$$\text{Let } Y = U(X)$$

Monotonic transf.  
increasing or decreasing

define a (1-1) transformation  
 from  $A$  onto  $B$  [space of  $Y$ ]

$$X = \bar{U}'(Y) \in A$$

Then the p.d.f. of  $Y$  can be define as  
 below

$$g(y) = f[\bar{U}'(y)] \cdot |J| \quad \text{Jacobian determinate}$$

$$\text{where } J = [\bar{U}'(y)]' = X' = \frac{dX}{dY}$$

$$\underline{\text{or}} \quad g(y) = \frac{f[\bar{U}'(y)]}{\left| \frac{dy}{dx} \right|}$$

● In case when the transformation  
 not monotonic

( $\bar{U}' \rightarrow$ )



ex Given p.d.f of  $X$  as below

(4)

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

← uniform  
dist-  
on  $(0,1)$

Find the p.d.f of  $Y = -2 \ln X$

Sol.  $Y = -2 \ln X = u(X)$

$$X = e^{-Y/2} = u^{-1}(Y)$$

$$|J| = \left| \frac{dX}{dY} \right| = \left| -\frac{1}{2} e^{-Y/2} \right| = \frac{1}{2} e^{-Y/2}$$

$$0 < X < 1$$

$$-\infty < \ln X < 0$$

$$0 < -2 \ln X < \infty$$

$$0 < Y < \infty$$

$$\begin{aligned} e^0 &= 1 \Rightarrow \ln 1 = 0 \\ e^{-\infty} &= 0 \Rightarrow \ln 0 = -\infty \end{aligned}$$

$$\therefore g(Y) = f[u^{-1}(Y)] \cdot |J|$$

$$= 1 \cdot \frac{1}{2} e^{-Y/2}$$

$$\therefore g(Y) = \begin{cases} \frac{1}{2} e^{-Y/2} & \text{for } 0 < Y < \infty \\ 0 & \text{o.w.} \end{cases}$$

Remark If  $X$  is d.r.v. with p.m.f  
 $f(x)$  under space  $A$ ; s.t.  $A = \{x_1, \dots, x_n\}$   
 Let  $Y = U(X)$  is not (1-1) transformation  
 under space  $B$  since there exists  
 $x_i \in A$  such that  $y = U(x_i)$  for some  $i=1, 2, \dots, r$   
 $r < n$

Then the p.m.f. of  $Y$  under space  $B$  will be

$$g(y) = \sum_{i=1}^r f[u^{-1}(y)] = \sum_{i=1}^r f[x_i] \quad \checkmark$$

ex. Given a p.m.f of  $X$  as below

$$f(x) = \begin{cases} 1/6 & \text{for } x = -1 \\ 4/6 & \text{for } x = 0 \\ 1/6 & \text{for } x = 1 \end{cases}$$

Find p.m.f. of  $Y = X^2$ .

Sol.  $A = \{-1, 0, 1\}$

$$Y = U(X) = X^2 \longrightarrow B = \{0, 1\}$$

$$\therefore g(y) = f[u^{-1}(y)] = f(x)$$

$$g(y) = \begin{cases} 4/6 & \text{for } y = 0 \\ (1/6) + (1/6) = 2/6 & \text{for } y = 1 \end{cases}$$

$$P(X \geq 0) = \sum_{x=0}^1 P(X=x) = f(0) + f(1) = \frac{4}{6} + \frac{1}{6} = \frac{5}{6} \in [0, 1]$$

$$P(Y > 0) = g(1) = P(Y=1) \\ = \frac{2}{6} \in [0, 1]$$



# Transformation of random variables

## [1] Transformation of discrete random variables

- If  $X$  is a r.v. with cdf  $F_X$ , then  $Y = g(X)$  is also an r.v.
- If we write  $y = g(x)$ , the function  $g(x)$  defines a mapping from the original sample space of  $X$ ,  $S$  to a new sample space  $\Psi$ , the sample space of the r.v.  $Y$ .

$$g(x): S \longrightarrow \Psi$$

- Let  $y = g(x)$  define a one-to-one transformation that is, the equation  $y = g(x)$  can be solved uniquely :  $x = g^{-1}(y)$ .

ex.1  $y = x - 1 \longrightarrow x = y + 1$   $[1-1]$   
 $y = g(x) \quad x = g^{-1}(y)$

ex.2  $y = x^2 \longrightarrow x = \pm \sqrt{y}$  [not (1-1)]

- When transformation is not (1-1), find disjoint partitions of  $S$  for which transformation is (1-1).
- If  $X$  is d.r.v., then  $S$  is countable.  
The sample space of  $Y = g(X)$  is  $\Psi = \{y; y = g(x); x \in S\}$  also countable. The pmf for  $Y$  is  
$$f_Y(y) = P(Y=y) = \sum_{x \in g^{-1}(y)} P(X=x) = \sum_{x \in g^{-1}(y)} f_X(x)$$



## Examples

① Let  $X \sim \text{Geo}(p) \Rightarrow f(x) = p(1-p)^{x-1}$  for  $x=1,2,\dots$   
Find the p.m.f. of  $Y = X-1$  [ $Y=g(X)$ ],  $[1-1]$

Sol.  $X = Y+1$  [ $X=g^{-1}(Y)$ ]

$\therefore f_Y(y) = f_X(y+1) = p(1-p)^{(y+1)-1}$  for  $y=0,1,2,\dots$

② Let  $X$  be a d.r.v. has the following p.m.f

$$f(x) = \begin{cases} 1/5 & \text{for } x=-2 \\ 1/6 & \text{for } x=-1 \\ 1/5 & \text{for } x=0 \\ 1/5 & \text{for } x=1 \\ 1/30 & \text{for } x=2 \end{cases}$$

Let  $Y = X^2$ ;  $A = \{-2, -1, 0, 1, 2\} \rightarrow B = \{0, 1, 4\}$

$$g(y) = \begin{cases} 1/5 & y=0 \\ 7/30 & y=1 \\ 17/30 & y=4 \end{cases}$$

$[X=0 \rightarrow f(0)=1/5]$   
 $[X=\pm 1 \rightarrow f(1)=1/5, f(-1)=1/6] \rightarrow \frac{1}{5} + \frac{1}{6} = \frac{10+4}{30} = \frac{14}{30}$   
 $[X=\pm 2 \rightarrow f(2)=1/30, f(-2)=1/5] \rightarrow \frac{1}{30} + \frac{1}{5} = \frac{1+6}{30} = \frac{7}{30}$

