

chapter Five

Some special probability distributions

بعض التوزيعات الاحتمالية الخاصة المتعلقة بتغير واهتمت

* تأتي خصوصية هذه التوزيعات لأهميتها في الحياة ومن أهمها ولا يقتصر على الحياة من جهة أخرى

* تحت خاصية بأسماء معينة نسبة لاسم عالم أو تجربة معينة أو صفة معينة كأن تكون منقول من الحدين أو اسم عالم أو سلسلة الهندسية... أو ظاهرة في المجتمع أو حالة صرفة... الخ

بعض التوزيعات المتقطعة

X is a.r.v.

بعض التوزيعات المستمرة

d.r.v.

- Bernoulli dist.
- Binomial dist.
- Geometric dist.
- poisson dist.
- = etc.

c.r.v.

- Gamma dist.
- Exponential dist.
- Chi-Square dist.
- Normal dist.
- Standard Normal dist.
- = Beta dist.
- = etc.

ملامح في كل توزيع من التوزيعات اعلاه يتطلب معرفة واهتمت (اهتمت؟)

- ① $f(x)$, $M_x(t)$, $E(X)$, $Var(X)$ (يتطلب جدول)
- ② Examples.

chapter Five

Some special probability distributions

بعض التوزيعات الاحتمالية الخاصة المتعلقة بمفرد واحد

* تأتي خصوصية هذه التوزيعات لأهميتها في الحياة ومن أهمها ولا يقتصر على الحياة من جهة أخرى

* تحت خاصية بأعداد معينة نسبة للأسم عالم أو تجربة معينة أو صفة معينة كأن تكون منقول زير الحدين أو اسم عالم أو ليلة الهندية ... أو ظاهرة في المجتمع أو دالة صرفة ... الخ

بعض التوزيعات المتقطعة

X is ar.v.

بعض التوزيعات المستمرة

d.r.v.

- Bernoulli dist.
- Binomial dist.
- Geometric dist.
- poisson dist.
- = etc.

c.r.v.

- Gamma dist.
- Exponential dist.
- Chi-Square dist.
- Normal dist.
- Standard Normal dist.
- = Beta dist.
- = etc.

ملامحة في كل توزيع من التوزيعات أعلاه يتطلب معرفة وصف (امتناعاً !!)

- ① $f(x)$, $M_x(t)$, $E(X)$, $Var(X)$ (يتطلب جدول)
- ② Examples.

chapter five [ch.5 - part 1]
Some special probability distributions

①

I Discrete Distributions [d.r.v.]

① Binomial Distribution (توزيع ذي الحدين)
(Bernoulli dist.) (توزيع برونولي)

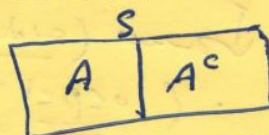
Bernoulli experiment (Bernoulli trials)

is an experiment that has only two possible outcomes event A (success) and A^c (Fail)

ex.1 Toss a coin

$$S = \{H, T\}$$

A : to get $H \rightarrow P(A) = 1/2$
 A^c : to get $T \rightarrow P(A^c) = 1/2$



ex.2 Roll a dice once

$$S = \{1, 2, \dots, 6\}$$

$A = \{1, 6\}$, $A^c = \{2, 3, 4, 5\}$

$$P(A) = \frac{2}{6} = \frac{1}{3}, \quad P(A^c) = \frac{4}{6} = \frac{2}{3}$$

Now, Suppose $P(A) = P$,

$$\text{so } P(A^c) = 1 - P$$

$$0 < P < 1$$

If we repeat the Bernoulli experiment
 n -times ($n \in \mathbb{I}^+$).

And if $X \equiv$ is a r.v. refer to number of
 A happens.

② Then the pr. Mass function of X is

هنا $f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x=0,1,2,\dots,n \\ 0 & \text{o.w.} \end{cases}$

هذه دالة احتمالية يجب اختبار شروطها

في محاور: $\left\{ \begin{array}{l} ① f(x) \geq 0 \\ ② \sum_{x=0}^n f(x) = 1 \end{array} \right.$

نلاحظ ان دالة تقع ايضاً بالاعتماد الى قيم X على n و p (كميات معلومة) فسنسجل

معالم التوزيع $(n \in \mathbb{I}^+, 0 < p < 1)$??

ثم نقول ان X يتوزع توزيع ذي الحدين (Binomial dist.) ونرمز له

$$X \sim b(n, p)$$

ex. If $X \sim b(4, \frac{1}{3})$, then write ① $f(x)$ p.m.f.

② Find $P(X \geq 1)$

Sol. ① $f(x) = \begin{cases} \binom{4}{x} (\frac{1}{3})^x (\frac{2}{3})^{4-x} & \text{for } x=0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

② $P(X \geq 1) = 1 - P(X < 1)$
 $= 1 - P(X=0)$
 $= 1 - f(0)$
 $= 1 - \binom{4}{0} (\frac{1}{3})^0 (\frac{2}{3})^4$
 $=$

الاحتمال
لقيم
[0,1]

Th.1 If $X \sim b(n, p)$, then the m.g.f. of X is $M_X(t) = (1 - p + pe^t)^n$ (3)

proof فرد مطلوب Hint $M_X(t) = E(e^{tx})$

$$= \sum_{x=0}^n e^{tx} f(x)$$

ex. If a r.v. X has the following probability function,

$$f(x) = \begin{cases} \binom{3}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

- Then اكتب اسم التوزيع
مع معادلة
- (1) Which dist. that a r.v. X has. (write the name of dist.)
 - (2) Write the $M_X(t)$
 - (3) Find $P(X > 2)$

Sol.

(1) $X \sim b(3, \frac{1}{2})$
 Binomial dist. (توزيع ثنائي التكرار)
نقطة ذي التكرار

(2) $M_X(t) = (1 - p + pe^t)^n$

$$= \left(\frac{1}{2} + \frac{1}{2}e^t\right)^3$$

(3) $P(X > 2) = P(X = 3)$

$$= f(3)$$

$$= \binom{3}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{3-3} = 1$$

$$= 1 \cdot \frac{1}{8} \cdot 1 = \frac{1}{8}$$

Th. 2 If $X \sim b(n, p)$, then

$$\mu_x = E(X) = np \quad \text{and} \quad \sigma_x^2 = \text{Var}(X) = np(1-p)$$

Proof
Hint

$$M_x(t) = (1-p + pe^t)^n$$

Compute $M'_x(t)$, $M''_x(t)$ at $t=0$

ex. Given a m.g.f. of X as below

$$M_x(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^6$$

- ② Find $f(x)$, $E(X)$, $\text{Var}(X)$
- ✓ ① Write the distribution (متوزع التوزيع)
- ③ Find $P(X=1)$, $P(X < 1)$

Sol. ① Binomial dist. $X \sim b\left(6, \frac{3}{4}\right)$

$$② f(x) = \begin{cases} \binom{6}{x} \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{6-x} & x=0, 1, 2, \dots, 6 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = np = 6 \left(\frac{3}{4}\right) = \frac{9}{2} = 4.5$$

$$\text{Var}(X) = np(1-p) = 6 \left(\frac{3}{4}\right) \left(\frac{1}{4}\right) = \frac{9}{2} \cdot \frac{1}{4} = \frac{9}{8}$$

$$③ P(X=1) = f(1) = \binom{6}{1} \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^{6-1} = \dots$$

$$P(X < 1) = P(X=0)$$

$$= f(0)$$

$$= \binom{6}{0} \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^{6-0}$$

$$= \dots$$

5

Remark $X \sim b(n, p)$ [Binomial dist.]

خاصه

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0, 1, \dots, n \\ 0 & \text{o.w.} \end{cases}$$

$$M_X(t) = (1-p + pe^t)^n$$

$$E(X) = np$$

$$V(X) = np(1-p)$$

When $n=1$, we obtain a

Bernoulli dist. (توزيع برنولي)

i.e. $X \sim b(1, p)$

مقدار
واقعی
p

$$f(x) = \begin{cases} \binom{1}{x} p^x (1-p)^{1-x} & x=0, 1 \\ 0 & \text{o.w.} \end{cases}$$

or

$$= \begin{cases} p^x (1-p)^{1-x} & x=0, 1 \\ 0 & \text{o.w.} \end{cases} \quad \begin{matrix} \binom{1}{0} = 1 \\ \binom{1}{1} = 1 \end{matrix}$$

&
$$M_X(t) = (1-p + pe^t)$$

$$E(X) = p$$

$$V(X) = p(1-p)$$

ex. If $X \sim b(1, \frac{3}{4})$, then ⑥

① Write $f(x)$, $M_x(t)$, $E(x)$ and $V(x)$

② Find $\underline{P(X=0)}$, $\underline{P(X=1)}$, $\underline{P(X=2)}$

Sol. ① $f(x) = \begin{cases} (\frac{3}{4})^x (\frac{1}{4})^{1-x} & x=0,1 \\ 0 & \text{o.w} \end{cases}$ توزيع
برنولی
از
طوب است

$$M_x(t) = (1-p + pe^t) = (\frac{1}{4} + \frac{3}{4}e^t)$$

$$E(x) = p = \frac{3}{4}$$

$$V(x) = p(1-p) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$$

$$\textcircled{2} \quad P(X=0) = f(0) = (\frac{3}{4})^0 (\frac{1}{4})^{1-0} = \frac{1}{4}$$

$$P(X=2) = 0$$

Remark (parameters) فیلد
است

It is a fixed value (quntum) exists
in a function to determined the form
(shape) of the function

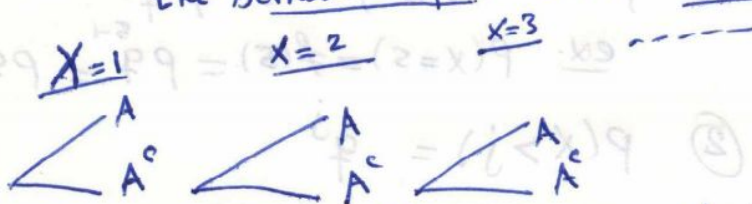
or a quntum control the shape
of the function

Chapter Five [Some special probability distribution]

part 2 [Geometric Distribution] التوزيع الهندسي

Suppose we have Bernoulli experiment.

Let $X \equiv$ r.v. number of repeat the Bernoulli exper- until A happens.



$R_X = \{1, 2, 3, \dots\} \rightarrow X$ is d.r.v.

then the probability function of X is

ie $f(X) = \begin{cases} p q^{x-1} & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases} \quad (p.m.f.)$

We say X followed Geometric dist.

i.e. $X \sim \text{Geo}(p)$ (p) and also (q = 1 - p)

Remark We have to verify that $f(x)$ is p.m.f.

i.e. ① $f(x) \geq 0 \quad \forall x$
② $\sum_{x=1}^{\infty} f(x) = 1$

Th.3 If $X \sim \text{Geo}(p)$, then the m.g.f. of X is

ie $M_X(t) = \frac{p e^t}{1 - q e^t} \quad \text{for } t < \ln\left(\frac{1}{q}\right)$

proof

Hint

$M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} \cdot \underbrace{\left(\frac{p q^{x-1}}{f(x)}\right)}_{f(x)}$

ie $\sum_{x=1}^{\infty} e^{tx} \cdot p q^{x-1}$

Th. 4 If $X \sim \text{Geo}(p)$, then

$$E(X) = \frac{1}{p} \text{ and } \text{Var}(X) = \frac{q}{p^2}$$

(2)

proof

Hint We use m.g.f. ($M_X(t)$) from th. 3 and derive it one and two times and put ($t=0$) to obtain $E(X)$, $E(X^2)$, $\text{Var}(X)$

Remark

① $P(X=i) = f(i) = pq^{i-1}$ (p.m.f.)
ex: $P(X=5) = f(5) = pq^{5-1} = pq^4$

② $P(X > j) = q^j$

ex.1 $P(X > 1) = 1 - P(X \leq 1) = 1 - P(X=1)$
 $= 1 - f(1)$
 $= 1 - pq^0$
 $= 1 - p$
 $= q$

ex.2 $P(X > 2) = 1 - P(X \leq 2)$
 $= 1 - P(X=1 \cup X=2)$
 $= 1 - [P(X=1) + P(X=2)]$
 $= 1 - [f(1) + f(2)]$
 $= 1 - [pq^0 + pq^1]$
 $= 1 - [p + pq]$
 $= (1-p) - pq$
 $= q - pq$
 $= q(1-p)$
 $= q \cdot q = q^2$

ملاحظة

لذا،

→ ③ $P(X \geq i) = P(X > i-1) \Rightarrow [P(X \geq 2) = P(X \geq 3)]$
 $= q^{i-1}$

→ ④ $P(X \leq i) = 1 - P(X > i) = 1 - q^i$

→ ⑤ $P(X < i) = 1 - P(X \geq i) = 1 - q^{i-1}$

X is d.i.v.

Th. 5 If $X \sim \text{Geo}(p)$, then

$$P(X > \underline{k+j} \mid X > \underline{k}) = P(X > j)$$

Memory
property
الذاكرة

ex. If $X \sim \text{Geo}(\frac{1}{3})$, then find

① $P(X > \underline{5} \mid X > \underline{3})$ ② $P(X > 4)$

Sol. ① $P(X > \underline{5} \mid X > \underline{3}) = P(X > 2)$

$$= q^2 \quad (\text{صباح الله})$$

$$= (\frac{2}{3})^2 = \frac{4}{9} \in [0,1]$$

$P = \frac{1}{3}$
 $q = \frac{2}{3}$

② $P(X > 4) = q^4$
 $= (\frac{2}{3})^4 = \frac{16}{81} \in [0,1]$

Examples

ex. 1 Suppose the probability to choose one defective tube is (0.4)

① Find the pr. to get one defective tube after (4) good tubes (أي في المرة)

② Find the pr. to get one defective tube after at least (3) good tubes.

Sol. ① A : to choose defective tube (أي) $P(A) = 0.4$

A^c : to choose good tube (أي) $P(A^c) = q = 1 - p = 1 - 0.4 = 0.6$

$$P(\text{ggggd}) = P(X=5) = ?$$

$$X \sim \text{Geo}(0.4) \Rightarrow f(x) = \begin{cases} (0.4)(0.6)^{x-1} & x=1,2,\dots \\ 0 & 0.4 \end{cases}$$

$$P(X=5) = P(5) = (0.4)(0.6)^{5-1} = (0.4)(0.6)^4$$

② $P(X \geq 4) = P(X > 3) = (q)^3 = (0.6)^3$

(4)

ex.2 If $X \sim \text{Geo}(\frac{3}{4})$, then write $f(x)$, $M(t)$, $E(X)$ and $V(X)$.

Sol.

$$f(x) = \begin{cases} \left(\frac{3}{4}\right) \left(\frac{1}{4}\right)^{x-1} & \text{for } x=1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$M(t) = \frac{pe^t}{1-qe^t} \text{ for } t < \ln\left(\frac{1}{q}\right)$$

$$= \frac{\left(\frac{3}{4}\right)e^t}{1-\left(\frac{1}{4}\right)e^t} \text{ for } t < \ln\left(\frac{1}{\frac{1}{4}}\right) \rightarrow (4)$$

$$E(X) = \frac{1}{p} = \frac{4}{3}$$

$$V(X) = \frac{q}{p^2} = \frac{(1/4)}{(3/4)^2} = \dots$$

ex.3 Throw the dice continuously until the number (5) happens (appears). If X represents the number of throws, then

- ① what is the probability dist.
- ② what is the pr. that we throw the dice (4) times
- ③ what is the average number of throws?

Sol. $A = \{5\} \xrightarrow{p(A)} p(A) = p = 1/6$

$A^c = \{1, 2, 3, 4, 6\} \xrightarrow{p(A^c)} p(A^c) = q = 1-p = 5/6$

$X \equiv$ No. of throws until $A = \{5\}$ appears

① $\therefore X \sim \text{Geo}(p) \rightarrow X \sim \text{Geo}\left(\frac{1}{6}\right)$

p.m.f $f(x) = \begin{cases} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^{x-1} & \text{for } x=1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$

② $p(X=4) = f(4) = \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^3$

③ Average of $X = E(X) = \frac{1}{p} = \frac{1}{1/6} = 6$

ex.4 A hunter ^{shoot} throws at the target and repeat throw until the target is hit. If the pr. of hitting the target is (0.7) and X represent the number of ^{throws} the target then

① Find the pr. that he needs (5) times to hit the target

② $=$ $=$ $=$ $=$ $=$ $=$ at least (5) times to hit the target

③ $=$ $=$ $=$ $=$ $=$ $=$ at most (5) $=$ $=$ $=$ $=$ $=$ $=$

Sol. $X \sim \text{Geo}(0.7) \rightarrow f(x) = \begin{cases} (0.7) (0.3)^{x-1} & \text{for } x=1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$

$p=0.7, q=0.3$

$p(X=5) = f(5)$

$p(X \geq 5) = p(X > 4) = (q)^4$

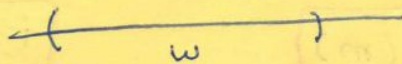
or $x \leq 5 = 1 - p(X > 5) = 1 - q^5$

Chapter Five - part 3

(1)

3- Poisson distribution (من التوزيع المستقل)

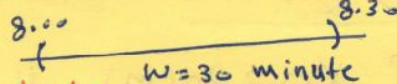
Given an interval of (time, space, distance ...) called w , $w > 0$



Let A be an event happen in interval w

Let $X \equiv$ No. of event A happens in w

ex. 1



w فترة زمنية

A : cars accident happens in

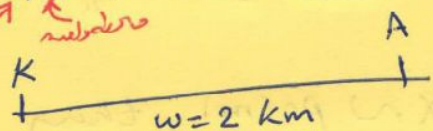
$w = 30 \text{ min.}$

$X \equiv$ No. of cars accident in w

$X = 0, 1, 2, \dots$

أرقام لا تقبل صفر

ex. 2



w فترة مسافة

A : police cars

$X \equiv$ No. of police cars in w

$X = 0, 1, 2, \dots$

Now, If $w = 1$ and $\lambda \equiv$ number of A happens when $w = 1$

$\lambda > 0$

$X \equiv$ No. of A happens in w

$$f(x) = \begin{cases} \frac{e^{-\lambda w} (\lambda w)^x}{x!} & x = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

$x = 0, 1, 2, \dots$

توزيع بواسون

$\Rightarrow X \sim \text{po}(\lambda w)$

Let $m = \lambda w$, $\boxed{w > 0}$ $\lambda > 0 \rightarrow \boxed{m > 0}$ (2)

Def $f(x) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x=0,1,2,\dots \\ 0 & \text{o.w} \end{cases}$

$\therefore \boxed{X \sim \text{po}(m)}$

$\boxed{\text{Poisson mean} = m}$

$f(x)$ is p.m.f $\begin{cases} \textcircled{1} f(x) \geq 0 \quad \forall x \in \mathbb{R}_x \\ \textcircled{2} \sum_{x=0}^{\infty} f(x) = 1 \end{cases}$ نتحقق

Th. 6 If $X \sim \text{po}(m)$, then the m.g.f. of X is $M(t) = e^{m(e^t - 1)}$ for $-\infty < t < \infty$

proof Hint $e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ maclaurin series

Th. 7 If $X \sim \text{po}(m)$, then $E(X) = m$ and $\text{Var}(X) = m$

proof Hint by th. 6 $\boxed{M'_X(0), M''_X(0) \Rightarrow \begin{bmatrix} E(X) \\ \text{Var}(X) \end{bmatrix}}$

ex. 1 If $X \sim \text{po}(2)$, then write $f(x)$, $M(t)$, $E(X)$, $\text{Var}(X)$, $P(X \geq 1)$ (compute)

Sol $\boxed{m=2}$

① $f(x) = \begin{cases} \frac{e^{-2} 2^x}{x!} & x=0,1,\dots \\ 0 & \text{o.w} \end{cases}$

② $M_X(t) = e^{2(e^t - 1)}$

③ $E(X) = \text{Var}(X) = 2$

④ $P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X=0)$

$= 1 - f(0)$

$= 1 - \frac{e^{-2} 2^0}{0!}$

$= 1 - 0.135 = 0.87$ $\in [0,1]$

ex. 2 The average number of customers entering a store is (4) per minute ^{دقيقة}

Assuming the number of customers has a poisson dist., Find the pr. that

- ① Two customers enter the store in half minute ^{نصف دقيقة}
- ② At least two customers $\geq$ half minute

Sol. $X = \text{No. of customers}$

$$E(X) = m = 4 \text{ in } w = 1$$

$$m = \lambda \cdot w$$

$$4 = \lambda \cdot 1$$

$$\lambda = 4$$

① when $w = \frac{1}{2} \Rightarrow P(X=2) = ?$

$$m = \lambda \cdot w \Rightarrow m = 4 \cdot \frac{1}{2} = 2$$

$$\Rightarrow X \sim \text{po}(2)$$

$$P(X) = \left\{ \frac{e^{-2} \cdot 2^x}{x!} \quad x = 0, 1, 2, \dots \right.$$

$$\therefore P(X=2) = f(2) = \frac{e^{-2} \cdot 2^2}{2!}$$

② when $w = \frac{1}{2} \Rightarrow P(X \geq 2) = ?$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - P[(X=0) \cup (X=1)]$$

$$= 1 - [f(0) + f(1)]$$

فوقها بالواحد
فوقها بالواحد

(4)

ex3 The average number of phone calls is (5) per minute [$w=1$]

If the phone calls follows Poisson dist then Find the pr. that there are

① 3 calls in minute ($w=1$) ←

② 3 calls in (15 seconds) ^{ثانية}

Sol.

$$m=5 \rightarrow w=1$$

① $X = \text{No. of phone calls}$

$$m = \lambda w = 5$$

$$X \sim \text{Po}(5)$$

$$f(x) = \frac{e^{-5} 5^x}{x!} \quad \text{for } x=0,1,2,\dots$$

$$p(X=3) = f(3) = \frac{e^{-5} 5^3}{3!}$$

② when $w = 15 \text{ seconds} \rightarrow p(X=3) = ?$

$$m = \lambda w = 5 \rightarrow 5 = \lambda \cdot 1 \rightarrow \lambda = 5 \quad \checkmark$$

$$w = 15 \text{ (Sec)} \Rightarrow w = \frac{15}{60} = \frac{1}{4} \text{ minute}$$

$$m = \lambda \cdot w \Rightarrow m = 5 \cdot \frac{1}{4} = 1.25$$

$$\Rightarrow X \sim \text{Po}(1.25) \Rightarrow f(x) = \frac{e^{-1.25} (1.25)^x}{x!} \quad \text{for } x=0,1,2,\dots$$

$$p(X=3) = f(3) = \frac{e^{-1.25} (1.25)^3}{3!} = ??$$

Chapter Five - Part 4

①

II Continuous Distribution

① Gamma Distribution توزيع كالا

We say a r.v. X follows Gamma dist.
 $(X \sim G(\alpha, \beta))$ with parameters α and β ,
 if the p.d.f. $f(x)$ has the form below

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^{\alpha}} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

① $f(x) \geq 0 \quad \forall x$ ← pdf
 ② $\int_0^{\infty} f(x) dx = 1$

② α, β are the parameters of the dist.

③ $\Gamma(\alpha) = (\alpha-1)!$ → $\Gamma(3) = 2! = 2 \times 1 = 2$

Th. 8 If $X \sim G(\alpha, \beta)$, then the m.g.f. of X
 is $M_X(t) = (1 - \beta t)^{-\alpha}$ for $t < \frac{1}{\beta}$

proof $M_X(t) = E[e^{tx}] = \dots$

Th. 9 If $X \sim G(\alpha, \beta)$, then $E(X) = \alpha\beta$
 and $\text{Var}(X) = \alpha\beta^2$

proof by th. 8 above
 $M'_X(0), M''_X(0) \Rightarrow E(X), V(X)$ ← نتیجه

ex.1 If $X \sim G(\alpha, \beta)$, then find
 $f(x)$, $M_X(t)$, $E(X)$ and $Var(X)$

Sol. $f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^\alpha} & \text{for } x \in (0, \infty) \\ 0 & \text{o.w} \end{cases} \quad 0 < x < \infty$

$M_X(t) = (1 - \beta t)^{-\alpha}$, $E(X) = \alpha \beta = 12$, $Var(X) = \alpha \beta^2 = 3 \cdot (4)^2 = 48$

ex.2 Find the p.d.f. and $E(X)$, $Var(X)$
 if the m.g.f. of X is

$M_X(t) = (1 - 3t)^{-2}$

Sol. We see that

$\Rightarrow X \sim G(2, 3)$

$\therefore f(x) = \begin{cases} \frac{x^{2-1} e^{-x/3}}{\Gamma(2) 3^2} & 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$

$E(X) = 2 \times 3 = 6$
 $Var(X) = 2 \times 3^2 = 18$

$M_X(t) = (1 - \beta t)^{-\alpha}$
 تذكر
 لو $\beta = 1$

$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^\alpha}$
 لو $\beta = 1$

$E(X) = \alpha \beta$
 $Var(X) = \alpha \beta^2$

ex.3 If the p.d.f. of X has the form below
 then find m.g.f. of X

Sol. $f(x) = \begin{cases} \frac{x^{2-1} e^{-x/2}}{\Gamma(2) 2^2} & 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$
 $\therefore X \sim G(2, 2)$
 $\therefore M_X(t) = (1 - 2t)^{-2}$

$f(x) = \frac{x^{\alpha-1} e^{-x/2}}{4}$ $0 < x < \infty$
 $0 & \text{o.w}$

$\Gamma(2) = 1! = 1$

$\Gamma(\alpha) = (\alpha-1)!$

Chapter Five - part 5

(1)

(2) Chi-Square Distribution

توزيع مربع كاي

It is special case of Gamma dist.

where (1) $\alpha = \frac{r}{2}$, $r \in \mathbb{I}^+$

(2) $\beta = 2$

∴ The new dist. is called chi-square dist.

denoted by $\chi^2(r)$ [r is called the degree of freedom]

∴ When $X \sim \chi^2(r)$, then pdf of X is

$$f(x) = \begin{cases} \frac{x^{\frac{r}{2}-1} e^{-x/2}}{\Gamma(\frac{r}{2}) 2^{r/2}} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^\alpha}$$

lob

⇒ Also

$$M_X(t) = (1-2t)^{-\frac{r}{2}}$$

$$E(X) = \left(\frac{r}{2}\right)(2) = r$$

$$\text{Var}(X) = \left(\frac{r}{2}\right)(2^2) = 2r$$

ex. 1

$$f(x) = \int_0^x \frac{x^{\frac{r}{2}-1} e^{-x/2}}{4} \quad 0 < x < \infty$$

o.w.

Find $M_X(t)$

Sol. $X \sim G(2, 2)$

→ or $X \sim \chi^2(4)$

∴ $M_X(t) = (1-2t)^{-2} \cdot \frac{4}{2}$

$$M_X(t) = (1-\beta t)^{-\alpha}$$

$$E(X) = \alpha \cdot \beta$$

$$\text{Var}(X) = \alpha \cdot \beta^2$$

lob is applied

$$\frac{r}{2} = 2$$

$$r = 4$$

$$M_X(t) = (1-\beta t)^{-\alpha}$$

$$M_X(t) = (1-2t)^{-r/2}$$

ex. 2 If X has a.m.g.f.

(2)

$$M_X(t) = (1 - 2t)^{-7} \text{ for } t < \frac{1}{2}$$

then write $f(x)$, $E(X)$ and $\text{var}(X)$

Sol.

$$M_X(t) = (1 - \beta t)^{-\alpha} \rightarrow \text{for Gamma}$$

$$M_X(t) = (1 - 2t)^{-7/2} \rightarrow \text{for } \chi^2(r)$$

$$\therefore \beta = 2, \alpha = 7$$

$$\text{i.e., } \beta = 2, \frac{r}{2} = 7 \Rightarrow r = 14$$

$$\therefore X \sim \chi^2(14) \rightarrow \text{or } X \sim G(7, 2)$$

$$\therefore f(x) = \frac{1}{\Gamma(7) 2^7} e^{-x/2} x^{6/2} \quad 0 < x < \infty \quad \beta = 2$$

$$E(X) = \alpha \cdot \beta = 7 \cdot 2 = 14$$

$$\text{or } E(X) = r = 14$$

$$\text{var}(X) = \alpha \cdot \beta^2 = 7 \cdot 4 = 28$$

$$\text{or } \text{var}(X) = 2r = 2 \cdot (14) = 28$$

Remark

① If $X \sim G(\alpha, \beta)$, then

$$Y = \frac{2X}{\beta} \sim \chi^2(2\alpha)$$

② If $X \sim G(\frac{r}{2}, \beta)$, then

$$Y = \frac{2X}{\beta} \sim \chi^2(r)$$

ex. 3

If $X \sim G(3, 4)$, then show that the dist of $\frac{2X}{4} \sim \chi^2(6)$

$$\text{Sol. } Y = \frac{2X}{\beta} \sim \chi^2(2\alpha) \rightarrow Y \sim \chi^2(6)$$

کون کا
ای کی
مکرو

$$r = \frac{r}{2}$$

$$r = 2\alpha$$

مقول کا
ای کی
مکرو

Ch 5 - part 4 - Beta Distribution

توزيع بيتا

A C.R.V. X is said it is follows (distribute) as a "Beta dist." if the p.d.f. of X has the forms below :

$$f(x; a, b) = \begin{cases} \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} & \text{for } 0 < x < 1; a, b > 0 \\ 0 & \text{constant. o.w.} \end{cases}$$

- ↑ p.d.f
- ① $f(x) \geq 0$
 - ② $\int_0^1 f(x) dx = 1$

denoted by $X \sim B(a, b)$

Where; $B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$

And, $B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$ [constant]

2

Th. If $X \sim B(a, b)$, then the m.g.f. of X is

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \frac{1}{B(a, b)} \cdot B(a+n, b)$$

Th. If $X \sim B(a, b)$, then $E(X) = \frac{a}{a+b}$ and

$$\text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$$

Remarks ① If $a=b=1$, then the Beta distribution reduces to uniform dist. (0,1) $f(x) = 1$ $0 < x < 1$

$$E(X) = \frac{1}{2}, \text{Var}(X) = \frac{1}{12}, M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \frac{1}{B(1,1)} \cdot B(1+n, 1)$$

② If $a=1$ and $b=2$, then Beta dist. known as triangular dist. $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

$$E(X) = \frac{1}{3} \text{ and } \text{Var}(X) = \frac{1}{18}, M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \frac{1}{B(1,2)} \cdot B(1+n, 2)$$

3

Example ^{المراهبين} The percentage of ^{الباحمين} betterers who make money of the race on a given day has Beta distribution with parameters $a=1$ and $b=5$. Find the pr. that fewer than ten percent comment winners on given day.

Sol.

Let $X \equiv$ pr. of betterers of race $\Rightarrow X \sim B(1, 5) \Rightarrow f(x) = \frac{1}{B(1,5)} x^0 (1-x)^{5-1}$ $0 < x < 1$

$$P(X < 0.1) = \int_0^{0.1} \frac{P(1+5)}{P(1)P(5)} x^0 (1-x)^{5-1} dx$$

$$5! = 5 \times 4! \\ P(1) = 0! = 1$$

$$= \frac{5!}{0!4!} \int_0^{0.1} (1-x)^4 dx$$

$$= -5 \left[\frac{(1-x)^5}{5} \right]_0^{0.1} = -[(0.9)^5 - 1]$$

$$= 1 - (0.9)^5 = \textcircled{0} \in [0, 1]$$

H.w If $X \sim B(1,1)$, then Find $E(Y)$
where $Y = X - (X - \frac{1}{2})^2$

4

Sol. $E(Y) = E\left\{X - (X - \frac{1}{2})^2\right\}$
 $= E(X) - E(X - \frac{1}{2})^2$

Hint ① $X \sim B(1,1) \rightarrow X \sim U(0,1) \rightarrow E(X) = 1/2$
 $V(X) = 1/12$

② $V(X) = E[(X - E(X))^2] = E[(X - 1/2)^2] = \frac{1}{12}$

$\therefore E(Y) =$

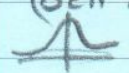
"Normal Distribution"

(التوزيع الطبيعي)

□

مقدمة

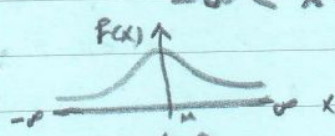
يعتبر التوزيع الطبيعي من أهم وأشهر التوزيعات الاحتمالية لأنه يصف معظم الظواهر الطبيعية مثل : درجة الحرارة ، مستوى الذكاء ، الوزن ، الطول ، درجات الطلبة ، ... الخ
ويستخدم التوزيع الطبيعي كتقريب لكثير من التوزيعات الاحتمالية ولذلك يستخدم في مجالات عديدة من فروع الإحصاء والاستدلال الإحصائي لكثير من البحوث الطبية والاجتماعية وعلوم الفلك وأكثرها من العلوم الأخرى
كما يسمى أيضاً توزيع غاوس (Gauss dist.) نسبة للعلماء الألماني (غاوس) والياً يسمى في بعض الأحيان (التوزيع الجرسى) (Bell dist.)
لأن منحنى التوزيع (الناله) يشبه بشكل الجرس (جرس متلوب) .



يُتبع
Now, we say a C.r.v. X Follow (distribute) as
a normal dist. with two parameters (μ) and (σ^2) if
 X has the following p.d.f $f(x)$

$$f(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma^2 > 0$$

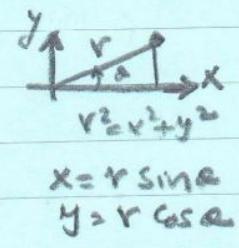
i.e; $X \sim N(\mu, \sigma^2)$



To be ensure $f(x)$ is p.d.f we need to prove ① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x) dx = 1$

Hint ① $I = \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sqrt{2\pi}$



② $I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r dr d\alpha$

③ $\int_{-\infty}^{\infty} y e^{-y^2/2} dy = 0$

④ $\int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy = \sqrt{2\pi}$

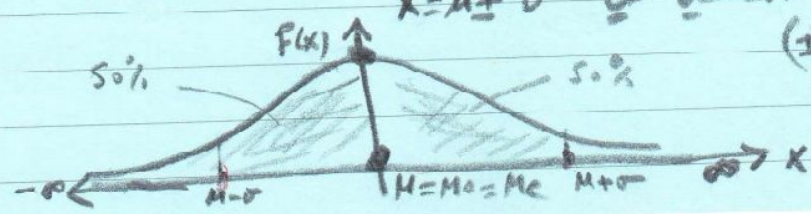
Some properties of Normal dist. بعض خصائص التوزيع الطبيعي

① التوزيع الطبيعي يتميز بخاصيتين هما μ (الوسط) وتنتج بعض القيم في المنحنى الطبيعي حول الوسط للتوزيع ونفس (التباين) ويبلغ مقياس لمدى انتشار القيم الممكنة للتغير العشوائي حول الوسط.

② في المنحنى الطبيعي تتساوى قيم التوزيع المركزية الثلاثة (الوسط والوسط والمعدل) حيث ستكون في نفس النقطة وهي مركز أو منتصف التوزيع.