Ex Let  $F(x) = 1 - e^{-2x}$  for  $x > 0$   
Consider  $x = e^y$ . Find  $f(y) = ?$

Sol.  $f(x) = F'(x) = 2e^{-2x}$  for  $x > 0$   
 $x = \ln y = u^{-1}(y)$

(طريقة إيجاد  $g(y)$  باستخدام  $f(x)$ )  
(using transformation)

$$|J| = |x'| = |(Lny)'| = \left| \frac{1}{y} \right| = \frac{1}{y}$$

$$\therefore g(y) = f[u^{-1}(y)] \cdot |J|$$

$$= 2e^{-2 \ln y} \cdot \left| \frac{1}{y} \right|$$

$$= 2e^{\ln y^{-2}} \cdot \left| \frac{1}{y} \right|$$

$$= \frac{2y^{-2}}{y}$$

$$= \frac{2}{y^3} = 2y^{-3} \text{ for } y > 1$$

$$\begin{pmatrix} e^\infty = \infty \\ e^{-\infty} = 0 \\ e^0 = 1 \end{pmatrix}$$

$$A = \{x; 0 < x < \infty\}$$

$$B = \{y; 1 < y < \infty\}$$

$$0 < x < \infty$$

$$e^0 < e^x < e^\infty$$

$$1 < y < \infty$$

موضح

## 2 Transformation of continuous random variables

- Let  $X$  be a c.r.v. with p.d.f. " $f$ "  
 Let  $y = g(x)$  be differentiable for all  $x$  and non-zero  
 Then  $Y = g(X)$  is also c.r.v. with p.d.f. as below

$$h(y) = \begin{cases} f(\bar{g}'(y)) \cdot \left| \frac{d}{dy} \bar{g}'(y) \right| & \text{for } y \in \psi \\ 0 & \text{o.w.} \end{cases}$$

ex.1 Let  $X$  have the following density

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\boxed{\log_e = \ln}$$

$$\text{Let } y = e^x$$

$$x = \bar{g}'(y) = \log y \Rightarrow dx = \frac{1}{y} dy$$

$$\therefore h(y) = 1 \cdot \left| \frac{1}{y} \right| \quad \text{for } 0 < \log y < 1$$

$$\therefore h(y) = \begin{cases} \frac{1}{y} & \text{for } 1 < y < e \\ 0 & \text{o.w.} \end{cases}$$

توضيح

$$\begin{pmatrix} 0 < x < 1 \\ e^0 < e^x < e^1 \\ 1 < y < e \end{pmatrix}$$

( $e = 2.7$  العدد النابسي)

ex.2 If  $X$  have the following density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad -\infty < x < \infty$$

Let  $y = x^2$ . Find the pdf of  $Y$ .

Sol.  $x = \pm \sqrt{y} \Rightarrow dx = \pm \frac{1}{2\sqrt{y}} dy$

$$h(y) = \frac{(2)}{\sqrt{2\pi}} e^{-y/2} \left| \frac{1}{2\sqrt{y}} \right|$$

$$0 < y < \infty \quad \left(\frac{1}{2}\right)^{1/2} \left(\frac{1}{2}\right)$$

$$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$$

$$\begin{aligned} &= \frac{y^{\frac{1}{2}-1} e^{-y/2}}{\sqrt{\pi} 2^{1/2}} \\ &= \frac{y^{\frac{1}{2}-1} e^{-y/2}}{\Gamma(1/2) 2^{1/2}} \end{aligned}$$

$$\sim \chi^2(1) \quad [\sim G(\frac{1}{2}, 2)]$$

تم



Ex. Let  $F(x) = 1 - e^{-2x}$  for  $x > 0$

consider  $Y = e^X$ ,  $f_Y(y) = ?$

طريقة ايجاد  $g(y)$   
باستخدام  $F(x)$

Sol  
يمكن ايجاد  $g(y)$   
بطريقة اخرى  
عن طريق  $f(x)$   
 $f(x) = F'(x) = 2e^{-2x}$

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y)$$

$$= F_X(\ln y) = 1 - e^{-2 \ln y}$$

$$= 1 - y^{-2}$$

for  $y > 1$

الفكرة  
ايجاد  $F_Y(y)$  من  $F_X(x)$   
ثم نجد  $g(y)$  من  $F_Y(y)$   
 $0 < X < \infty$   
 $e < e^X < \infty$   
 $1 < Y < \infty$

$$\therefore f_Y(y) = \frac{d}{dy} F_Y(y) = 2y^{-3} \text{ for } y > 1$$

Ex. consider c.v.d.  $X$  and  $Y = X^2$ ,  $f_Y(y) = ?$

$$F_Y(y) = P\left(\frac{X^2}{Y} \leq y\right) = P(-\sqrt{y} \leq X \leq +\sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\therefore f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \frac{d}{dy} F_X(\sqrt{y}) - \frac{d}{dy} F_X(-\sqrt{y})$$

$$= f_X(\sqrt{y}) \cdot \frac{d}{dy}(\sqrt{y}) - f_X(-\sqrt{y}) \cdot \frac{d}{dy}(-\sqrt{y})$$

$$= f_X(\sqrt{y}) \cdot \left(\frac{1}{2\sqrt{y}}\right) - f_X(-\sqrt{y}) \cdot \left(-\frac{1}{2\sqrt{y}}\right)$$

$$= \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})]$$

$$\frac{d}{dy} \sqrt{y} = \frac{1}{2\sqrt{y}}$$

~~no~~ (no)

Note: If  $Y = u(X)$  is not (1-1), then

① when  $X$  is d.r.v., define p.m.f. of  $Y$  as:

$$g(y) = \sum_{\substack{X \\ u(X)=y}} f[u^{-1}(y)] = \sum_{\substack{X \\ u(X)=y}} f[X]$$