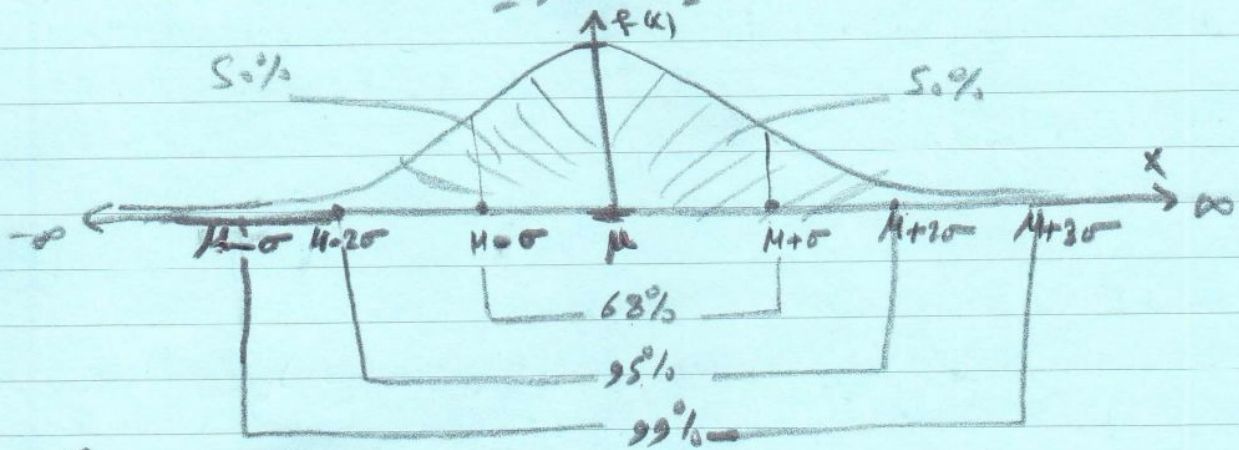
③ المنحنى الطبيعي متماثل (Symm) حول المحل العمودي المار من أعلى نقطة في المنحنى (قيمة التوزيع) حيث تقسم المنحنى الى نصفين متطابقين وتكون مساحة كل قسم (50%) من المساحة الكلية تحت المنحنى (والتي تسمى 1).

④ نقاط انقلاب المنحنى هي $x = \mu \pm \sigma$ (Inflection points)



④

من الملاحظة الكلية للتوزيع الطبيعي نسبة نسب معينة وفقاً لعدد الانحرافات المياريّة عن الوسط الحسابي وكما يأتي

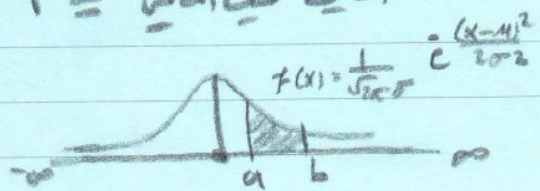


$$\int_{-\infty}^{\infty} f(x) dx = 1$$

المساحة الكلية تحت المنحنى = 1

صالح
عبدال
سليم

$$P(a < x < b) = \int_a^b f(x) dx$$



⑤

⑤

Th. 10 If a c.r.v. X has a normal dist. with the two parameters μ and σ^2 , then the M.g.f. of X is

البيان
في المفرد

$$M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \quad -\infty < t < \infty$$

Th. 11 If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$ & $Var(X) = \sigma^2$

Standard Normal dist.

«التوزيع الطبيعي القياسي»
هو التوزيع الطبيعي الذي وسطه الحسابي $\mu = 0$ وتباينه $\sigma^2 = 1$
وسمى أحياناً توزيع $(\bar{X})$ أو (Zero-one) ويرمز للمتغير العشوائي
الذي يتبع هذا التوزيع $N(0, 1)$ (التوزيع الطبيعي القياسي)

6

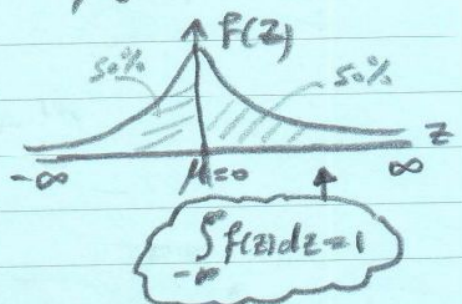
i.e; $Z \sim N(0,1)$, $\mu=0$, $\sigma^2=1$

* $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ $-\infty < z < \infty$

* $M(t) = e^{t^2/2}$ $-\infty < t < \infty$

* $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-w^2/2} dw$

* $P(a < Z < b) = \int_a^b f(z) dz$ ← ضمانه جدارى ضابط



ملاحظة يمكن تحويل التوزيع الطبيعي الى التوزيع القياسي عند ضابطى

Theorem If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$.
 كبرهان

7

proof $\because X \sim N(\mu, \sigma^2) \Rightarrow M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$Z = \frac{X-\mu}{\sigma} = \underbrace{\left(\frac{1}{\sigma}\right)}_a X + \underbrace{\left(\frac{-\mu}{\sigma}\right)}_b$ $a = \frac{1}{\sigma}$, $b = \frac{-\mu}{\sigma}$

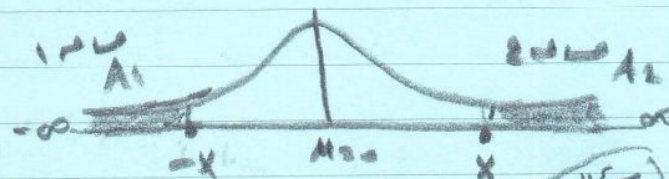
$M(t) = e^{bt} \cdot M_x(at)$ (مبدأ نظرية سلافا)
 $= e^{-\frac{\mu}{\sigma}t} \cdot \left[e^{\mu \frac{1}{\sigma}t + \frac{\sigma^2}{2} \left(\frac{1}{\sigma}t\right)^2} \right]$
 $= e^{-\frac{\mu}{\sigma}t + \frac{\mu}{\sigma}t + \frac{\sigma^2}{2} \cdot \frac{t^2}{\sigma^2}}$
 $= e^{\frac{t^2}{2}}$ $-\infty < t < \infty$

تذكير
 $M(t) = E[e^{tZ}]$

$\therefore Z \sim N(0,1)$

حساب الاحتمالات من الجدول الخاص بالتوزيع الطبيعي المعياري

$Z \sim N(0,1)$



مسألة
خاصة بالاحتمالات
بواسطة A_1 و A_2
 $P(X < x) = P(X > x)$

① $\Phi(z) = N(z) = P(Z \leq z)$

$= \int_{-\infty}^z f(w) dw = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$

② $\Phi(\infty) = N(\infty) = 1$

③ $\Phi(-a) = N(-a) = P(Z \leq -a) = P(Z \geq a) = 1 - P(Z \leq a) = 1 - \Phi(a)$

④ $P(Z > a) = 1 - P(Z \leq a) = 1 - \Phi(a)$

⑤ $P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a) = \Phi(b) - \Phi(a) = N(b) - N(a)$



توزيع جدول التوزيع الطبيعي المعياري

$P(Z \leq z) = \Phi(z) = N(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$

z	$N(z)$	z	$N(z)$	z	$N(z)$
0.0	0.5	1.1	0.864	2.05	0.980
0.1	0.540	1.2	0.885	2.15	
0.25	0.549	1.3	0.903	2.30	0.989
⋮		1.5	0.933		
0.85	0.802	1.85	0.968	2.80	0.997
⋮					
1.05	0.853	2.0	0.977	3.00	0.999
				∞	1

ex.1 If $Z \sim N(0,1)$, Find $P(Z < 0.85)$, $P(Z > 0.85)$ 10

Sol. by table of Z-dist.

$$P(Z < 0.85) = 0.802$$

$$P(Z > 0.85) = 1 - P(Z < 0.85) = 1 - 0.802 = \dots$$

ex.2 ^{↑ Compute} $P(Z < -1.85)$

(using Standard Normal dist. table)

Sol. $P(Z < -1.85) = 1 - P(Z < 1.85)$

$$= 1 - 0.968 = 0.032 \in [0,1]$$

ex.3 Find $P(2 < Z < 3)$ & $P(-2 < Z < 3)$

① $P(2 < Z < 3) = N(3) - N(2) = 0.999 - 0.977 = 0.022$

② $P(-2 < Z < 3) = N(3) - N(-2) = N(3) - [1 - N(2)]$
 $= 0.999 - [1 - 0.977]$
 $= 0.999 - 0.023$
 $= 0.976$

ex. If $X \sim N(\overset{\mu}{1}, \overset{\sigma^2}{4})$, then Find ① $P(X < 5)$ 11

② $P(5 < X < 7)$ using Z-tables [S.N.D.]

Sol. $X \sim N(\overset{\mu}{1}, \overset{\sigma^2}{4}) \rightarrow Z = \frac{X - \mu}{\sigma} \sim N(0,1)$ (نظرن)

① $P(X < 5) = P\left(\frac{X - \mu}{\sigma} < \frac{5 - \mu}{\sigma}\right) = P\left(Z < \frac{5 - 1}{2}\right)$
 $= P(Z < 2) = N(2) = 0.977$ (Z-table)

② $P(5 < X < 7) = P\left(\frac{5 - 1}{2} < Z < \frac{7 - 1}{2}\right) = P(2 < Z < 3)$
 $= N(3) - N(2)$
 $= 0.999 - 0.977 = 0.022$

12
مدار الفيزياء

H.W ① If $X \sim N(5, 16)$, then
Find $P(X \leq 13)$ using table of S.N.D.

② If $X \sim N(5, 4)$, then Find
 $P(1 < X < 8)$ using S.N.D. table

③ If $X \sim N(2, 4)$, then Compute
 $P(6 < X < 8)$ using table of S.N.D.

3) Geometric Distribution التوزيع الهندسي

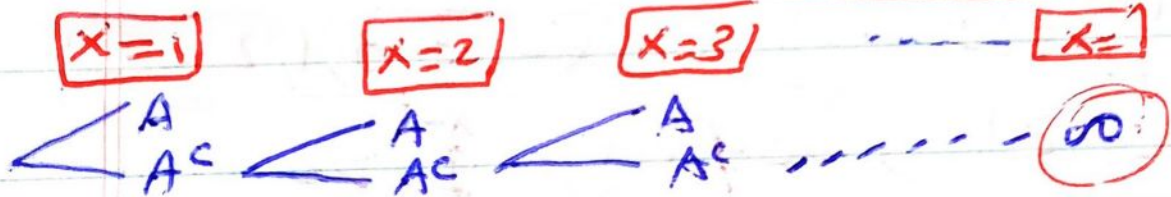
Given a Bernoulli experiment
with only two events A & A^c

let $P(A) = p$, $0 < p < 1$
 $P(A^c) = 1 - p = q$, $0 < q < 1$

A	A^c
نجاح	فشل

Let X be a r.v. refer to the number
of experiment until (A) happens at the
first time

عدد مراتب تكرار التجربة حتى يحصل على
الحدث A



$X = \{1, 2, 3, 4, \dots\}$

ex. Toss a coin until $\{H\}$ happens

$A = \{H\} \rightarrow P(A) = \frac{1}{2} = p$

$A^c = \{T\} \rightarrow P(A^c) = \frac{1}{2} = q$

$X =$ no. of exper. until H happens

what is the prob. that $X=4$

توزيع

$X=4$

$A^c A^c A^c A$

$P(X=4) = P(A^c A^c A^c A)$

احتمال وقوع هذا التسلسل

$= P(A^c) P(A^c/A^c) P(A^c/A^c A^c)$

$= P(A/A^c A^c A^c)$

$$\begin{aligned}
 P(X=4) &= P(A^c A^c A^c A) = P(A^c) P(A^c/A^c) P(A^c/A^c A^c) P(A/A^c A^c A^c) \\
 &= \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = q \cdot q \cdot q \cdot p \\
 &= [P(A^c)]^3 \cdot P(A) \\
 &= q^3 \cdot p
 \end{aligned}$$

نلاحظ

$X-1 = 3$

$X=4$ نجاح

Thus, the function (p.m.f.) of x defined as below

$$f(x; p) = \begin{cases} p q^{x-1} & \text{for } x=1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

It is called Geometric distribution

denoted by $X \sim \text{Geo}(p)$; $p = P(A)$

To show that $f(x; p)$ satisfy the conditions of p.m.f

Cond. ① T.P. $f(x) > 0$

$$x = 1, 2, \dots$$

$$x-1 = 0, 1, 2, \dots$$

$$0 < p < 1, 0 < q < 1, q^{x-1} > 0$$

$$\therefore p \cdot q^{x-1} > 0 \quad \leftarrow$$

$$\therefore f(x; p) > 0 \quad \text{for } x = 1, 2, \dots$$

Cond ② T.P. $\sum_{x=1}^{\infty} f(x; p) = 1$

$$\begin{aligned} \sum_{x=1}^{\infty} f(x; p) &= \sum_{x=1}^{\infty} p q^{x-1} = p \sum_{x=1}^{\infty} q^{x-1} = p \underbrace{[1 + q + q^2 + \dots]}_{\text{Geom. Series}} \\ &= p \frac{1}{1-q} \quad (0 < q < 1) \\ &= p \cdot \frac{1}{p} = 1 \end{aligned}$$

Convergent $|q| < 1$

$\therefore f(x; p)$ is p.m.f

Note

$$P(X > 1) = 1 - P(X \leq 1)$$

$$= 1 - P(X = 1) = 1 - f(1)$$

$$= 1 - p = q$$

H.W.

$$P(X > j) = q^j$$

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - [P(X=1) + P(X=2)]$$

$$= 1 - [p + pq]$$

ex: Suppose the prob. to choose one defective tube is (0.4)

3

① Find the pr. to get ~~the~~ defective tube after at most (4) good tubes

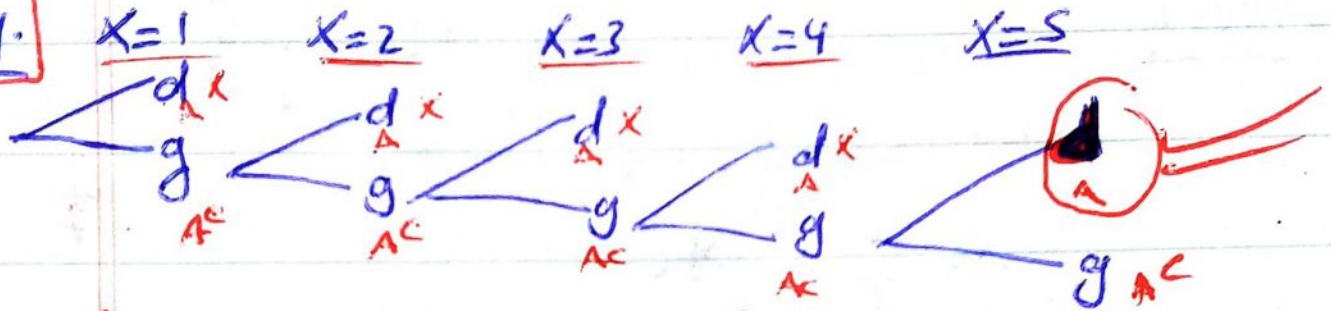
def.	good
A عَلَف	A ^c جيد

$$p = P(A) = 0.4$$

$$q = P(A^c) = 0.6$$

② after greater than 4 good tubes

Sol.



Let $X \equiv$ no. of experiments until A happens
[until A [def.] happens]

$$P(ggggd) = P(X=5)$$

$$\therefore X \sim \text{Geo}(0.4)$$

$$f(x, p) = \begin{cases} p \cdot q^x & x=1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$f(x, 0.4) = \begin{cases} (0.4)(0.6)^{x-1} & \text{for } x=1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore P(X=5) = f(5) = (0.4)(0.6)^{5-1} = (0.4)(0.6)^4$$

or by conditional

$$P(ggggd) = P(g_1)P(g_2|g_1)P(g_3|g_1g_2)P(g_4|g_1g_2g_3) \cdot P(d|g_1g_2g_3g_4)$$

$$= (0.6)(0.6)(0.6)(0.6)(0.4)$$

$$= (0.6)^4 \cdot (0.4)$$

$$= q^4 \cdot p$$

$$(2) P(X > 4) = 1 - P(X \leq 4) \\ = q^4$$

تکامل از ۱ تا ۴ (4)

Th.3 If $X \sim \text{Geo}(p)$, then the m.g.f. of X is

$$M(t) = \frac{pe^t}{1-qe^t} \text{ for } t < \ln\left(\frac{1}{q}\right)$$

proof $\because X \sim \text{Geo}(p) \rightarrow f(x) = p q^{x-1} \quad x=1, 2, \dots$

$$M(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} \cdot p q^{x-1}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q} \cdot qe^t \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

$$= pe^t [1 + qe^t + (qe^t)^2 + \dots]$$

$$= pe^t \cdot \frac{1}{1-qe^t}$$

$$= \frac{pe^t}{1-qe^t} \text{ for } t < \ln\left(\frac{1}{q}\right)$$

Geometric series
converges $|r| = |qe^t| < 1$
 $qe^t < 1 \rightarrow t < \ln \frac{1}{q}$

Th.4 If $X \sim \text{Geo}(p)$, then $E(X) = \frac{1}{p}$ & $V(X) = \frac{1}{p^2}$

proof $M(t) = \frac{pe^t}{1-qe^t}$

$$M'(t) = \frac{(1-qe^t)pe^t - (pe^t)(-qe^t)}{(1-qe^t)^2} = \frac{pe^t - (pe^t)(-qe^t) + pe^t(qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t}{(1-qe^t)^2}$$

$$E(X) = M'(0) = \frac{pe^0}{(1-qe^0)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Find $M''(t) \rightarrow E(X^2) = \frac{(1+q)}{p^2}$

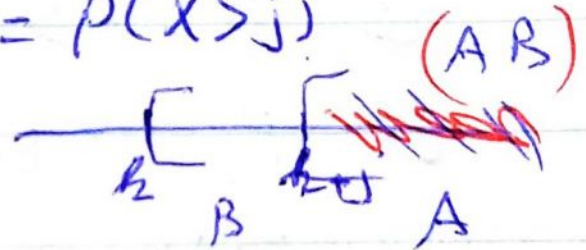
$$V(X) = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{1+q-1}{p^2} = \frac{q}{p^2}$$