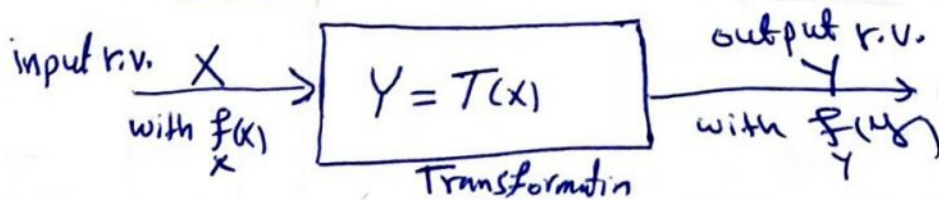
② when  $X$  is c.r.v., define p.d.f. of  $Y$  as:

$$g(y) = \sum_{\substack{X \\ u(X)=y}} f[u^{-1}(y)] \cdot |J| = \sum_{\substack{X \\ u(X)=y}} f[X] \cdot |X'|$$



# Transformation of Random Variables

2



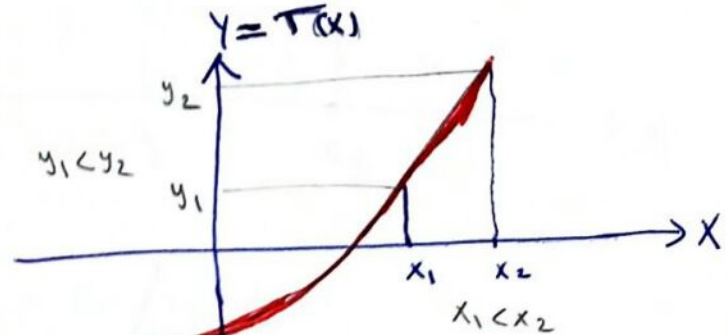
**Case 1**

If transformation is monotonically increasing

$$Y = T(x)$$

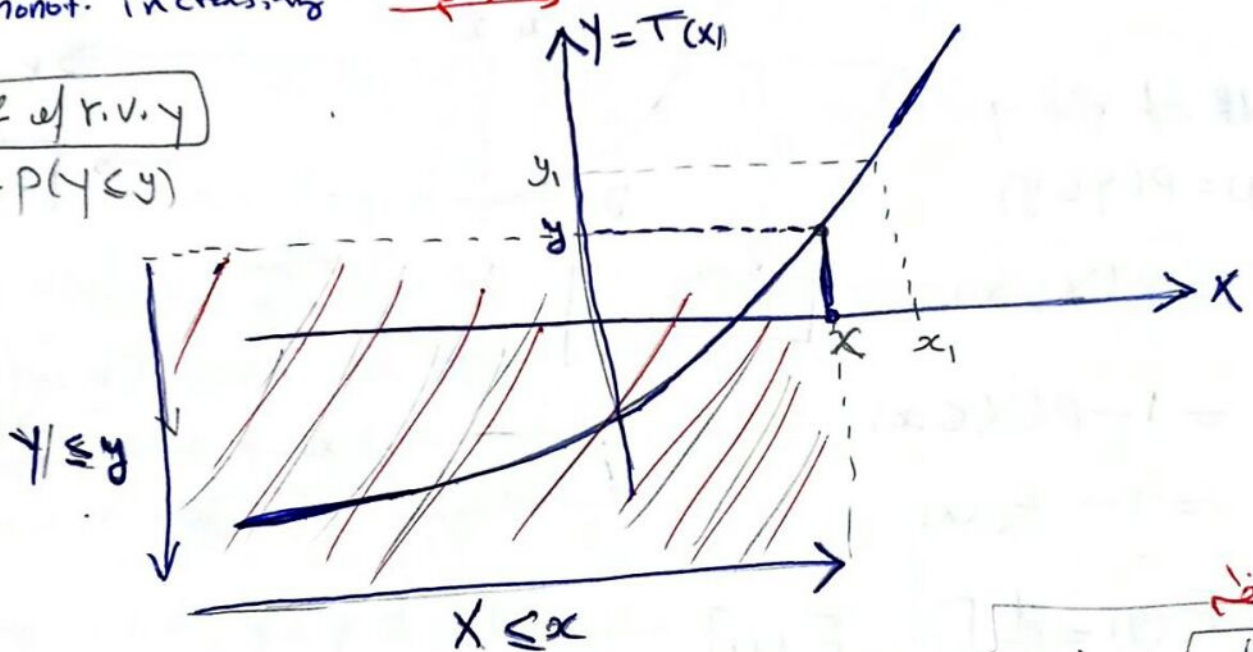
for  $x_1 \rightarrow y_1 = T(x_1)$   
 $x_2 \rightarrow y_2 = T(x_2)$

if  $x_1 < x_2 \rightarrow y_1 < y_2$   
 (monot. increasing)



cdf of r.v.  $Y$

$$F_Y(y) = P(Y \leq y)$$



$$F_Y(y) = P(Y \leq y) = P(X \leq x)$$

$$F_Y(y) = \int_{-\infty}^x f_X(x) dx$$

$$F_Y(y) = F_X(x)$$

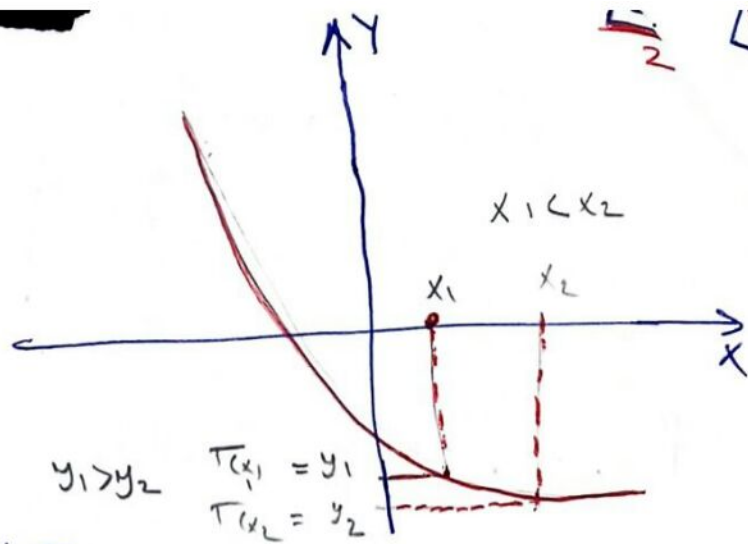
$$\frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(x) = \left[ \frac{d}{dx} F_X(x) \right] \cdot \frac{dx}{dy} = f_X(x) \cdot \frac{dx}{dy} = f_X(x) \cdot \left( \frac{dy}{dx} \right)^{-1}$$

ملاحظة  $\frac{dy}{dx} > 0$

ملاحظة  
 إذا كان  $X \leq x$  يجب أن  $Y \leq y$   
 لو أن  $x < x_1$  فإن  $y < y_1$  حيث  $y_1 = T(x_1)$

Case 2 If transformation is monotonically decreasing

$Y = T(X)$   
 for  $x_1 \rightarrow y_1 = T(x_1)$   
 $x_2 \rightarrow y_2 = T(x_2)$   
 if  $x_1 < x_2 \rightarrow y_1 > y_2$



cdf of r.v.  $Y$   
 $F_Y(y) = P(Y \leq y)$

$$= P(X > x)$$

$$= 1 - P(X \leq x)$$

$$= 1 - F_X(x)$$

$$\frac{d}{dy} F_Y(y) = \frac{d}{dy} [1 - F_X(x)]$$

$$= - \frac{d}{dx} F_X(x) \cdot \frac{dx}{dy}$$

$$= \frac{d}{dx} F_X(x) / \left( - \frac{dy}{dx} \right)$$

$$= f_X(x)$$

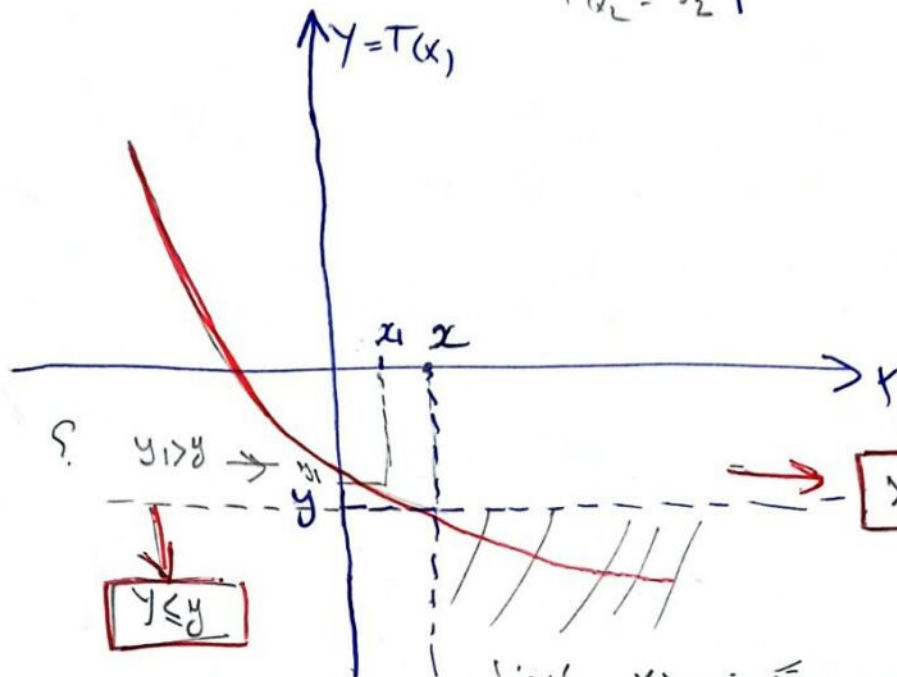
$$= \frac{f_X(x)}{- \left( \frac{dy}{dx} \right)}$$

$$\frac{dy}{dx} < 0$$

لا يتغير اتجاه

In general

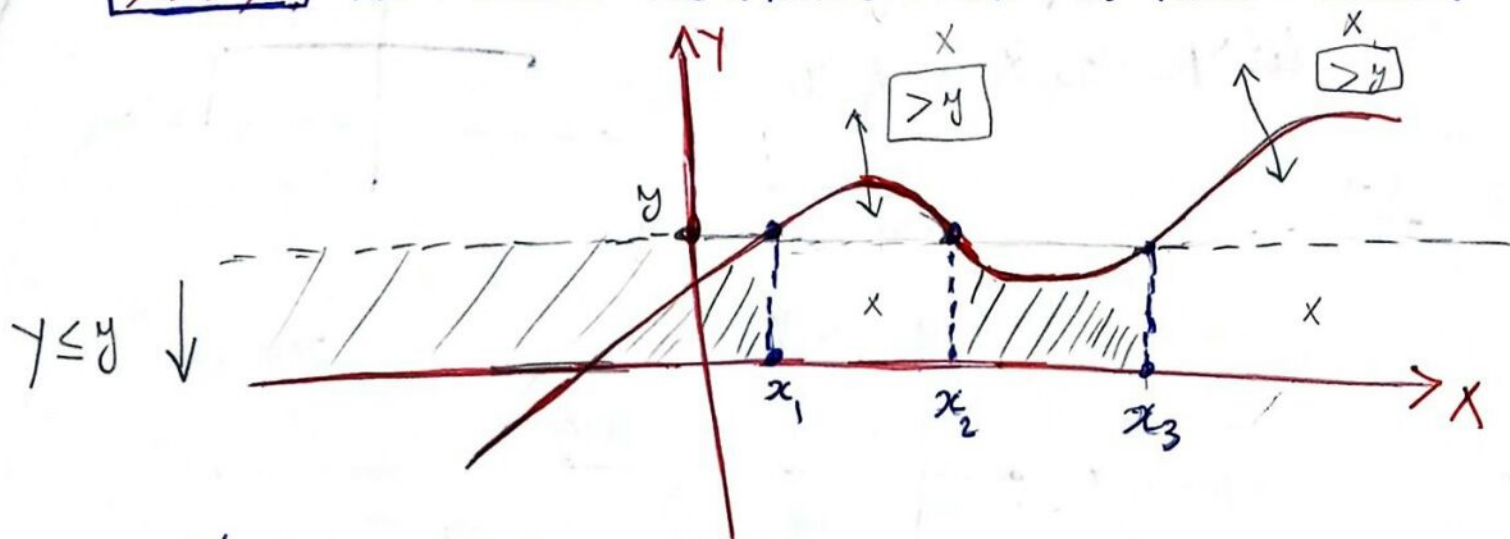
$$f_Y(y) = \frac{f_X(x)}{\left| \frac{dy}{dx} \right|} = f_X(x) \cdot \left| \frac{dx}{dy} \right|$$



ملاحظة: هنا يجب ان تكون  $X > x$  لماذا  
 لو افترضنا ان  $x_1$  اقل من  $x$  (انظر الرسم) انك  
 $x_1 < x \leftarrow y_1 > y$  و  $y_1 > y$  و  $y \leq y$