Th. 5 If $X \sim \text{Geo}(p)$, then

المبدأ
فقدان
الذاكرة
memoryless
property

$$P(X > k+j | X > k) = P(X > j)$$

proof



$$P(X > k+j | X > k) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(X > k+j)}{P(X > k)} = \frac{q^{k+j}}{q^k} = q^j$$

أو
نوعاً آخر
الذاكرة
المفقودة
في كل تجربة $X=1, 2, 3, \dots$
 $\Rightarrow P(X > k) = [P(X > k-1)]^2 = q^k$

$$\begin{aligned} \textcircled{1} P(X > 1) &= 1 - P(X \leq 1) = 1 - P(X=1) \\ &= 1 - p = q \end{aligned}$$

$$\begin{aligned} \textcircled{2} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - [P(X=1) + P(X=2)] \\ &= 1 - [p + pq] \\ &= (1-p) - pq \\ &= q - pq \\ &= q(1-p) \\ &= q \cdot q = q^2 \end{aligned}$$

$$\begin{aligned} \textcircled{3} P(X > 3) &= 1 - P(X \leq 3) \\ &= 1 - [P(X \leq 2) + P(X=3)] \\ &= 1 - [(1-q^2) + pq^2] \\ &= q^2 - pq^2 = q^2(1-p) \\ &= q^3 \end{aligned}$$

$$\begin{aligned} \textcircled{4} P(X > 4) &= 1 - P(X \leq 4) \\ &= 1 - [P(X \leq 3) + P(X=4)] \\ &= 1 - [(1-q^3) + pq^3] \\ &= q^3 - pq^3 \\ &= q^3(1-p) = q^4 \end{aligned}$$

$$\begin{aligned} \therefore P(X > j) &= q^j \\ P(X > j+k) &= q^{j+k} \end{aligned}$$

6

Remark If $X \sim \text{Geo}(p)$, then

$$(1) P(X=i) = f(i) = q^{i-1} p$$

$$(2) P(X > i) = q^i$$

$$(3) P(X \geq i) = P(X > i-1) = q^{i-1}$$

$$(5) P(X \leq i) = 1 - P(X > i) = 1 - q^i$$

$$(6) P(X < i) = 1 - P(X \geq i) = 1 - q^{i-1}$$

ex. If $X \sim \text{Geo}(\frac{1}{4})$, then
Find $P(X > 8 | X > 3)$

$$p = \frac{1}{4}$$

$$\Rightarrow q = \frac{3}{4}$$

Sol. by Memoryless Property

$$P(X > 8 | X > 3) = P(X > 5) = q^5 \\ = \left(\frac{3}{4}\right)^5$$

ex.2 Find the pr. that the student

will pass the Exam after taking 5 exams

given that he was already fail in (3) exams

Sol. $A = \text{pass}$ ($P(A) = \frac{1}{3}$), $A^c = \text{fail}$ ($P(A^c) = \frac{2}{3}$)

$X \equiv$ no. of Exams until pass

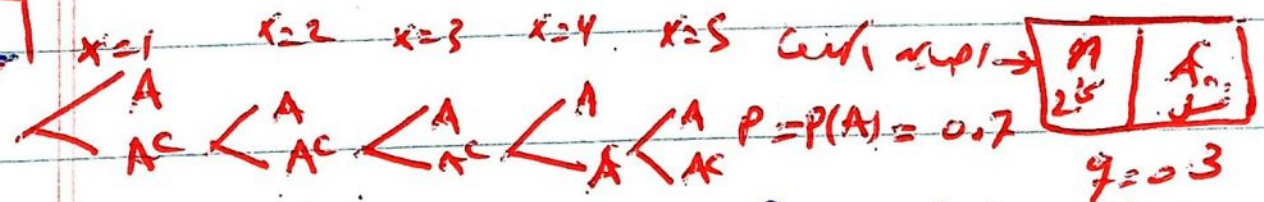
$$P(X > 5 | X > 3) = P(X > 2) = q^2 \\ = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

shoot

ex. A hunter throws at the target and repeat thrown (shooting) until the target is hit (infect). If the pr. of hitting the target is (0.7) and if X refer to the number of throws at the target until the target hitting.

- (1) Find the pr. that he needs (5) times to hit the target.
- (2) " " " " " he need at least (5) times to hit the target
- (3) " " " " " he need at most (5) times to hit the target.

Sol.



$X \sim \text{Geo}(0.7) \rightarrow f(x) = (0.7)(0.3)^{x-1}, x=1, 2, \dots$

$P(X=5) = f(5) = (0.7)(0.3)^4$

$P(X \geq 5) = P(X > 4) = q^4 = (0.3)^4$

$P(X \leq 5) = 1 - P(X > 5) = 1 - q^5 = 1 - (0.3)^5$

$$P(X > j) = \sum_{x=j+1}^{\infty} p q^{x-1} = p [q^j + q^{j+1} + \dots]$$

$$= p q^j [1 + q + q^2 + \dots]$$

$$= p \cdot q^j \frac{1}{(1-q)} = q^j$$

④ Poisson Distribution

Given an interval of (time, ^{width} ~~length~~, space, distance, height, ...) ~~Length~~ Length w, w_0

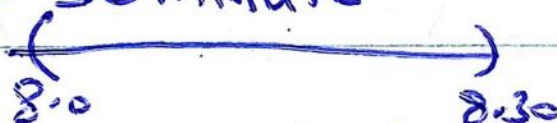


Let A be an event happens in interval w

~~Let~~ $X \equiv$ No. of ~~times~~ times that (A) happens in w

ex.1

$w \equiv$ time interval, $w = [8.0 \text{ Am} - 8.30 \text{ Am}]$
 $w = 30 \text{ minute}$



تو زنی

Let $A \equiv$ car accident happens in $w = 30 \text{ minute}$

$X \equiv$ No. of times that A happens in w

$R_X = \{0, 1, 2, \dots\}$

ex.2

$w \equiv$ distance from A to K

$w \equiv 2 \text{ km}$ $A \left(\begin{array}{c} w = 2 - \text{km} \\ w \end{array} \right) K$

A : police cars in $w = 2 \text{ km}$

$X \equiv$ No. of times that A happens in w

ex.3

$w =$ length of 5 meters

$X \equiv$ No. of A happens in $w \rightarrow R_X = \{0, 1, \dots\}$

$A \equiv$ Jumper players pass 5 meters

W = interval of space

2

ex3 A: typing error in one page $w=1$

$X \equiv$ number of typing error in w (one page)

$$R_x = \{0, 1, \dots\}$$

Know, if $w=1$, $A \equiv$ Event happens in w
if $\lambda \equiv$ No. of A happens when $w=1$ ($\lambda > 0$)
and $X \equiv$ No. of A happens in (λw) [$\lambda > 0, w > 0$
 $\lambda w > 0$]
then

$$\Rightarrow f(X; \lambda w) = \begin{cases} \frac{e^{-\lambda w} (\lambda w)^x}{x!} & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

p.m.f

$$X \sim \text{po}(\lambda w)$$

If we assume $m = \lambda w \rightarrow m > 0$

the p.m.f of X will be

$$f(X; m) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$X \sim \text{po}(m)$$

$m > 0$ (مقدار التوزيع)
ممكن تسميته $\alpha, \beta, \gamma, \dots$

To show that

$f(x; m)$ is p.m.f

Cond. ① $f(x) \geq 0$
Cond. ② $\sum_{x=0}^{\infty} f(x) = 1$

Cond. ① T.P. $f(x, m) > 0$, $x=0, 1, \dots$

$$\because m > 0 \Rightarrow e^{-m} > 0, m^x > 0, x! > 0$$

$$\therefore f(x, m) = \frac{e^{-m} m^x}{x!} > 0 \text{ for } x=0, 1, \dots$$

= 0 o.w

Cond. ② T.P. $\sum_{x=0}^{\infty} f(x, m) \stackrel{?}{=} 1$

$$\sum_{x=0}^{\infty} f(x, m) = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} = e^{-m} \left(\sum_{x=0}^{\infty} \frac{m^x}{x!} \right) e^m$$

$$= e^{-m} e^m = 1$$

$\therefore f(x, m)$ is p.m.f. [Poisson dist.]

Th. 6 If $X \sim \text{po}(m)$, then the m.g.f. of X is

$$M_X(t) = e^{m(e^t - 1)} \text{ for } -\infty < t < \infty$$

Proof

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-m} m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!}$$

$$= e^{-m} e^{me^t} = e^{m(e^t - 1)}$$

Th. 7 If $X \sim \text{po}(m)$, then $E(X) = V(X) = m$

Proof

$$M_X'(t) = e^{m(e^t - 1)} \cdot me^t \Rightarrow E(X) = M_X'(0) = e^0 \cdot m(1) = m$$

$$M_X''(t) = e^{m(e^t - 1)} \cdot me^t + me^t \cdot me^t \cdot e^{m(e^t - 1)} = e^{m(e^t - 1)} (m + m^2 e^t)$$

$$E(X^2) = M_X''(0) = e^0 (m + m^2 e^0) = m + m^2$$

$$\Rightarrow \text{Var}(X) = E(X^2) - [E(X)]^2 = m + m^2 - m^2 = m$$

ex If $X \sim \text{po}(2)$, then write $f(x)$, $E(X)$, $V(X)$ and compute $P(X \geq 1)$

Sol. $m=2 \rightarrow f(x) = \begin{cases} \frac{e^{-2} 2^x}{x!} & \text{for } x=0,1,\dots \\ 0 & \text{otherwise} \end{cases}$

$E(X) = V(X) = m = 2$

$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X=0) = 1 - f(0)$

$= 1 - \frac{e^{-2} 2^0}{0!} = 1 - e^{-2} = 1 - 0.13 = 0.87$

ex. The average number of customers entering a store is (4) per minute. Assuming the number of customers has poisson dist. Find the pr. that at least two customers entering the store in (1) minute (2) half minute

Sol. let $X = \text{No. of customers entering the store}$

(1) $E(X) = m = 4$ in $w=1 \rightarrow f(x) = \frac{e^{-4} 4^x}{x!}$, $\lambda = 4$
 $\rightarrow m = \lambda \cdot w = 4 \rightarrow 4 = \lambda \cdot 1 \rightarrow \lambda = 4$
 $\therefore P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)] = \dots$

(2) $E(X) = m = 4$ in $w=1 \rightarrow m = \lambda \cdot w \rightarrow 4 = \lambda \cdot 1$
 $\lambda = 4$
 know if $w = \frac{1}{2} \rightarrow m = ? \rightarrow P(X \geq 2) = ?$

$\therefore m = \lambda \cdot w \Rightarrow m = (4) \cdot (\frac{1}{2}) = 2$

$\therefore X \sim \text{po}(2) \rightarrow f(x) = \frac{e^{-2} 2^x}{x!}$ for $x=0,1,\dots$

$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)] = \dots$

ex. The average number of phone calls is (5) per minute ($w=1$). If the phone calls follows poisson dist. then find the pr. that ① 3 calls in minute ($w=1$)

② 3 calls in 15 seconds ($w = \frac{15}{60} = \frac{1}{4}$ min.)

Sol. $m=5 \rightarrow w=1$

① $X \equiv$ No. of phone calls

$$m = \lambda w = 5 \rightarrow f(x) = \frac{e^{-5} 5^x}{x!} \quad x=0,1,\dots$$

$$P(X=3) = f(3) = \frac{e^{-5} 5^3}{3!}$$

② When $w = 15 \text{ Sec} \rightarrow w = \frac{15}{60} = \frac{1}{4} \text{ minute}$

$$m = \lambda \cdot w \Rightarrow 5 = \lambda \cdot \frac{1}{4} \rightarrow \boxed{\lambda = 5}$$

$$m = \lambda \cdot w = 5 \cdot \frac{1}{4} = 1.25$$

$$X \sim P(1.25) \rightarrow f(x) = \frac{e^{-1.25} (1.25)^x}{x!} \quad x=0,1,2,\dots$$

$$P(X=3) = f(3) = \frac{e^{-1.25} (1.25)^3}{3!} = \dots$$

ex The average number of defects on tape (6) is $\overset{m}{(4)}$ per $\overset{w}{(1000)}$ feet. Assuming that the number of defects has poisson dist.,

① Find the pr. that a tape of long (2400) feet has 2-defects ($X=2$)

② Find the pr. that a tape of Long (1200) feet has no-defect ($X=0$)

Sol. Let $X \equiv$ No. of defects on a tape
 $E(X) = m = 4$ per $w = 1000$ feet
 $m = \lambda w \rightarrow 4 = \lambda \cdot 1000 \rightarrow \lambda = \frac{4}{1000} = \frac{1}{250} = 0.004$

① if $w = 2400$ Find $P(X=2)$

$$m^* = \lambda \cdot w = \underbrace{(0.004)}_{\frac{4}{1000}} (2400) = 4(24) = 9.6$$

$$f(x; m) = \begin{cases} \frac{e^{-9.6} (9.6)^x}{x!} & x=0, 1, \dots \end{cases}$$

$$\therefore P(X=2) = f(2) = \frac{e^{-9.6} (9.6)^2}{2!}$$

② $w = 1200 \rightarrow$ Find $P(X=0)$

$$\lambda = \frac{4}{1000} \rightarrow m = \lambda \cdot w \Rightarrow m = \frac{4}{1000} \cdot 1200 = 4.8$$

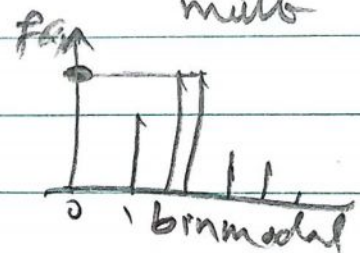
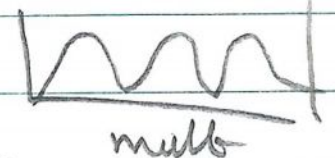
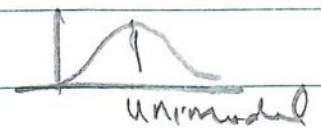
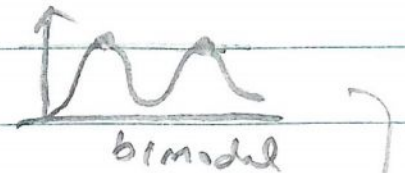
$$f(x) = \begin{cases} \frac{e^{-4.8} (4.8)^x}{x!} & x=0, \dots \end{cases}$$

$$P(X=0) = f(0) = \frac{e^{-4.8} (4.8)^0}{0!} = e^{-4.8}$$

ex. If $X \sim \text{po}(2)$ → then $f(x) = \frac{e^{-2} 2^x}{x!}$ Find Mode

المسألة هي إيجاد x في التوزيع
الذي له أكبر احتمال
عند $x=0$
يمكن أن يوجد أكثر من مثال

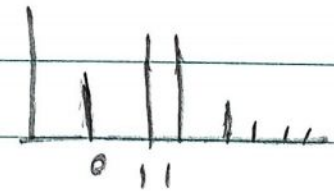
المسألة هي إيجاد x في التوزيع
الذي له أكبر احتمال
عند $x=0$
يمكن أن يوجد أكثر من مثال



$x=0 \rightarrow f(0) = 1 \cdot e^{-2}$
 $x=1 \rightarrow f(1) = 2e^{-2}$
 $x=2 \rightarrow f(2) = 2e^{-2}$
 $x=3 \rightarrow f(3) = \frac{4}{3}e^{-2}$
 $x=4 \rightarrow f(4) = \frac{2}{3}e^{-2}$
 ↓ decrease

$x=1, 2$ are the mode

ex. If $X \sim \text{po}(2.1)$



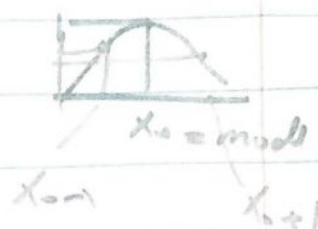
$f(0) = e^{-2.1}$
 $f(1) = (2.1) e^{-2.1}$
 $f(2) = (2.1 \cdot 2.05) e^{-2.1}$
 $f(3) = (1.54) e^{-2.1}$
 $f(4) = (0.81) e^{-2.1}$
 ↓ decrease

$\text{Mode} = [m] = [2.1] = 2$

المسألة هي إيجاد x في التوزيع الذي له أكبر احتمال

$$X \sim \text{Po}(m) = \frac{e^{-m} m^x}{x!}$$

proof let x_0 be the mode
 $\Rightarrow f(x_0) \geq f(x_0 - 1)$ ①



$$\frac{f(x_0)}{f(x_{0-1})} \geq 1$$

$$\frac{\frac{e^{-m} m^{x_0}}{x_0!}}{\frac{e^{-m} m^{x_0-1}}{(x_0-1)!}} = \frac{m^{x_0}}{m^{x_0-1}} \cdot \frac{(x_0-1)!}{x_0!} = \frac{m}{x_0} \geq 1$$

$$\frac{m}{x_0} \geq 1 \rightarrow \boxed{x_0 \leq m}$$

$\Rightarrow f(x_0) \geq f(x_{0+1})$

$$\frac{f(x_0)}{f(x_{0+1})} \geq 1$$

$$\frac{\frac{e^{-m} m^{x_0}}{x_0!}}{\frac{e^{-m} m^{x_0+1}}{(x_0+1)!}} = \frac{m^{x_0}}{m^{x_0+1}} \cdot \frac{(x_0+1)!}{x_0!} = \frac{x_0+1}{m} \geq 1$$

$$(x_0+1)(x_0)! / x_0! = x_0+1$$

$$\frac{x_0+1}{m} \geq 1$$

$$x_0+1 \geq m$$

$$\boxed{x_0 \geq m-1}$$

$$m-1 \leq x_0 \leq m$$

$$\Rightarrow x_0 = m, m-1$$

① $X \sim \text{Po}(2)$

$$1 \leq x_0 \leq 2$$

② $X \sim \text{Po}(4.2)$

$$3.2 \leq x_0 \leq 4.2$$

$$x_0 = [4.2]$$

$$\boxed{x_0 = 4}$$

$$f(x|m) = \frac{e^{-m} m^x}{x!}$$

$$F_x(x) = P(X \leq x) = \sum_{x=0}^x \frac{e^{-m} m^x}{x!} = \frac{1}{2} \quad \text{for } x = \text{median}$$

ex. $m=1$ $f(x) = \frac{e^{-1} 1^x}{x!}$

$$F(x) = \sum_{x=0}^{\infty} \frac{e^{-1} 1^x}{x!} = 1$$

1.372

Else m
mode = $m-1, m$

$$E(X) = \frac{3 \text{Med} - m_0}{2}$$

$$2E(X) = 3 \text{Med} - m_0$$

$$\text{Med} = \frac{1}{3}(2E(X) + m_0)$$

either $= \frac{1}{3}(2m + m) = \frac{3m}{3} = m$ (1)

or $= \frac{1}{3}(2m + (m-1)) = \frac{3m-1}{3} = m - \frac{1}{3}$ (2)

$$m - \frac{1}{3} \leq \text{median} \leq m$$

If $m=4$
 $4 - \frac{1}{3} \leq \text{median} \leq 4$

$4 - \frac{1}{3}$

$3 \leq \text{median} \leq 4 \rightarrow \text{Med} = 4$

$m \neq 4$

$$m = 4 \cdot \frac{13}{3} = 4.33$$

$$4.33 - \frac{1}{3} \leq \text{median} \leq 4.33$$

$$\frac{12}{3} \leq \text{median} \leq \frac{13}{3} = 4.33$$

$4 \leq \text{median} \leq 4.33$
 $\rightarrow \frac{12}{3} = 4$

II Continuous distribution

(i)