# Case 3 Non Monotonic transformation of random variables



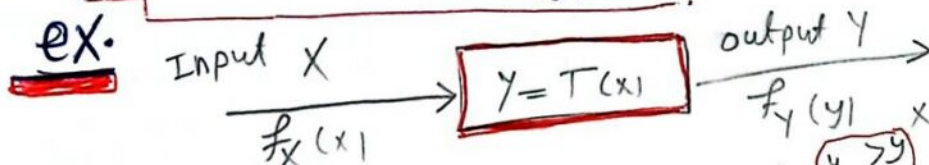
$$P(Y \leq y) = P(X \leq x_1) + P(x_2 \leq X \leq x_3)$$

$$= F_X(x_1) + F_X(x_3) - F_X(x_2)$$

$$= f_X(x_1) / \frac{dy}{dx_1} + f_X(x_3) / \frac{dy}{dx_3} - f_X(x_2) / \frac{dy}{dx_2}$$

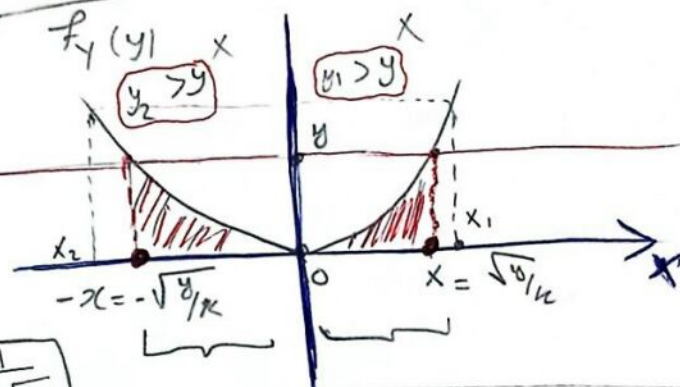
$$P(Y \leq y) = \sum_{i=1}^n \frac{f_X(x_i)}{\left| \frac{dy}{dx_i} \right|}$$

Non-mont. trans.



$$\neq y = kx^2 = T(x)$$

$$x = \pm \sqrt{\frac{y}{k}} = T^{-1}(y)$$



$$\frac{dy}{dx} = 2kx$$

$$= 2k\sqrt{\frac{y}{k}}$$

$$= 2\sqrt{ky}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{ky}}$$

$$F_Y(y) = P(Y \leq y)$$

$$= P(kx^2 \leq y)$$

$$= P(-\sqrt{\frac{y}{k}} \leq X \leq \sqrt{\frac{y}{k}})$$

$$= F_X(\sqrt{\frac{y}{k}}) - F_X(-\sqrt{\frac{y}{k}})$$

$$\frac{dF_Y(y)}{dy} = \frac{dF_X(\sqrt{\frac{y}{k}})}{dx} \frac{dx}{dy} - \frac{dF_X(-\sqrt{\frac{y}{k}})}{dx} \frac{d(-x)}{dy}$$

$$= f_X(\sqrt{\frac{y}{k}}) \cdot \frac{d\sqrt{\frac{y}{k}}}{dy} + f_X(-\sqrt{\frac{y}{k}}) \cdot \frac{d\sqrt{\frac{y}{k}}}{dy}$$

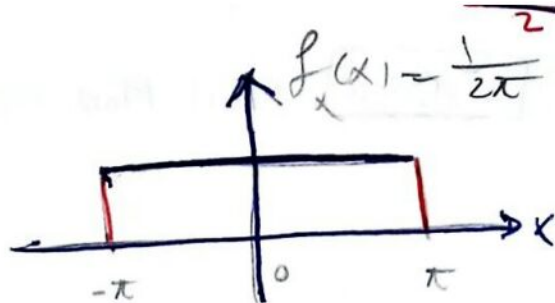
$$= \left| \frac{d\sqrt{\frac{y}{k}}}{dy} \right| [f_X(\sqrt{\frac{y}{k}}) + f_X(-\sqrt{\frac{y}{k}})]$$

$$= \frac{[f_X(\sqrt{\frac{y}{k}}) + f_X(-\sqrt{\frac{y}{k}})]}{\left| \frac{dy}{dx} \right|}$$

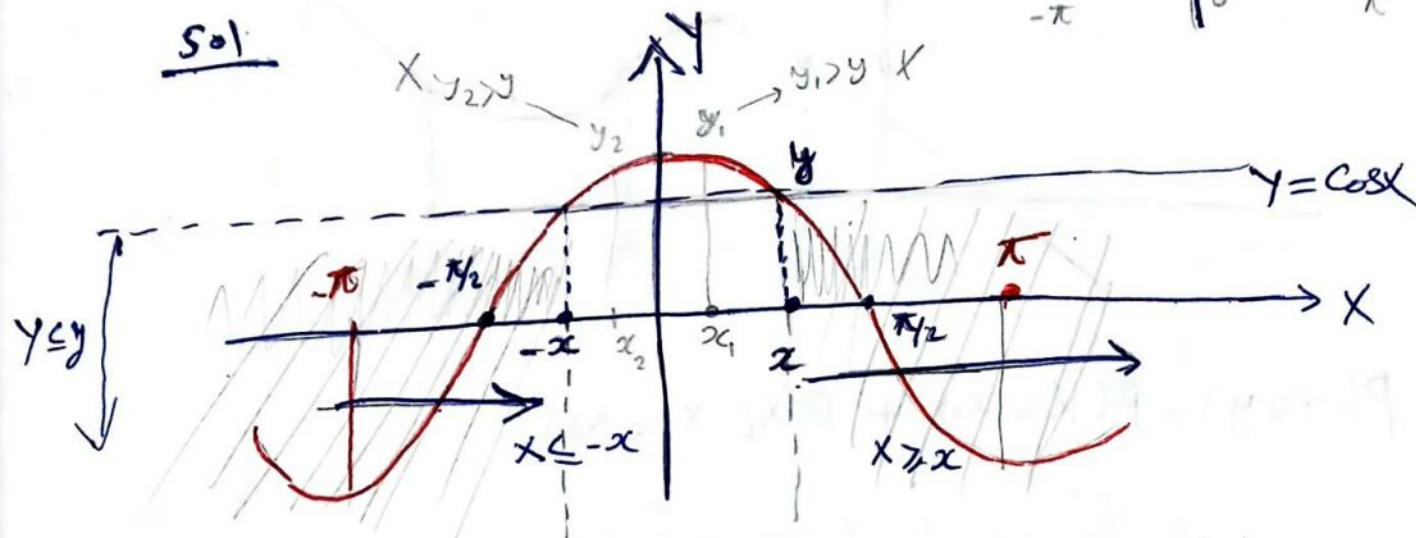
→ alt



Ex  $X \sim \text{unif}(-\pi, \pi)$   
 Let  $Y = \cos X$  ;  $f_Y(y) = ?$



Sol



$$F_Y(y) = P(Y \leq y) = 1 - P(X \leq x)$$

$$= P(X \leq -x) + P(X > x)$$

$$= F_X(-x) + 1 - F_X(x)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(-x)}{dx} \cdot \frac{d(-x)}{dy} + \frac{dF_X(x)}{dx} \cdot \frac{dx}{dy}$$

$$= f_X(-x) \cdot \left(-\frac{dx}{dy}\right) + f_X(x) \cdot \frac{dx}{dy}$$

$$= - \frac{[f_X(x) + f_X(-x)]}{\frac{dy}{dx}}$$

$$y = \cos x$$

$$\frac{dy}{dx} = -\sin x$$

$$x = \cos^{-1}(y)$$

$$\frac{dy}{dx} = -\sin(\cos^{-1}y)$$

$$= \frac{-\left[\frac{1}{2\pi} + \frac{1}{2\pi}\right]}{-\sin(\cos^{-1}y)}$$

$$= \left\{ \frac{\frac{1}{\pi}}{\sin(\cos^{-1}(y))} \right\} \text{ for } -1 \leq y \leq 1$$

H.w  $X \sim \text{unif}(-1, 1)$   
 $Y = X^2 \rightarrow$  Find  $f_Y(y) = ??$

$$= \left( \frac{1}{2\sqrt{y}} \right)$$

o.w

o.c.c

$$\frac{dy}{dx} = 2x \quad x = \pm \sqrt{y}$$

1. Find  $f(x)$  and sketch  $F(x)$  &  $f(x)$ .

2. Find  $p(x \leq 4)$ ,  $p(x > 8)$ ,  $p(2 < x \leq 8)$ ,  $p(x = 2)$ .

### Transformation of random variables:

1. Transf. of d. r. v.  $x$ :

If  $x$  be a d. r. v. have a p.m.f  $f(x)$ , and let  $A$  be the space of  $x$  for  $f(x) > 0$ , and let  $y = u(x)$  define a (1-1) transformation that maps  $A$  onto  $B$  (The space of  $y$ ).

If we solve  $y = u(x)$  for  $x$  in term of  $y$ .

Say  $x = u^{-1}(y) \in A$  then for  $y = u(x) \in B$ , we have  $x = u^{-1}(y) \in A$

Know if we consider the r.v.  $Y = u(x)$  if  $y \in B$ , then  $x = u^{-1}(y) \in A$  and the events  $Y = y$  [or  $u(x) = y$ ] and  $x = u^{-1}(y)$  are equivalent.

Then the p.m.f  $g(y)$  of r.v.  $Y$  is define as follows:

$$g(y) = p[Y = y] = p[Y = u(x)] = p[x = u^{-1}(y)] = f[u^{-1}(y)]$$

$$\therefore g(y) = f[u^{-1}(y)]$$

Ex. "1": Given a p.m.f  $f(x) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w} \end{cases}$

Find the p.m.f of  $Y = 2x$

Sol.:  $Y = 2x = u(x)$

$$A = \{x; x = 0, 1, 2, \dots\}, \quad X = \frac{y}{2} = u^{-1}(y)$$

$$B = \{y; y = 0, 2, 4, 6, \dots\}, \quad g(y) = f[u^{-1}(y)] = f\left[\frac{y}{2}\right]$$

$$\therefore g(y) = \begin{cases} \frac{e^{-m} m^{y/2}}{(y/2)!} & \text{for } y = 0, 2, 4, \dots \\ 0 & \text{o.w} \end{cases}$$

توزيع بواسون (Poisson)  
 $X \sim P_0(m)$   
 عملية توزيع بواسون  
 كمية ثابتة معلومة

Ex. "2": Let  $x$  have the following p.M.f

$$f(x) = \begin{cases} \binom{5}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{5-x} & x = 0, 1, \dots, 5 \\ 0 & \text{o.w} \end{cases}$$

توزيع ثنائي الحدين  
 $X \sim b(n, \frac{1}{4})$   
 $X \sim b(n, p)$

Find the p.M.f of  $y = x^2$

Sol.:  $y = x^2 = u(x)$

$$A = \{x; x = 0, 1, 2, 3, 4, 5\}, \quad x = u^{-1}(y) = \sqrt{y}$$

$$B = \{y; y = 0, 1, 4, 9, 16, 25\}$$

$$\therefore g(y) = f[u^{-1}(y)] = f(\sqrt{y})$$

$$= \begin{cases} \binom{5}{\sqrt{y}} \left(\frac{1}{4}\right)^{\sqrt{y}} \left(\frac{3}{4}\right)^{5-\sqrt{y}} & \text{for } y = 0, 1, 4, 9, 16, 25 \\ 0 & \text{o.w} \end{cases}$$

H. W. 1. If  $x$  has a uniform dist. on  $\{1, 2, 3\}$  then find the p.m.f of  $y = 4x + 2$ .

$$2. \text{ Given a p.m.f. } f(x) = \begin{cases} \left(\frac{1}{8}\right)^x, \frac{512}{73} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$$

Find the p.m.f of  $y = x^3$ .

$$= \begin{cases} \binom{3}{x} \left(\frac{1}{8}\right)^x \left(\frac{7}{8}\right)^{3-x} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

or  $f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^x & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$

استخدمنا قاعدة ج.S.

## 2. Transfor of c.r.v. x:

Let  $x$  be a c.r.v. have a p.d.f  $f(x)$ ,  $f(x) > 0$  for each point of the space  $A$  of  $x$ , let  $y = u(x)$  define a (1-1) transf. from  $A$  to space  $B$  of  $y$ .