Gamma distribution

Def Gamma function ("Γ")

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy \quad ; \quad \boxed{\alpha > 0}$$

Use integration by parts

$$u = y^{\alpha-1} \quad \& \quad dv = e^{-y} dy$$

$$du = (\alpha-1) y^{\alpha-2} \quad , \quad v = -e^{-y}$$

$$\int_0^{\infty} \Gamma(\alpha) = - \underbrace{y^{\alpha-1} e^{-y}}_{\text{Zero}} \Big|_0^{\infty} + \int_0^{\infty} e^{-y} (\alpha-1) y^{\alpha-2} dy$$

$$= (\alpha-1) \int_0^{\infty} e^{-y} y^{\alpha-2} dy$$

$$= (\alpha-1) \underline{\Gamma(\alpha-1)}$$

$$\Gamma(\alpha-1) = \int_0^{\infty} e^{-y} y^{\alpha-2} dy$$

نعم صارت المتكامل بالجزء الثاني من المتكامل

$$= (\alpha-2) \underline{\Gamma(\alpha-2)}$$

وهكذا نقاد العملية مرة أخرى

$$\Gamma(\alpha-2) = \int_0^{\infty} e^{-y} y^{\alpha-3} dy = (\alpha-3) \Gamma(\alpha-3) \dots$$

حتى نستنتج أن

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2) \dots 3 \cdot 2 \cdot 1 = \underline{(\alpha-1)!}$$

ex. $\Gamma(1) = 0! = 1$ $\Gamma(3) = 2! = 2$
 $\Gamma(2) = 1! = 1$ $\Gamma(4) = 3! = 6$

Know, $\Gamma(y) = \int_0^{\infty} y^{y-1} e^{-y} dy$

let $y = \frac{x}{\beta} \Rightarrow \beta > 0 \Rightarrow dy = \frac{1}{\beta} dx$

$$\begin{aligned} 0 < y < \infty \\ 0 < \frac{x}{\beta} < \infty \\ 0 < x < \infty \end{aligned}$$

$$\therefore \Gamma(y) = \int_0^{\infty} \left(\frac{x}{\beta}\right)^{y-1} e^{-\frac{x}{\beta}} \cdot \frac{1}{\beta} dx$$

$$\Gamma(y) = \int_0^{\infty} \frac{x^{y-1} e^{-\frac{x}{\beta}}}{\beta^y} dx \quad \div, \Gamma(y)$$

$$1 = \int_0^{\infty} \frac{x^{y-1} e^{-\frac{x}{\beta}}}{\Gamma(y) \beta^y} dx$$

دالة تكاملية
يأخذها

$$\therefore f(x; \alpha, \beta) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^{\alpha}}, & 0 < x < \infty, \alpha, \beta > 0 \\ 0, & \text{elsewhere} \end{cases}$$

cond. (i) holds $x > 0, \alpha > 0 \rightarrow x^{\alpha-1} > 0$
 $e^{-x/\beta} > 0, \Gamma(\alpha) > 0, \beta^{\alpha} > 0$

cond. (ii) holds (above) *

(مشتق متوالي)
 $X \sim G(\alpha, \beta)$

مطابقة التوزيع ($\beta, \alpha > 0$)

$\alpha \equiv$ shape parameter شكل

$\beta \equiv$ scale parameter مقياس

③

Th. 8 If $X \sim G(\alpha, \beta)$, then the m.g.f. of X

$$M_X(t) = (1 - \beta t)^{-\alpha}$$

proof

$$\begin{aligned} M_X(t) &= E[e^{tx}] \\ &= \int_0^{\infty} e^{tx} \cdot \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^{\alpha}} dx \\ &= \int_0^{\infty} \frac{x^{\alpha-1} e^{-x(\frac{1}{\beta} - t)}}{\Gamma(\alpha) \beta^{\alpha}} dx \\ &= \int_0^{\infty} \frac{x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})}}{\Gamma(\alpha) \beta^{\alpha}} dx \end{aligned}$$

let $y = x(\frac{1-\beta t}{\beta}) \rightarrow x = \frac{\beta}{1-\beta t} y$
 $dx = \frac{\beta}{1-\beta t} dy$
 $0 < y < \infty$

$$M_X(t) = \int_0^{\infty} \frac{\left(\frac{\beta y}{1-\beta t}\right)^{\alpha-1} \cdot e^{-y}}{\Gamma(\alpha) \beta^{\alpha}} \left(\frac{\beta}{1-\beta t}\right) dy$$

$$\beta^{-\alpha} \cdot \left(\frac{\beta}{1-\beta t}\right)^{\alpha} \left\{ \int_0^{\infty} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy \right\} = 1 \quad \text{[by } \int_0^{\infty} y^{\alpha-1} e^{-y} dy = \Gamma(\alpha) \text{]}$$

$$\begin{aligned} \beta^{-\alpha} \cdot \left(\frac{\beta}{1-\beta t}\right)^{\alpha} &= \left(\frac{1}{1-\beta t}\right)^{\alpha} = (1-\beta t)^{-\alpha} \\ &= (1-\beta t)^{-\alpha} \end{aligned}$$

Th. 2 If $X \sim G(\alpha, \beta)$, then $E(X) = \alpha\beta$
and $Var(X) = \alpha\beta^2$

proof $M(t) = (1 - \beta t)^{-\alpha}$

$$M'_x(t) = (-\alpha)(1 - \beta t)^{-\alpha-1}(-\beta) = \alpha\beta(1 - \beta t)^{-(\alpha+1)}$$

$$E(X) = M'_x(0) = \alpha\beta \frac{(1-0)^{-(\alpha+1)}}{1} = \alpha\beta$$

$$M''_x(t) = -(\alpha+1)\alpha\beta(1 - \beta t)^{-(\alpha+2)}(-\beta) = (\alpha+1)\alpha\beta^2(1 - \beta t)^{-(\alpha+2)}$$

$$E(X^2) = (\alpha+1)\alpha\beta^2(1-0)^{-(\alpha+2)} = \alpha^2\beta^2 + \alpha\beta^2$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= \cancel{\alpha^2\beta^2} + \alpha\beta^2 - [\cancel{\alpha\beta}]^2 = \alpha\beta^2$$

ex. 1 If $X \sim G(\overset{\alpha}{3}, \overset{\beta}{4})$, then find $f(x)$, $M(t)$, $E(X)$, $Var(X)$

Sol. $f(x) = \begin{cases} \frac{x^2 e^{-x/4}}{\Gamma(3) 4^3} & 0 < x < \infty \\ 0 & o.w \end{cases}$ مطلوب

$$M(t) = (1 - 4t)^{-3}, E(X) = 3 \cdot 4 = 12, Var(X) = 3 \cdot 4^2 =$$

ex. 2 If $M(t) = (1 - 3t)^{-2}$, write $f(x)$, $E(X)$ and $Var(X)$

Sol. $X \sim G(2, 3)$

$$\therefore f(x) = \begin{cases} \frac{x^1 e^{-x/3}}{\Gamma(2) 3^2} & 0 < x < \infty \\ 0 & o.w \end{cases} \quad E(X) = 2 \cdot 3 = 6$$

$$Var(X) = 2 \cdot 3^2 = 18$$

(H.W.)

ex. 3 If $X \sim G(2, 2)$, then write $f(x)$, $M(t)$, $E(X)$ and $Var(X)$

مطلوب

2 - Chi-Square distribution $[X^2(r)]$

توزيع مربع کای

it's a special case of Gamma dist. when

① $\alpha = \frac{r}{2}$, $r \in \mathbb{I}^+$ ($r = 2\alpha$)

② $\beta = 2$

denoted by $X^2(r)$



$r \equiv$ refer to degree of freedom (d.f.)

$$f(x; \alpha, \beta) = \begin{cases} \frac{\alpha^{\alpha} e^{-x/\beta}}{\Gamma(\alpha) \beta^{\alpha}} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

So $X \sim X^2(r)$, then the p.d.f. of X will be

$$f(x; r) = \begin{cases} \frac{x^{r/2-1} e^{-x/2}}{\Gamma(r/2) 2^{r/2}} & 0 < x < \infty, r > 0 \\ 0 & \text{o.w.} \end{cases}$$

$\alpha = r/2$
 $\beta = 2$

also, $M_X(t) = (1 - 2t)^{-r/2}$

$E(X) = \alpha\beta = \left(\frac{r}{2}\right)(2) = r$

$V(X) = \alpha^2\beta = \left(\frac{r}{2}\right)(4) = 2r$

Note If $X \sim \Gamma(\alpha, \beta)$, then $X \sim X^2(r)$

$\alpha = \frac{r}{2}$
 $\beta = 2$

$$f(x, \alpha) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^{\alpha}} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

ex.

$X \sim \Gamma(2, 2)$

$\equiv X^2(4)$

$M_X(t) = (1 - 2t)^{-2}$ $\frac{4}{2} = \frac{r}{2}$

$M_X(t) = (1 - 2t)^{-4/2}$

$$r=2\alpha$$

$$r=4$$

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^\alpha} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$X \sim G(2, 2)$$

$$X \sim \chi^2(4)$$

$$E(X) = \alpha\beta = r = 4 \quad \rightarrow \Gamma(2) = 1$$

$$V(X) = \alpha\beta^2 = 2r = 8$$

ex.1 If X has a m.g.f

$$M_X(t) = (1 - 2t)^{-7} \quad \text{for } t < \frac{1}{2}$$

then write $f(x)$, $E(X)$ and $\text{Var}(X)$

Sol. $M_X(t) = (1 - \beta t)^{-\alpha} \rightarrow$ is gamma

$$M_X(t) = (1 - 2t)^{-r/2} \rightarrow \text{for } \chi^2(r)$$

$$\beta = 2, \alpha = 7$$

$$\beta = 2, \frac{r}{2} = 7 \rightarrow \boxed{r = 14}$$

$$X \sim \chi^2(14)$$

$$\text{or } X \sim G(7, 2)$$

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^\alpha} \quad 0 < x < \infty$$

$$E(X) = \alpha\beta = r = 7 \cdot 2 = 14$$

$$V(X) = \alpha\beta^2 = 2r = 7 \cdot 4 = 28$$

Table of chi-square dist. 3

$$X \sim \chi^2_{(r)} \quad P(X \leq x) = \int_0^x \frac{x^{r/2-1} e^{-x/2}}{\Gamma(r/2) 2^{r/2}} dx = F(x)$$

d.f.	0.01	0.025	0.05	0.1	0.95	0.975
1	—	0.001	0.005	3.84	5.02	—
2	—	0.051	0.103	—	—	—
3	0.115	—	0.352	7.81	9.35	—
4	0.554	0.484	0.711	0.49	11.1	—
5	—	—	1.64	12.16	—	—
10	2.56	3.25	3.94	18.3	20.5	—

Ex. $X \sim \chi^2_{(3)}$, Find $P(X \leq 0.352)$

Sol. from table $P(X \leq 0.352) = 0.05$

Remark $P(a \leq X \leq b) = P(X \leq b) - P(X \leq a)$

ex. If $X \sim \chi^2_{(10)}$, then Find $P(3.25 \leq X \leq 20.5)$ using table (r=10)

Sol $P(3.25 \leq X \leq 20.5) = P(X \leq 20.5) - P(X \leq 3.25)$

$$= 0.975 - 0.025$$

$$= 0.950$$



ex 3. If $X \sim \chi^2(4)$ using table of χ^2_{crit} (4)

Find (1) $P(X \leq 11.1)$ (2) $P(0.484 \leq X \leq 0.49)$
 (3) $P(X > 0.711)$ (4) $P(X=2)$

Sol. (1) $P(X \leq 11.1) = 0.975$



(2) $P(0.484 \leq X \leq 0.49) = P(X \leq 0.49) - P(X \leq 0.484)$
 $= 0.950 - 0.025 = 0.925$



(3) $P(X > 0.711) = 1 - P(X \leq 0.711)$

$= 1 - 0.05 = 0.95$



(4) $P(X=2) = 0$?? Is it (c.v.)

ex. If $X \sim \chi^2(4) = (1-2t)^{-5}$ for $t < \frac{1}{2}$

Find (1) $P(X \leq 3.94)$, (2) $P(X > 18.3)$ using table χ^2_{crit}
 (3) Find $E(X)$ and $V(X)$

(4) write $f(x)$

Sol. $X \sim G(5, 2) \equiv X \sim \chi^2(10)$ $r=10$
 $\alpha = \frac{r}{2} = 5$
 $\beta = 2\alpha = 10$
 so by table of $\chi^2_{(10)}$, we find

(1) $P(X \leq 3.94) = 0.05$ (2) $P(X > 18.3) = 1 - P(X \leq 18.3)$

(3) $E(X) = \alpha\beta = r = 10$ $= 1 - 0.95 = 0.05$

$V(X) = \alpha\beta^2 = 2r = 20$

(4) $f(x) = \frac{1}{\Gamma(5)} \frac{e^{-x/2}}{x^{5/2}}$ $\rightarrow \beta$

$24 = 4H$

$= \frac{1}{768} \frac{e^{-x/2}}{x^{5/2}}$

$0 < X < \infty$

$0 < X < \infty$

Lemma If $X \sim G(\alpha, \beta)$; $\alpha = \frac{r}{2}$ & $\beta \neq 2$

then the r.v. $Y = \frac{2X}{\beta} \sim \chi^2(r)$ or $\chi^2(2\alpha)$

proof $\because X \sim G(\alpha, \beta)$ & $\alpha = \frac{r}{2}, \beta \neq 2$

$$\Rightarrow M_X(t) = (1 - \beta t)^{-\alpha} = (1 - \beta t)^{-r/2}$$

$$Y = \underbrace{\left(\frac{2}{\beta}\right)X}_{(a)} + \underbrace{0}_{(b)}$$

$$M_Y(t) = e^{bt} \cdot M_X(at)$$

$$\therefore M_Y(t) = e^{0t} \cdot M_X\left(\frac{2}{\beta}t\right)$$

$$= \left[1 - \beta\left(\frac{2}{\beta}t\right)\right]^{-r/2} = (1 - 2t)^{-r/2}$$

$$r = 2\alpha$$

$$\therefore Y \sim \chi^2(r) \text{ or } Y \sim \chi^2(2\alpha)$$

ex. $X \sim G(3, 4)$; $\alpha = 3, \beta \neq 2$

then find $P(3.28 \leq X \leq 25.2)$

χ^2 distribution
interval
find

از صحت این نتیجه یادداشت کنید $\beta \neq 2$

$$\Rightarrow P(3.28 \leq X \leq 25.2)$$

$$= P(1.64 \leq \underbrace{\left(\frac{2X}{4}\right)}_Y \leq 12.6)$$

$$Y \sim \chi^2(6)$$

$2\alpha \rightarrow r$

$$= P(Y \leq 12.6) - P(Y \leq 1.64)$$

$$= 0.95 - 0.05$$

$$= 0.90$$

(6)

ex. If X has a p.d.f as below

$$f(x) = \begin{cases} \frac{x^2 e^{-x/2}}{2^3} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

$2 = 2^3$ $\frac{16}{2^3} = 8$ o.w.

Find ① $P(X \leq 0.872)$ ② $P(1.24 \leq X \leq 16.8)$

Sol. $\because X \sim G(3, 2) \equiv X \sim \chi^2(6)$

$r = 2\alpha = 6$ $\leftarrow \alpha \uparrow \beta$

& by table of $\chi^2(6)$ we find

$$\begin{aligned} \text{① } P(X \leq 0.872) &= \\ \text{② } P(1.24 \leq X \leq 16.8) &= \end{aligned} \quad \left. \begin{array}{l} \text{تقریبی} \\ \text{داده شده} \end{array} \right\}$$

ex. If $M_X(t) = (1 - 2t)^{-2} = (1 - 2t)^{-\frac{4}{2}}$

then Find $P(X > 9.35)$ using χ^2 table

Sol. $X \sim \chi^2(4)$ or $X \sim G(2, 2)$

so by table of $\chi^2(4)$

$$P(X > 9.35) = 1 - P(X \leq 9.35)$$

$$= 1 - 0.975$$

$$= 0.025$$

(7)

Lemma If $x_i \sim \chi^2_{(1)}$ are iid r.v.s
 then show that $y = \sum_{i=1}^n x_i \sim \chi^2_{(n)}$

proof [Using Mgf]

$$\left[\begin{array}{l} x_1 \sim \chi^2_{(1)} \rightarrow M_{x_1}(t) = (1-2t)^{-1/2} = E[e^{x_1 t}] \\ x_2 \sim \chi^2_{(1)} \rightarrow M_{x_2}(t) = (1-2t)^{-1/2} = E[e^{x_2 t}] \\ \vdots \\ x_n \sim \chi^2_{(1)} \rightarrow M_{x_n}(t) = (1-2t)^{-1/2} = E[e^{x_n t}] \end{array} \right]$$

$$\begin{aligned} \& M_y(t) &= E[e^{yt}] = E[e^{(x_1 + x_2 + \dots + x_n)t}] \\ &= E[e^{x_1 t + x_2 t + \dots + x_n t}] \\ &= E[e^{x_1 t} \cdot e^{x_2 t} \cdot \dots \cdot e^{x_n t}] \quad \text{Ind. قانون} \\ &= E[e^{x_1 t}] \cdot E[e^{x_2 t}] \cdot \dots \cdot E[e^{x_n t}] \quad \text{مستقل} \\ &= M_{x_1}(t) \cdot M_{x_2}(t) \cdot \dots \cdot M_{x_n}(t) \\ &= (1-2t)^{-1/2} \cdot (1-2t)^{-1/2} \cdot \dots \cdot (1-2t)^{-1/2} \\ &= (1-2t)^{-n/2} \rightarrow \boxed{(1-2t)^{-n/2}} \\ &\therefore y \sim \chi^2_{(n)} \quad \text{ناتج} \end{aligned}$$

8

Note $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

بعد البصحة على لتوزيع الطبيعي التي لم نتطرق اليه كحد الآن

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

$0 < x < \infty$
 $0 < y < \infty$

let $y = \frac{x^2}{2}$

$x^2 = 2y$

$x = \sqrt{2y}$

$dx = \sqrt{2} \cdot \frac{1}{2\sqrt{y}} dy$

$dx = \frac{1}{\sqrt{2}} y^{-1/2} dy$

(2) $\int_0^{\infty} e^{-y} \cdot \frac{1}{\sqrt{2}} y^{-1/2} dy = \sqrt{2\pi}$

$\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}} \int_0^{\infty} y^{\frac{1}{2}-1} e^{-y} dy = \sqrt{2\pi}$

$\sqrt{2} \int_0^{\infty} y^{\frac{1}{2}-1} e^{-y} dy = \sqrt{2} \sqrt{\pi}$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

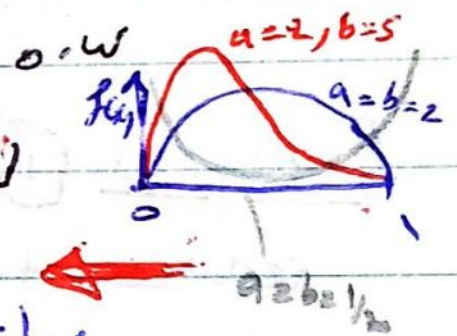
$\Gamma(x) = \int_0^{\infty} y^{x-1} e^{-y} dy$

④ Beta dist.

A c.r.v. X is said to have (follow) a "Beta dist." if its p.d.f. has the following form

$$f(X; a, b) = \begin{cases} \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

where $a > 0$ and $b > 0$



denoted by $X \sim B(a, b)$

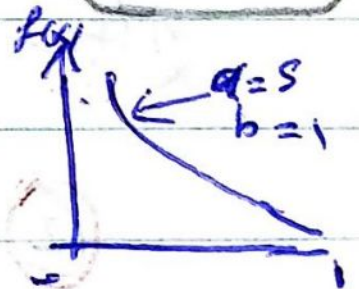
$$\text{where } B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$$

$$\text{And } B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

تحت تابع موزون
توزيع مع طمان

Note ① Beta Function is symm

$$\text{i.e. } B(a, b) = B(b, a)$$



$$\textcircled{2} \frac{B(a, b+1)}{b} = \frac{B(a+1, b)}{a}$$

$$\textcircled{3} B(a, b) = B(a+b, b) * B(a, b+1)$$

ex. $B(3, 4) = \int_0^1 x^{2} (1-x)^{3} dx$

$$= \frac{\Gamma(3) \Gamma(4)}{\Gamma(7)} = \frac{2! \times 3!}{6!} = \frac{1}{60}$$

2

Th. If $X \sim B(a, b)$, then the m.g.f. of X is

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{B(a, b)} B(a+n, b)$$

(برهان)

Th. If $X \sim B(a, b)$, then

$$E(X) = \frac{a}{a+b} \text{ \& \; } Var(X) = \frac{ab}{(a+b)^2 (a+b+1)}$$

Remarks (1) If $a=b=1$, then the Beta dist. reduces to uniform dist. over $(0, 1)$

i.e. $f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

$$E(X) = \frac{1}{2}, \text{ Var}(X) = \frac{1}{12} \text{ and}$$

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{B(1, 1)} \cdot B(1+n, 1) = 1$$

$$\begin{aligned} B(1, 1) &= \frac{P(1)P(1)}{P(2)} \\ &= \frac{1}{1} \\ B(1, n, b) &= \frac{P(n+1)}{P(n+2)} \\ &= \frac{n!}{(n+1)n!} \\ &= \frac{1}{n+1} \end{aligned}$$

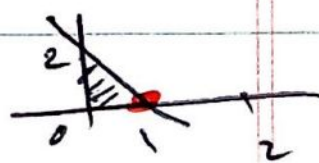
(2) If $a=1$, $b=2$, then Beta dist. known as triangular dist.

$$\begin{aligned} B(1, 2) &= \frac{P(1)P(2)}{P(3)} \\ &= \frac{1}{2} \end{aligned}$$

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} B(1, 2) &= \frac{1}{2} \\ \frac{1}{B(1, 2)} &= 2 \end{aligned}$$

$$E(X) = \frac{1}{3}, \text{ Var}(X) = \frac{1}{18}, \text{ } M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{B(1+n, 2)}{B(1, 2)}$$



(3)

ex. Suppose a r.v. Y has the following p.d.f

$$f(y) = \begin{cases} c y^3 (1-y)^2 & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value of c ② Find $F(y)$

Sol.

" $f(y)$ is p.d.f

$$\therefore c \int_0^1 y^{3-1} (1-y)^{2-1} dy = 1$$

$$\therefore y \sim B(4, 3)$$

$$f(y) = \frac{1}{B(a, b)} y^{a-1} (1-y)^{b-1}$$

$$\therefore c = \frac{1}{B(a, b)} = \frac{1}{B(4, 3)}$$

$$c \cdot B(4, 3) = 1$$

$$c = \frac{1}{B(4, 3)}$$

$$c = \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} = \frac{3! \times 4! \times 3!}{3! \times 2!} = 60$$

$$F(y) = \int_0^y 60 t^3 (1-t)^2 dt$$

$$= 60 \int_0^y [t^3 - 2t^4 + t^5] dt$$

$$= 60 \left[\frac{t^4}{4} - \frac{2t^5}{5} + \frac{t^6}{6} \right]_0^y$$

$$= 60 \left[\frac{y^4}{4} - \frac{2y^5}{5} + \frac{y^6}{6} \right]$$

$$0 \leq y < 1$$

$$60 \left[\frac{30}{60} - \frac{2(24)}{60} + \frac{20}{60} \right] = 1$$

$$y \geq 1$$

$$\frac{1}{4} - \frac{2}{5} + \frac{1}{6} = \frac{15}{60} - \frac{24}{60} + \frac{10}{60} = \frac{1}{60}$$

(4)

E(X)H.W. If $X \sim B(1, 1)$, then find E(Y)

$$\text{where } Y = X - (X - \frac{1}{2})^2$$

$$\text{Sol. } E(Y) = E\left[X - (X - \frac{1}{2})^2\right]$$

$$= E(X) - E(X - \frac{1}{2})^2$$

$$\textcircled{1} \because X \sim B(1, 1) \rightarrow X \sim U(0, 1) \xrightarrow{V(X)} E(X) = \frac{1}{2}$$

$$X \sim \text{unif}(0, 1) \quad V(X) = \frac{1}{12}$$

$$\textcircled{2} E(X) = \frac{1}{2}$$

$$\therefore V(X) = E\left[(X - E(X))^2\right] = E\left[\left(X - \frac{1}{2}\right)^2\right] = \frac{1}{12}$$

$$\therefore E(Y) = E(X) - V(X)$$

$$= \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$$

$$\textcircled{\text{or}} \quad E(X) = \frac{a}{a+b} = \frac{1}{1+1} = \frac{1}{2}$$

$$V(X) = \frac{a \cdot b}{(a+b)^2 (a+b+1)} = \frac{1 \cdot 1}{(1+1)^2 (1+1+1)}$$

$$= \frac{1}{4 \cdot 3} = \frac{1}{12}$$

5
Beta

Example ^{المراقبين} the percentage of ^{المتسابقين} betterers who ^{في سباق} make money of the race on a given day

has Beta dist with parameters $a=1, b=5$.

Find the pr. that fewer than ^{أقل} ten percent ^{١٠٪ = ٠.١} comment winners on given day

Sol. let $X \equiv$ per. of bettors of race

$$X \sim B(1, 5) \Rightarrow f(x) = \frac{1}{B(1, 5)} x^{1-1} (1-x)^{5-1} \quad 0 < x < 1$$

$$P(X < 0.1) = \int_0^{0.1} \frac{\Gamma(6)}{\Gamma(1)\Gamma(5)} x^0 (1-x)^4 dx$$

$$= \frac{5!}{4!} \int_0^{0.1} (1-x)^4 dx$$

$$= -\frac{(1-x)^5}{5} \Big|_0^{0.1} = -\frac{(1-x)^5}{5} \Big|_0^{0.1}$$

$$= -\left[(0.9)^5 - 1\right] = 1 - (0.9)^5$$

$$= 1 - 0.59049 = 0.40951 \in [0, 1]$$

⑥

example

$$① \int_0^1 \sqrt{x} (1-x)^2 dx = \int_0^1 x^{\frac{3}{2}-1} (1-x)^{3-1} dx = B(\frac{3}{2}, 3)$$

$$\text{or} = B(\frac{1}{2}+1, 2+1)$$

$$② \int_0^1 x^{-\frac{1}{3}} (1-x)^{-\frac{1}{3}} dx = B(-\frac{1}{3}+1, -\frac{1}{3}+1) \\ = B(\frac{2}{3}, \frac{2}{3})$$

$$\Gamma(\frac{2}{3}) = ??$$

$$\Gamma(\frac{n+1}{2}) = \frac{(2n-1)!! \sqrt{\pi}}{2^n}$$

$$0!! = \sqrt{\frac{2}{\pi}}$$

$$\Gamma(\frac{4}{3}) = \Gamma(1 + \frac{1}{3})$$

$$n!! = 2^{n/2} \cdot (\frac{n}{2})!$$

$$\Gamma(n + \frac{1}{3}) = \Gamma(\frac{1}{3}) \frac{(3n-2)!!!}{3^n}$$

$$\Gamma(1 + \frac{1}{3}) = \Gamma(\frac{1}{3}) \frac{(3-2)!!!}{3} = \frac{1}{3} \Gamma(\frac{1}{3})$$

$$\Gamma(\frac{1}{3}) \approx 2.679$$

$$3.6276$$

$$\prod_{k=1}^2 \Gamma(\frac{k}{3}) = \frac{2\pi}{\sqrt{3}}$$

$$\Gamma(\frac{1}{3}) \cdot \Gamma(\frac{2}{3}) = \frac{2\pi}{\sqrt{3}} \Rightarrow \Gamma(\frac{2}{3}) = \frac{\frac{2\pi}{\sqrt{3}}}{\Gamma(\frac{1}{3})} \\ = \frac{3.6276}{2.679}$$

Remark

(7)

$$\Gamma(n+1) = n \Gamma(n)$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\textcircled{1} \quad \Gamma(-\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = \underline{\underline{-2\sqrt{\pi}}}$$

$$\textcircled{2} \quad \Gamma(0) = \frac{\Gamma(1)}{0} = \frac{1}{0} = \infty$$

$$\textcircled{3} \quad \Gamma(-1) = \frac{\Gamma(0)}{-1} = -\infty$$

ex $\int_0^{\infty} x^6 e^{-2x} dx = ?$ Using Gamma

$$\int_0^{\infty} \left(\frac{y}{2}\right)^6 e^{-y} \cdot \frac{1}{2} dy$$

$$= \left(\frac{1}{2}\right)^7 \int_0^{\infty} y^6 e^{-y} dy$$

$$= \left(\frac{1}{2}\right)^7 \Gamma(7)$$

$$y = 2x \Rightarrow x = \frac{y}{2}, dx = \frac{dy}{2}$$

$$\begin{aligned} 0 < x < \infty \\ 0 < 2x < \infty \\ 0 < y < \infty \end{aligned}$$

ex $\int_0^{\infty} x^2 e^{-2x^2} dx$

$$\int_0^{\infty} \frac{y}{2} e^{-y} \frac{1}{2\sqrt{2}} y^{\frac{1}{2}} dy$$

$$\frac{1}{4\sqrt{2}} \int_0^{\infty} y^{\frac{3}{2}} e^{-y} dy = \frac{1}{4\sqrt{2}} \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{8\sqrt{2}}$$

$$\text{let } y = 2x^2 \Rightarrow x = \sqrt{\frac{y}{2}}$$

$$dx = \frac{1}{2} \frac{1}{\sqrt{y}} dy$$

$$= \frac{\sqrt{\pi}}{2}$$

8

Beta

ex. $\int_0^1 x^{\frac{1}{2}} (1-x)^3 dx = B(\frac{3}{2}, 4) = B(4, \frac{3}{2})$

$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(x\pi)}$

Theorem

ex. $\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3}) = \frac{\pi}{\sin(\frac{\pi}{3})} = \frac{2\pi}{\sqrt{3}}$

ex. $\Gamma(\frac{1}{4}) \Gamma(\frac{7}{4})$

المجموع لا يساوي 1

$= \Gamma(\frac{1}{4}) \cdot \frac{3}{4} \Gamma(\frac{3}{4}) = \frac{3}{4} \Gamma(\frac{1}{4}) \Gamma(\frac{3}{4})$

$= \frac{3}{4} \cdot \frac{\pi}{\sin(\frac{\pi}{4})} = \frac{1}{\sqrt{2}}$

ex Evaluate $\int_0^2 \frac{x^2}{\sqrt{2-x}} dx$

$= \int_0^2 x^2 (2-x)^{-1/2} dx$

let $x = 2y$, $y = \frac{x}{2}$
 $dx = 2dy$

$\int_0^2 (2y)^2 (2-2y)^{-1/2} \cdot 2 dy$

$\rightarrow 0 < x < 2$

$\rightarrow 0 < \frac{x}{2} < 1$

$0 < y < 1$

$4(2)(2)^{-1/2} \int_0^1 y^2 (1-y)^{-1/2} dy$

2.

$\Gamma(3) \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\frac{8}{\sqrt{2}} B(3, \frac{1}{2}) = \frac{8}{\sqrt{2}}$

$\Gamma(\frac{7}{2}) = \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi}$

$= \frac{16 \times 8}{15 \sqrt{2}}$

التوزيع الطبيعي

④ Normal distribution [Gauss dist. و Bell dist.]

يعتبر التوزيع الطبيعي من أهم وأشهر التوزيعات الاحتمالية

لأنه يصف معظم الظواهر الطبيعية مثل : درجة الحرارة ،

مستوى الذكاء ، الوزن ، الطول ، درجات إكلية ... الخ

ويستخدم التوزيع الطبيعي كتقريب لكثير من التوزيعات

الاحتمالية ولذلك يستخدم في مجالات عديدة من فروع

الإحصاء والاستدلال الإحصائي لكثير من البحوث الطبية

والاجتماعية وعلوم الفلك والكثير من العلوم الأخرى.

وسمى أيضاً توزيع كاوس (Gauss dist) نسبة للعلم

الرياضي (Gauss) واليه أيضاً يسمى بالتوزيع الجبرسي

(Bell dist) لأن منحنى التوزيع (الدالة) يشبه

كل الجرس المقلوب  متغير عشوائي متصل.

Now, we say a c.r.v. X follow

Normal dist. with two parameters

μ [mean] and σ^2 [variance] iff X has

the following p.d.f. $f(x)$ [denoted by $X \sim N(\mu, \sigma^2)$]

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

P.d.f.