If we can write  $x = u^{-1}(y)$  as the inverse of the function  $u$ , then the d.p.f. of  $y$  is:

$$g(y) = f[u^{-1}(y)] \cdot [J] \quad \text{where } J = [u^{-1}(y)]' = x' = \frac{dx}{dy}$$

Jacobian

$$\text{Ex. "1": } f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w} \end{cases} \quad \text{Find the p.d.f of } y = 8x^3$$

Sol.:  $Y = 8x^3 = u(x)$



$$A = \{x; 0 < x < 1\}, \quad x = \frac{\sqrt[3]{y}}{2} = u^{-1}(y)$$

$$B = \{y; 0 < y < 8\}$$

$$P[u^{-1}(y)] = P\left[\frac{\sqrt[3]{y}}{2}\right] = 2 \cdot \frac{\sqrt[3]{y}}{2} = \sqrt[3]{y} \quad (\text{since } f(x) = 2x)$$

$$J = \frac{dx}{dy} = \frac{1}{6\sqrt[3]{y^2}}$$

$$\therefore g(y) = f[u^{-1}(y)] \cdot |J| = \begin{cases} \frac{\sqrt[3]{y}}{6\sqrt[3]{y^2}} & \text{for } 0 < y < 8 \\ 0 & \text{o.w} \end{cases}$$

Ex. "2": If  $x \sim \text{unif}(0, 1)$  then find the p.d.f of  $y = -2\ln x$ .

sol.:  $\because x \sim \text{unif}(0, 1)$

$$\therefore f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

$$y = u(x) = -2 \ln x, \quad x = u^{-1}(y) = e^{-y/2}$$

$$|J| = \left| \frac{dx}{dy} \right| = \left| -\frac{1}{2} e^{-y/2} \right| = \frac{1}{2} e^{-y/2}$$

$$g(y) = f(u^{-1}(y)) \cdot |J| = 1 \cdot \frac{1}{2} e^{-y/2} \quad (\text{since } f[u^{-1}(y)] = f[e^{-y/2}] = 1, \text{ since } f(x) = 1)$$

The space of  $y = \{y; 0 < y < \infty\}$ ;  $\left( \begin{matrix} y(0) = -2\ln(0) = \infty \\ y(1) = -2\ln(1) = 0 \end{matrix} \right)$

$$g(y) = \begin{cases} \frac{e^{-y/2}}{2} & \text{for } 0 < y < \infty \\ 0 & \text{o.w} \end{cases}$$

H. W 81.  $f(x) = \begin{cases} \frac{x^2}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w} \end{cases}$  Find p.d.f of  $y = x^3$

2. Given a p.d.f  $f(x) = \begin{cases} 2xe^{-x^2} & \text{for } 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$  Find p.d.f of  $y = x^2$

ex: Let  $X$  be a d.r.v. has the following p.m.f.

$$f(x) = \begin{cases} 1/5 & \text{for } x = -2 \\ 1/6 & \text{for } x = -1 \\ 1/5 & \text{for } x = 0 \\ 1/15 & \text{for } x = 1 \\ 11/30 & \text{for } x = 2 \end{cases} \Rightarrow A = \{-2, -1, 0, 1, 2\}$$

Let  $Y = X^2$ , then Find the p.m.f. of  $Y$

Sol.

|    |    |   |   |   |   |
|----|----|---|---|---|---|
| -2 | -1 | 0 | 1 | 2 | A |
| 4  | 1  | 0 | 1 | 4 | B |

Remark

|         |                          |
|---------|--------------------------|
| $Y = 0$ | from $X = 0$             |
| $Y = 1$ | from $X = -1$ & $X = +1$ |
| $Y = 4$ | from $X = -2$ & $X = +2$ |

$$g(y) = \begin{cases} 1/5 & y=0 \\ 7/30 & y=1 \\ 17/30 & y=2 \end{cases}$$

$[X=0 \rightarrow f(0) = 1/5]$   
 $[X=\pm 1 \rightarrow f(-1) = 1/6, f(1) = 1/15 \Rightarrow \frac{7}{30}]$   
 $[X=\pm 2 \rightarrow f(-2) = 1/5, f(2) = 11/30]$

$$g(y) = \sum_{y=U(x_i)} P[U^{-1}(x)] = \sum_{\substack{y=x \\ y=U(x)}} P(x)$$

$$* P(Y < 2) = \sum_{y=0}^1 g(y) = \frac{1}{5} + \frac{7}{30} = \frac{13}{30} \in [0, 1]$$

$$* P(X < 0) = \sum_{x=-2}^{-1} f(x) = \frac{1}{5} + \frac{1}{6} = \frac{11}{30} \in [0, 1]$$

Ex 7

$$\textcircled{1} g(y=1) = \sum_{\substack{x=-1, 1 \\ 1=U(\pm 1)=(\pm 1)^2}} f(x) = f(-1) + f(1) = \frac{1}{6} + \frac{1}{15} = \frac{7}{30}$$

$$\textcircled{2} g(y=2) = \sum_{\substack{4=U(\pm 2)=(\pm 2)^2}} f(x) = f(-2) + f(2) = \left(\frac{1}{5}\right) + \left(\frac{11}{30}\right) = \frac{17}{30}$$

ex.2 Given p.d.f.  $f(x)$  as below

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ for } -\infty < x < \infty$$

Let  $y = x^2 \rightarrow$  Find p.d.f. of  $y$

Sol. A:  $-\infty < x < \infty$

B:  $0 < x^2 = y < \infty$

We know,  $g(y) = f[\bar{u}'(y)] \cdot |J|$

$$\therefore y = u(x) = x^2$$

$$x = \bar{u}'(y) = \pm \sqrt{y} \rightarrow J = \frac{dx}{dy} = \pm \frac{1}{2\sqrt{y}}$$

Note  $y = x^2$  not (1-1) since  $x$  has positive and negative values.

$$\therefore g(y) = \sum_{x=\pm\sqrt{y}} f[\bar{u}'(y)] \cdot |J| = \sum_{x=\pm\sqrt{y}} f(x) |J|$$

$$\rightarrow g(y) = [f(-\sqrt{y}) + f(\sqrt{y})] \cdot |J|$$

$$\begin{aligned} \Rightarrow g(y) &= \sum_{y=u(x)=(\pm\sqrt{y})^2} f(x) \cdot |J| \\ &= [f(-\sqrt{y}) + f(\sqrt{y})] |J| \\ &= \left[ \frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \right] \cdot \frac{1}{2\sqrt{y}} \\ &= \frac{2}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}} \\ &= \begin{cases} \frac{y^{-1/2} e^{-y/2}}{\sqrt{\pi} \cdot 2^{1/2}} & \text{for } 0 < y < \infty \\ 0 & \text{o.w} \end{cases} \end{aligned}$$

Gamma( $\frac{1}{2}$ , 2)

chi square with  $r=1$



# "Transformations"

Q1: If  $X$  has a uniform dist. on  $\{1, 2, 3\}$  then find the p.m.f. of  $Y = 4X + 2$ .