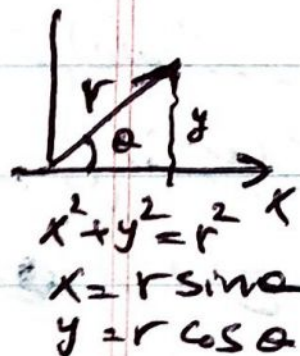
$-\infty < \mu < \infty$
 $\sigma^2 > 0$



We have to show $f(x)$ is p.d.f
 i.e. ① $f(x) \geq 0$, ② $\int_{-\infty}^{\infty} f(x) dx = 1$
holds

Hint ① $I = \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sqrt{2\pi}$

② $I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} dr d\theta = 2\pi$



$x^2 + y^2 = r^2$
 $x = r \sin \theta$
 $y = r \cos \theta$

✓ ③ $\int_{-\infty}^{\infty} y e^{-y^2/2} dy = 0$

✓ ④ $\int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy = \sqrt{2\pi}$

③ odd
 ④ even

proof ③ $\int_{-\infty}^{\infty} y e^{-y^2/2} dy = -e^{-y^2/2} \Big|_{-\infty}^{\infty} = -[e^{-\infty} - e^{-\infty}] = \underline{\underline{zero}}$

proof ④ $\int_{-\infty}^{\infty} y \cdot y e^{-y^2/2} dy$ let $u = y \rightarrow du = dy$
 $= -y e^{-y^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-y^2/2} dy$
 $du = y e^{-y^2/2} dy$
 $v = -e^{-y^2/2}$
 from ①

$= zero + \sqrt{2\pi}$

$= \underline{\underline{\sqrt{2\pi}}}$

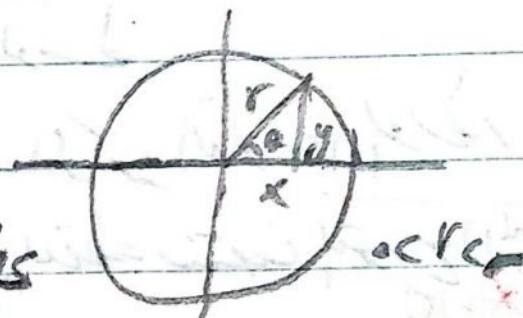
proof
①, ②

3

T.P $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

proof $I^2 = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \times \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dy dx$



Transfer to polar coordinates

by $\sin \theta = \frac{y}{r} \rightarrow y = r \sin \theta$
 $\cos \theta = \frac{x}{r} \rightarrow x = r \cos \theta$; $0 < \theta < 2\pi$
 $r > 0$

or $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$

Jacobian $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$

so $I^2 = \int_0^{\infty} \int_0^{2\pi} e^{-\frac{1}{2}(r^2 \cos^2 \theta + r^2 \sin^2 \theta)} \cdot r d\theta dr$

$= \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{2}} r d\theta dr = \int_0^{\infty} \left[\int_0^{2\pi} d\theta \right] e^{-\frac{r^2}{2}} r dr$
 $= \int_0^{\infty} [2\pi] \cdot e^{-\frac{r^2}{2}} r dr = 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} r dr$
 $= 2\pi \left[-e^{-\frac{r^2}{2}} \right]_0^{\infty} = -2\pi [e^{-\infty} - e^0]$

so $I^2 = 2\pi \rightarrow I = \sqrt{2\pi}$

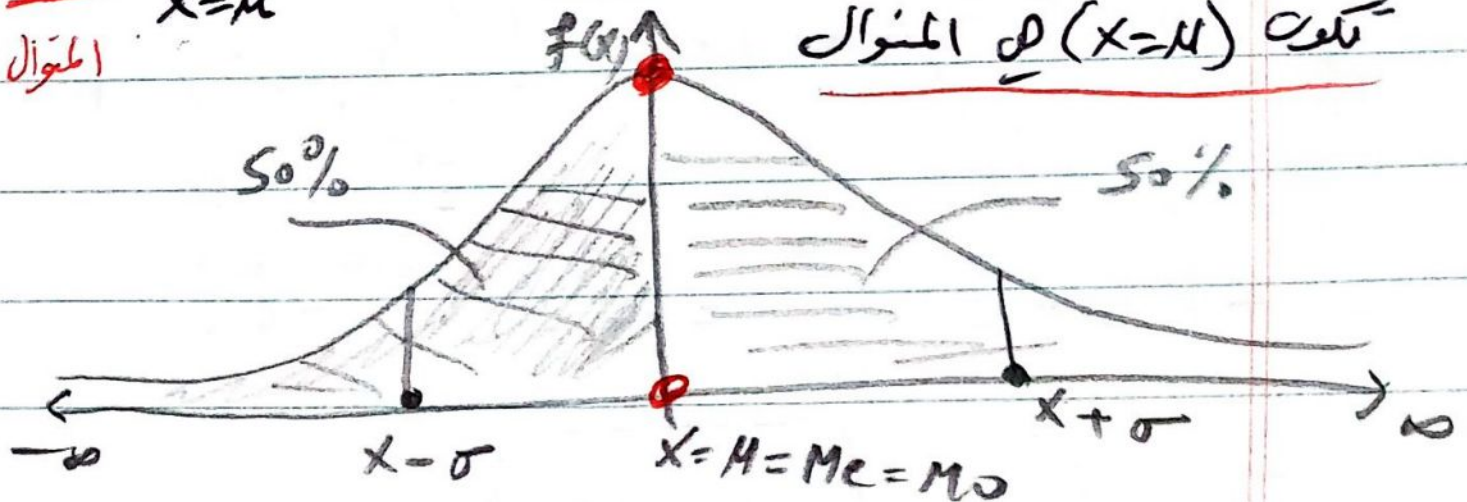
Some properties of Normal dist.

بعض خصائص التوزيع الطبيعي

- ① يتميز التوزيع الطبيعي بمعلمتين هما الوسط (μ) وتنتج معظم قيم التوزيع في المنحنى حول وسط التوزيع (μ) والمعلمة الثانية هي التباين (σ^2) ويعطى مقياساً لمعدل انتشار القيم الممكنة لتوزيع المتغير العشوائي حول الوسط.
- ② في المنحنى الطبيعي تتساوى قيم التوزيع المركزية الثلاثة (الوسط، الوسط، المتوال) حيث ستكون في نفس النقطة منتصف التوزيع (المتساوي).
- ③ المنحنى الطبيعي متماثل (Symmetric) حول الخط العمودي المار من أعلى نقطة في المنحنى (قمة التوزيع) ويقطع محور السينات إلى نصفين متساويين (أي يقسم المنحنى إلى نصفين متساويين) وبذلك تكون مساحة كل نصف (50%) من المساحة الكلية للمنحنى التي هي 1 (100%). (نقطة μ هي الوسط).
- ④ نقاط انقلاب المنحنى (inflection points) هي

$$X = \mu \pm \sigma$$

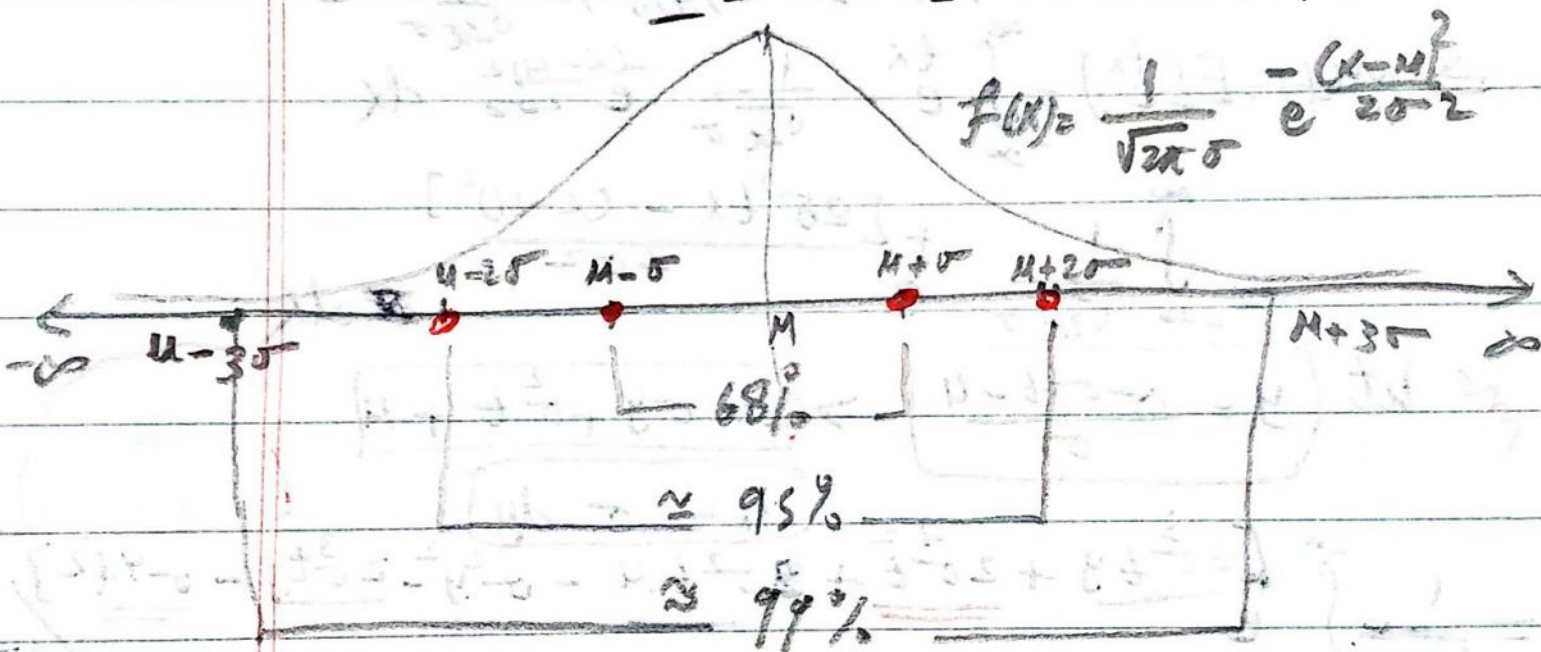
- ⑤ أعلى نقطة للمنحنى هي عند ($X = \mu$) وهي تسمى نقطة Mode (المتوال) تكون ($X = \mu$) هي المتوال.



⑥ من الممارسة الكلية لتوزيع العينة (البالغة 100%) تم
حساب نسب خصبة قدرته ونمطاً لعدد الانحرافات لمعياره

عن الوسط الحسابي وكما يلي

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



$$\int_{\mu-\sigma}^{\mu+\sigma} f(x) dx \approx 0.68$$

$$\int_{\mu-2\sigma}^{\mu+2\sigma} f(x) dx \approx 0.95$$

$$\int_{\mu-3\sigma}^{\mu+3\sigma} f(x) dx \approx 0.99$$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

وطبيعاً

⑦ يمكن حساب الاحتمالات لارتفاعه تحت المنحنى من نقطة a إلى b
حسب التعريف الاحتمالية وذلك مع وجود جميع ايجاد اغلب
هذه الحسابات باستخدام جدول خاصه

i.e. $P(a < X < b) = \int_a^b f(x) dx$



Thm 10 If $X \sim N(\mu, \sigma^2)$, then m.g.f. of X

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \quad -\infty < t < \infty$$

proof $M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

let $y = \frac{x-\mu}{\sigma} \rightarrow x = \sigma y + \mu \Rightarrow dx = \sigma dy$

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(\sigma y + \mu)} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2}} \cdot \sigma dy$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma t y - \frac{y^2}{2}} dy$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2 - 2\sigma t y}{2}} dy \quad \leftarrow \text{[u] uL[u]}$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2 - 2\sigma t y + \sigma^2 t^2}{2} + \sigma^2 t^2} dy$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y - \sigma t)^2}{2}} dy$$

let $z = y - \sigma t \rightarrow y = z + \sigma t \rightarrow dy = dz$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \right) = 1$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Th. 11 If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$, $Var(X) = \sigma^2$

proof $\because X \sim N(\mu, \sigma^2)$

$$\therefore M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$\dot{M}_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)$$

$$\underline{E(X)} = \dot{M}_X(0) = e^0 (\mu + 0) = \mu$$

$$\ddot{M}_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} (\sigma^2) + (\mu + \sigma^2 t) e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)$$

$$\ddot{M}_X(0) = \underline{e^0 (\sigma^2)} + \mu^2 \underline{e^0}$$

$$\underline{E(X^2)} = \sigma^2 + \mu^2$$

$$\begin{aligned} \therefore Var(X) &= E(X^2) - [E(X)]^2 \\ &= \sigma^2 + \mu^2 - [\mu]^2 \\ &= \sigma^2 \end{aligned}$$

ex 1 If $X \sim N(2, 4)$, then write

① $f(x)$, $M_X(t)$, $E(X)$ and $Var(X)$

② Compute $P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$

$$\underline{i.e.} \Rightarrow P(2 - 4 < X < 2 + 4)$$

$$P(-2 < X < 6) \approx 0.95$$

or using Z $\rightarrow P(-2 < \frac{X - \mu}{\sigma} < 2) = P(-2 < Z < 2)$

$$= P(Z < 2) - P(Z < -2) \Rightarrow P(Z < 2) - [1 - P(Z < 2)]$$

$$= 2 P(Z < 2) - 1 = 2 * (0.977) - 1 \approx 0.95$$

المطلوب 7

ex. 2 If $X \sim N(3, 16)$ then
show that $P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.998$

i.e; $P(3 - 3(4) < X < 3 + 3(4)) = ?$
 $= P(-9 < X < 15)$

using z $P(\mu - 3\sigma < X < \mu + 3\sigma)$
 $= P(-3 < \frac{X - \mu}{\sigma} < +3)$

$= P(-3 < Z < +3)$

$= 2N(3) - 1$

$= 2(0.999) - 1$

≈ 0.998

⑤ Standard Normal dist. (S.N.d.) التوزيع
الطبيعي القياسي

Let $X \sim N(\mu, \sigma^2)$

and $\mu=0$ & $\sigma^2=1$, we obtain
a new dist. called "Standard Normal dist."
denoted by $X \sim N(0,1)$

وسمى هذا التوزيع (Zero-One) أو (توزيع 0 و 1)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

$$M_X(t) = e^{\frac{t^2}{2}} \quad -\infty < t < \infty$$

$E(X) = 0$ & $var(X) = \sigma^2$

حساب الاحتمالات للتوزيع الطبيعي القياسي

من خلال جدول يسمى جدول Z أو J, N الطبيعي القياسي

$$P(X \leq a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = N(a)$$

ملاحظة
① $P(X > a) = 1 - P(X \leq a) = 1 - N(a)$

② $P(X \leq -a) = 1 - P(X \leq a) = 1 - N(a)$

③ $P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = N(b) - N(a)$

وحيث أن الجدول الطبيعي القياسي لا يعطي a, b

Table of Standard Normal dist. (S.N.d)

$$F(x) = P(X \leq x) = \underline{N(x)} = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$$

X	N(x)	X	N(x)	X	N(x)
0.0	0.5	1.1	0.864	2.05	0.98
0.1	0.54	1.2	0.885	2.15	0.984
0.25	0.549	1.3	0.903	2.30	0.989
⋮		⋮		⋮	
0.85	0.802	1.85	0.968	2.8	0.997
⋮		⋮		⋮	
1.05	0.853	2	0.977	3	0.999

ex If $X \sim N(0,1)$, then

$$* P(X < 0.85) = 0.802 = N(0.85)$$

$$* P(X < -1.85) = 1 - P(X < 1.85) = 1 - N(1.85)$$

$$= 1 - 0.968 = \dots$$

$$P(1.05 < X < 2.15) = P(X < 2.15) - P(X < 1.05)$$

$$= N(2.15) - N(1.05)$$

$$= 0.989 - 0.853 = \dots$$

ex. If $X \sim N(\mu, \sigma^2)$

$$\begin{aligned}
 P(-2.15 < X < 3) &= P(X < 3) - P(X < -2.15) \\
 &= P(X < 3) - [1 - P(X < 2.15)] \\
 &= 0.999 - [1 - 0.984] \\
 &= 0.999 - 1 + 0.984 \\
 &= -0.001 + 0.984 = 0.983
 \end{aligned}$$

ex. 1 If $X \sim N(\mu, \sigma)$, then show that
 $P(\mu - \sigma < X < \mu + \sigma) \approx 0.68$
 using table of S.N.d.

$$\begin{aligned}
 P(\mu - \sigma < X < \mu + \sigma) &= P(0 - 1 < X < 0 + 1) \\
 &= P(-1 < X < 1) \\
 &= P(X < 1) - P(X < -1) \\
 &= P(X < 1) - [1 - P(X < 1)] \\
 &= N(1) - 1 + N(1) \\
 &= 2N(1) - 1 \\
 &= 2(0.84) - 1 = 1.68 - 1 \\
 &= 0.68
 \end{aligned}$$

⑤ Standard Normal dist. (S.N.d.) التوزيع الطبيعي القياسي

If $X \sim N(\mu, \sigma^2)$

and $\mu=0$ & $\sigma^2=1$, we obtain
a new dist. called "Standard Normal dist."
denoted by $X \sim N(0,1)$

وسمى في بعض الأحيان Zero-One (توزيع 0 و 1)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

$$M_X(t) = e^{-\frac{t^2}{2}} \quad -\infty < t < \infty$$

$$E(X) = 0 \quad \& \quad \text{var}(X) = \sigma^2$$

حساب الاحتمالات للتوزيع الطبيعي القياسي
من خلال جدول يسمى جدول Z، z ، μ ، σ ، σ^2

$$P(X \leq a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = N(a) \equiv F_X(a)$$

ملحق

$$① P(X > a) = 1 - P(X \leq a) = 1 - N(a)$$

$$② P(X \leq -a) = 1 - P(X \leq a) = 1 - N(a)$$

$$③ P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = N(b) - N(a)$$

و يسمى هذا الجدول الجدول الطبيعي القياسي، a ، b ، μ ، σ ، σ^2 ✓

Table of Standard Normal dist. (S.N.d)

$$F(x) = P(X \leq x) = \underline{N(x)} = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$$

X	N(x)	X	N(x)	X	N(x)
0.0	0.5	1.1	0.864	2.05	0.98
0.1	0.54	1.2	0.885	2.15	0.984
0.25	0.549	1.3	0.903	2.30	0.989
⋮		⋮		⋮	
0.85	0.802	1.85	0.968	2.8	0.997
⋮		⋮		⋮	
1.05	0.853	2	0.977	3	0.999

ex If $X \sim N(0,1)$, then

$$* P(X < 0.85) = 0.802 = N(0.85)$$

$$* P(X < -1.85) = 1 - P(X < 1.85) = 1 - N(1.85) = 1 - 0.968 = \dots$$

$$\begin{aligned} P(1.05 < X < 2.15) &= P(X < 2.15) - P(X < 1.05) \\ &= N(2.15) - N(1.05) \\ &= 0.989 - 0.853 = \dots \end{aligned}$$

ex. If $X \sim N(0,1)$ ^{μ σ^2} توزيع عادي

$$\begin{aligned}
 P(-2.05 < X < 3) &= P(X < 3) - P(X < -2.05) \\
 &= P(X < 3) - [1 - P(X < 2.05)] \\
 &= 0.999 - [1 - 0.984] \\
 &= 0.999 - 1 + 0.984 \\
 &= -0.001 + 0.984 = 0.983
 \end{aligned}$$

ex.1 If $X \sim N(0,1)$, then show that
 $P(\mu - \sigma < X < \mu + \sigma) \approx 0.68$
 using table of S.N.D.