Sol.  $X \sim \text{unif.}(3)$  ;  $X = 1, 2, 3$  (d.r.v.)

$$A = \{X; X = 1, 2, 3\}$$

$$B = \{Y; Y = 6, 10, 14\}$$

$$Y = 4X + 2 = u(X) \in B$$

$$X = \frac{Y-2}{4} = u^{-1}(Y) \in A$$

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

$$g(y) = f[u^{-1}(y)] = f\left(\frac{y-2}{4}\right)$$

$$g(y) = \begin{cases} \frac{1}{3} & \text{for } y = 6, 10, 14 \\ 0 & \text{o.w.} \end{cases} \Rightarrow Y \sim \text{unif.}(3)$$

#

Q2: Given a p.m.f. of  $X$ :

$$f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases} \leftarrow f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$Y = X^3 = u(X) \in B$$

$$X = \sqrt[3]{Y} = u^{-1}(Y) \in A$$

$$A = \{X; X = 1, 2, 3, \dots\}$$

$$B = \{Y; Y = 1, 8, 27, \dots\}$$

$$g(y) = f[u^{-1}(y)] = f(\sqrt[3]{y})$$

$$\therefore g(y) = \begin{cases} 7\left(\frac{1}{8}\right)^{\sqrt[3]{y}} & \text{for } y = 1, 8, 27, \dots \\ 0 & \text{o.w.} \end{cases}$$

Check  $g(y)$  is p.m.f.

#

(Transformations)

$$Q_3: f(x) = \begin{cases} \frac{x^2}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

Find p.d.f. of  $y = x^3$ .

Sol.  $y = x^3 = u(x) \in B$   
 $x = \sqrt[3]{y} = u^{-1}(y) \in A$

$$A = \{x; 0 < x < 3\}$$

$$B = \{y; 0 < y < 27\} \quad (y(0) = 0 \text{ \& } y(3) = 27)$$

$$J = [u^{-1}(y)]' = |x'| = \left| \frac{dx}{dy} \right|$$

$$J = (\sqrt[3]{y})' = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{y^2}}$$

$$|J| = \left| \frac{1}{3\sqrt[3]{y^2}} \right| = \frac{1}{3\sqrt[3]{y^2}}$$

$$\begin{aligned} f[u^{-1}(y)] &= f[\sqrt[3]{y}] \text{ \& } f(x) = \frac{x^2}{9} \\ &= \frac{(\sqrt[3]{y})^2}{9} = \frac{\sqrt[3]{y^2}}{9} \end{aligned}$$

$$\therefore g(y) = f[u^{-1}(y)] |J|$$

$$= \frac{(\sqrt[3]{y^2})}{9} \cdot \frac{1}{3\sqrt[3]{y^2}} = \frac{1}{27}$$

$$\therefore g(y) = \begin{cases} \frac{1}{27} & \text{for } 0 < y < 27 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore y \sim \text{unif}(0, 27)$$

#

Q4: Given p.d.f. of  $X$ :

$$f(x) = \begin{cases} 2xe^{-x^2} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find p.d.f. of  $Y = X^2$ .

Sol.  $Y = X^2 = u(X) \in A$   
 $X = \sqrt{Y} = u^{-1}(Y) \in B$

$$A = \{X; 0 < X < \infty\}$$

$$B = \{Y; 0 < Y < \infty\} \rightarrow \begin{cases} Y(0) = (0)^2 = 0 \\ Y(\infty) = (\infty)^2 = \infty \end{cases}$$

$$\begin{aligned} f[u^{-1}(y)] &= f[\sqrt{y}] ; f(x) = 2xe^{-x^2} \\ &= 2\sqrt{y} e^{-(\sqrt{y})^2} = 2\sqrt{y} e^{-y} \\ &= 2\sqrt{y} e^{-y} \end{aligned}$$

$$\begin{aligned} J &= [u^{-1}(y)]' = x' = \frac{dx}{dy} \\ &= (\sqrt{y})' = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}} \end{aligned}$$

$$|J| = \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2\sqrt{y}}$$

$$\begin{aligned} \therefore g(y) &= f[u^{-1}(y)] |J| \\ &= (2\sqrt{y} e^{-y}) \left( \frac{1}{2\sqrt{y}} \right) = e^{-y} \end{aligned}$$

$$\therefore g(y) = \begin{cases} e^{-y} & \text{for } 0 < y < \infty \\ 0 & \text{o.w.} \end{cases}$$

(check  $g(y)$  is p.d.f.)

$$\therefore Y \sim \text{exp}(\lambda=1) ; \lambda=1$$

← right side  
 ← left side



## (Transformations)

### Exercises (H.W.)

Q<sub>1</sub>:  $X \sim \text{unif}(0,1)$  &  $Y = -2 \ln X$ ; then show that:  
 $Y \sim \chi^2(2)$  or  $Y \sim G(1,2)$ .

Q<sub>2</sub>: If  $X \sim \text{unif}(0,1)$  &  $Y = -4 \ln X$ ; then find p.d.f. of  $Y$ .

Q<sub>3</sub>: If  $X \sim P(3)$ ;  $Y = 2X$ ; find p.d.f. of  $Y$ .

Q<sub>4</sub>: If  $X \sim b(3, \frac{1}{2})$ ;  $Y = X^2$ ; find p.f. of  $Y$ .

Q<sub>5</sub>: If  $X \sim f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

&  $Y = 8X^3$ , then find the p.d.f. of  $Y$ .

Q: Given a p.d.f. of  $X$ :

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

① Let  $Y = X(4-X)$  be a transformation.  
Find a p.d.f. of  $Y$  ( $f_Y(y)$ ) & C.d.f.  $G(y)$ .

② Let  $Y = X(2-X)$  be a transformation.  
Find a p.d.f. of  $Y$  ( $f_Y(y)$ ) & C.d.f.  $G(y)$ .

Sol. ①  $Y = X(4-X) = 4X - X^2 = u(x) \in B$

$$A = \{x; 0 < x < 2\}$$