$$\begin{aligned}
 P(\mu - \sigma < X < \mu + \sigma) &= P(0 - 1 < X < 0 + 1) \\
 &= P(-1 < X < 1) \\
 &= P(X < 1) - P(X < -1) \\
 &= P(X < 1) - [1 - P(X < 1)] \\
 &= N(1) - 1 + N(1) \\
 &= 2N(1) - 1 \\
 &= 2(0.84) - 1 = 1.68 - 1 \\
 &= 0.68
 \end{aligned}$$

Th (12) If X is c.r.v. such that $X \sim N(\mu, \sigma^2)$
then

① If $Y = aX + b$ then $Y \sim N(a\mu + b, a^2\sigma^2)$; $a, b \in \mathbb{R}$
 $a \neq 0$

Pr ② If $Z = \frac{X - \mu}{\sigma}$, $\sigma \neq 0 \rightarrow Z \sim N(0, 1)$ تحويل إلى طبيعى

③ If $X_1, X_2, \dots, X_n$ are Independent c.r.v.s such
that $X_i \sim N(\mu_i, \sigma_i^2)$, then

$$W = \sum_{i=1}^n X_i \sim N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$$

proof ① $\because X \sim N(\mu, \sigma^2)$
 $\therefore M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$Y = aX + b \rightarrow M_Y(t) = e^{bt} \cdot M_X(at)$$

$$\therefore M_Y(t) = e^{bt} \cdot e^{\mu at + \frac{\sigma^2 a^2 t^2}{2}}$$

$$= e^{(a\mu + b)t + \frac{(a^2\sigma^2)t^2}{2}}$$

$$\therefore Y \sim N(a\mu + b, a^2\sigma^2)$$

$$(2) Z = \frac{X - \mu}{\sigma} = \left(\frac{1}{\sigma}\right)X + \left(-\frac{\mu}{\sigma}\right)$$

$$M_Z(t) = e^{\frac{bt}{\sigma}} \cdot M_X\left(\frac{t}{\sigma}\right)$$

$$\therefore M_Z(t) = \left(e^{-\frac{\mu t}{\sigma}}\right) \cdot M_X\left(\frac{1}{\sigma}t\right)$$

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$= e^{-\frac{\mu}{\sigma}t} \cdot e^{\mu \frac{t}{\sigma} + \frac{\sigma^2}{2} \frac{t^2}{\sigma^2}}$$

$$= e^{-\cancel{\frac{\mu}{\sigma}t} + \cancel{\frac{\mu}{\sigma}t} + \frac{t^2}{2}} = e^{\frac{t^2}{2}} \text{ action}$$

$$(3) X_i (i=1, \dots, n) \sim N(\mu_i, \sigma_i^2)$$

$$M_{X_i}(t) = e^{\mu_i t + \frac{\sigma_i^2 t^2}{2}}$$

$$W = \sum_{i=1}^n X_i$$

$$M_W(t) = E[e^{Wt}] = E[e^{\sum_{i=1}^n X_i t}] = E[e^{(X_1 + X_2 + \dots + X_n)t}]$$

$$= E[e^{X_1 t + X_2 t + \dots + X_n t}] = E[e^{X_1 t} \cdot e^{X_2 t} \cdot \dots \cdot e^{X_n t}]$$

$$= E[e^{X_1 t}] \cdot E[e^{X_2 t}] \cdot \dots \cdot E[e^{X_n t}]$$

$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t)$$

$$= \left[e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \right] \left[e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} \right] \cdot \dots \cdot \left[e^{\mu_n t + \frac{\sigma_n^2 t^2}{2}} \right]$$

$$= e^{(\sum \mu_i) t + (\sum \sigma_i^2) \frac{t^2}{2}}$$

$$\therefore W \sim N(\sum \mu_i, \sum \sigma_i^2)$$

Note Try if $X_i \sim N(\mu_i, \sigma_i^2)$, $W = \sum_{i=1}^n X_i$

then $W \sim N(n\mu, n\sigma^2)$

$$\begin{pmatrix} 0,1 \\ Z \end{pmatrix}$$

Examples

تحويل Normal الى Standard Normal

تكون نتيجته اعداد (S.N.d) لـ (Z)

ex.1

If $X \sim N(5, 16)$, then

Find $P(X \leq 13)$ using the table of Standard Normal dist. (Z-table)

Sol.

$$X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

S.N.d

$$\text{So } P(X \leq 13) = P\left(\frac{X - \mu}{\sigma} \leq \frac{13 - \mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{13 - 5}{4}\right)$$

$$= P(Z \leq 2)$$

$$= N(2) = 0.977$$

من الجدول

ex.2

If $X \sim N(5, 4)$, then find

$P(1 \leq X \leq 8)$ using (Z-table)

Sol. $P(1 \leq X \leq 8) = P\left(\frac{1 - 5}{2} \leq \frac{X - 5}{2} \leq \frac{8 - 5}{2}\right)$

$$= P\left(-2 \leq Z \leq \frac{3}{2}\right)$$

$$= P\left(Z \leq \frac{3}{2}\right) - P(Z \leq -2)$$

$[1 - P(Z \leq 2)]$

$$= P\left(Z \leq \frac{3}{2}\right) + P(Z \leq 2) - 1$$

$$= N\left(\frac{3}{2}\right) + N(2) - 1$$

$$= 0.933 + 0.977 - 1$$

$$= 0.910$$

ex. 3 If $X \sim N(2, 4)$, then find $P(6 < X < 8)$ using (z-table.)

Sol. $P(6 < X < 8) = P\left(\frac{6-2}{2} < \frac{X-2}{2} < \frac{8-2}{2}\right)$
 $= P(2 < Z < 3)$
 $= N(3) - N(2)$
 $= 0.994 - 0.977$

ex. 4 If $X \sim N(3, 9)$, then find $P(|X-3| < 6)$
Sol. $P(|X-3| < 6)$

$P(|X-3| < 6) = P(-6 < X-3 < 6)$
 $= P\left(-\frac{6}{3} < \frac{X-3}{3} < \frac{6}{3}\right)$
 $= P(-2 < Z < 2)$
 $= N(2) - N(-2)$
 $= N(2) - [1 - N(2)]$
 $= 2N(2) - 1$
 $= 2[0.9772] - 1$
 $= 0.9544$

$P(|X-3| \geq 6) = 1 - P(|X-3| < 6)$
 $= 1 - 0.9544$

ex. 5 If $X \sim N(25, 36)$, and if

$$P(|X - 25| \leq \underline{c}) = 0.9544, \text{ Find } c?$$

Sol. $P(|X - 25| \leq c) = P(-c \leq X - 25 \leq c)$

$$0.9544 = P\left(-\frac{c}{6} \leq \frac{X - 25}{6} \leq \frac{c}{6}\right)$$

$$0.9544 = P\left(-\frac{c}{6} \leq Z \leq \frac{c}{6}\right)$$

$$= N\left(\frac{c}{6}\right) - N\left(-\frac{c}{6}\right) = [1 - N\left(-\frac{c}{6}\right)]$$

$$= 2N\left(\frac{c}{6}\right) - 1$$

$$\therefore N\left(\frac{c}{6}\right) = \frac{1 + 0.9544}{2} = 0.9772$$

$$\therefore \frac{c}{6} = 2$$

من جداول Z

$$\therefore \boxed{c = 12}$$

H.w. If the m.g.f. of X is

$$M_X(t) = e^{3t + 8t^2}$$

$$8 = \frac{16}{2}$$

then Find $P(-1 < X < 9)$

Sol Hint $\mu_X = 3 = \sigma_X^2 = 16$

$$\therefore X \sim N(3, 16)$$

$$\sigma = 4$$

$$\therefore P(-1 < X < 9) = P\left(\frac{-1 - 3}{4} < Z < \frac{9 - 3}{4}\right)$$

$$= P(-1 < Z < 1.5)$$

النتيجة

$$=$$

Theorem If $X \sim N(0, 1)$, then $Y = X^2 \sim \chi^2(1)$

proof $X \sim N(0, 1) \rightarrow f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$

طريقة 1
نموذج آخر

$$Y = X^2 \rightarrow X = \pm \sqrt{Y} \rightarrow \frac{dx}{dy} = J = \pm \frac{1}{2\sqrt{y}}$$

$$\begin{bmatrix} x_1 = \sqrt{y} \\ x_2 = -\sqrt{y} \end{bmatrix}$$

$$|J| = \frac{1}{2} y^{-1/2}$$

by transformation

$$g(y) = \sum_{i=1}^2 f(x_i) |J|$$

$$-\infty < x < \infty$$

$$0 < x^2 < \infty$$

$$0 < y < \infty$$

$$= [f(x_1) + f(x_2)] \cdot \frac{1}{2\sqrt{y}}$$

$$= \left[\frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \right] \cdot \frac{1}{2} y^{-1/2}$$

$$= \frac{2}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2} y^{-1/2}$$

$$= \frac{e^{-y/2}}{\sqrt{y} \sqrt{2\pi}}$$

$$= \frac{\frac{1}{2} - 1}{y} e^{-y/2}$$

$$= \frac{1}{\Gamma(\frac{1}{2}) 2^{1/2}} e^{-y/2} \quad 0 < y < \infty$$

$\therefore Y \sim \chi^2(1)$

2nd

using M.g.f

$$M_Y(t) = E[e^{ty}] = E[e^{tx^2}] = \int_{-\infty}^{\infty} e^{tx^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2(1-2t)}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} \cdot \frac{1}{\sqrt{1-2t}} dw = (1-2t)^{-1/2}$$

let $w = \sqrt{1-2t} x$
 $dw = \sqrt{1-2t} dx$

$\therefore Y \sim \chi^2(1)$

Thm If $X_i (i=1, \dots, n) \stackrel{i.i.d}{\sim} N(0,1)$ and if
 $Y = \sum_{i=1}^n X_i^2$, then $Y \sim \chi^2(n)$

proof

من الفرضية

$$\text{if } X_i \sim N(0,1) \rightarrow X_i^2 \sim \chi^2(1)$$

$$\text{So } X_i^2 \sim \chi^2(1) \rightarrow M_{X_i^2}(t) = (1-2t)^{-1/2}$$

$$\text{let } Y = \sum_{i=1}^n X_i^2$$

$$\begin{aligned} M_Y(t) &= E[e^{(\sum_{i=1}^n X_i^2)t}] = E[e^{X_1^2 t + X_2^2 t + \dots + X_n^2 t}] \\ &= E[e^{X_1^2 t}] E[e^{X_2^2 t}] \dots E[e^{X_n^2 t}] \\ &= M_{X_1^2}(t) \cdot M_{X_2^2}(t) \dots M_{X_n^2}(t) \end{aligned}$$

$$= (1-2t)^{-1/2} \cdot (1-2t)^{-1/2} \dots (1-2t)^{-1/2}$$

$$= (1-2t)^{-\frac{n}{2}}$$

عند ذلك
 جمعنا

$$\text{So } Y \sim \chi^2(n)$$

$$\beta = 2$$

$$r = n$$

$$\alpha = \frac{n}{2}$$

ex If $X_i (i=1, \dots, n)$ are independent r.v.

s.t. $X_i \sim \text{Exp}(\gamma)$, then

$$T = \frac{\sum_{i=1}^n X_i}{n} \sim G(n, \frac{1}{n\gamma})$$

Sol. $X_i \sim \text{Exp}(\gamma) \rightarrow M_{X_i}(t) = (1 - \frac{t}{\gamma})^{-1}$

$$\begin{aligned} M_T(t) &= E[e^{Tt}] = E[e^{\frac{\sum X_i}{n} t}] = E[e^{\frac{X_1}{n} t + \frac{X_2}{n} t + \dots + \frac{X_n}{n} t}] \\ &= E[e^{\frac{X_1}{n} t}] \dots E[e^{\frac{X_n}{n} t}] \end{aligned}$$

$$= M_{X_1}(\frac{t}{n}) \dots M_{X_2}(\frac{t}{n}) \dots M_{X_n}(\frac{t}{n})$$

$$= [(1 - \frac{t}{n\gamma})^{-1}] [(1 - \frac{t}{n\gamma})^{-1}] \dots [(1 - \frac{t}{n\gamma})^{-1}]$$

$$= (1 - \frac{t}{n\gamma})^{-n}$$

$$\therefore T \sim G(n, \frac{1}{n\gamma})$$

ex If $X_i (i=1, \dots, n)$ are independent r.v.

s.t. $X_i \sim \text{EXP}(\gamma)$, then

$$T = \frac{\sum_{i=1}^n X_i}{n} \sim G(n, \frac{1}{n\gamma})$$

Sol. $X_i \sim \text{EXP}(\gamma) \rightarrow M_{X_i}(t) = (1 - \frac{t}{\gamma})^{-1}$

$$\begin{aligned} M_T(t) &= E[e^{Tt}] = E[e^{\frac{\sum X_i}{n} t}] = E[e^{\frac{X_1}{n}t + \frac{X_2}{n}t + \dots + \frac{X_n}{n}t}] \\ &= E[e^{\frac{X_1}{n}t}] \dots E[e^{\frac{X_n}{n}t}] \end{aligned}$$

$$\begin{aligned} &= M_{X_1}(\frac{t}{n}) \dots M_{X_n}(\frac{t}{n}) \\ &= [(1 - \frac{t}{n\gamma})^{-1}]^n \end{aligned}$$

$$= (1 - \frac{t}{n\gamma})^{-n}$$

$$\therefore T \sim G(n, \frac{1}{n\gamma})$$

Q / At $X_i \sim \chi^2(1)$, $Y = \sum_{i=1}^5 X_i \sim \chi^2(5)$

proof $M_{X_i}(t) = (1-2t)^{-1/2} \quad \forall i=1, \dots, 5$

$$\begin{aligned}
 M_Y(t) &= E[e^{Yt}] = E[e^{(\sum X_i)t}] = E[e^{(X_1 + X_2 + \dots + X_5)t}] \\
 &= E[e^{X_1 t + X_2 t + \dots + X_5 t}] \\
 &= E[e^{X_1 t}] \cdot E[e^{X_2 t}] \cdot \dots \cdot E[e^{X_5 t}] \\
 &= M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_5}(t) \\
 &= (1-2t)^{-1/2} (1-2t)^{-1/2} \cdot \dots \cdot (1-2t)^{-1/2} \\
 &= (1-2t)^{-5/2} \\
 \therefore Y &\sim \chi^2(5)
 \end{aligned}$$

Q /

$$X \sim N(\mu, \sigma^2)$$

Q. 11

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$Z = \frac{X - \mu}{\sigma} = \underbrace{\left(\frac{1}{\sigma}\right)}_a X + \underbrace{\left(\frac{-\mu}{\sigma}\right)}_b$$

$$M_Z(t) = e^{bt} \cdot M_X(at) \Rightarrow$$

$$= e^{-\frac{\mu}{\sigma}t} M_X\left(\frac{1}{\sigma}t\right)$$

$$= e^{-\frac{\mu}{\sigma}t} \cdot e^{\frac{\mu}{\sigma}t + \frac{\sigma^2 t^2}{2\sigma^2}}$$

$$= e^{-\frac{\mu}{\sigma}t + \frac{\mu}{\sigma}t + \frac{t^2}{2}}$$

$$= e^{\frac{t^2}{2}}$$

$$\therefore Z \sim N(0, 1)$$

or by transformation

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad z = \frac{x-\mu}{\sigma} \rightarrow x = \sigma z + \mu$$

$$J = \frac{dx}{dz} = \sigma$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} \cdot \sigma$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \rightarrow Z \sim N(0, 1)$$

Q / If $X_i \sim \exp(\theta)$; $i=1, 2, 3$

$$W = \sum_{i=1}^3 X_i \sim G(3, \frac{1}{\theta})$$

proof $\because X_i \sim \exp(\theta) \rightarrow M_{X_i}(t) = (1 - \frac{t}{\theta})^{-1}$

$$M_W(t) = E[e^{(X_1+X_2+X_3)t}] = E[e^{X_1 t + X_2 t + X_3 t}]$$

$$= E[e^{X_1 t}] \cdot E[e^{X_2 t}] \cdot E[e^{X_3 t}]$$

$$= (1 - \frac{t}{\theta})^{-1} \cdot (1 - \frac{t}{\theta})^{-1} \cdot (1 - \frac{t}{\theta})^{-1}$$

$$= (1 - \frac{t}{\theta})^{-3} \sim G(3, \frac{1}{\theta})$$

copy the

Thm

Ex. If $X_i \sim \chi^2_{(1)}$ $i=1, \dots, n$ are indep. r.v.s
then $Y = \sum_{i=1}^n X_i \sim \chi^2_{(n)}$

Sol. $X_i \sim \chi^2_{(1)} \rightarrow M_{X_i}(t) = (1-2t)^{-1/2}$

$$\begin{aligned} M_Y(t) &= E[e^{(X_1 + \dots + X_n)t}] = E[e^{X_1 t} \dots e^{X_n t}] \\ &= E[e^{X_1 t}] \cdot E[e^{X_2 t}] \dots E[e^{X_n t}] \\ &= M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t) \\ &= [(1-2t)^{-1/2}] [(1-2t)^{-1/2}] \dots [(1-2t)^{-1/2}] \end{aligned}$$

$$= (1-2t)^{-n/2}$$

$$\therefore Y \sim \chi^2_{(n)}$$

$$\beta = 2$$

$$r = n$$

$$\alpha = \frac{n}{2}$$

Table of Discrete Probability Distributions

No.	Name of dist.	Parameters	function $f(x)$	M.g.f. $M_X(t)$	Mean μ_X	Variance σ_X^2
1	Bernoulli dist.	p	$X \sim b(1, p)$ $f(x; 1, p) = \begin{cases} p^x (1-p)^{1-x} & x=0,1 \\ 0 & \text{o.w.} \end{cases}$	$(1-p+pet)$	p	$p(1-p)$
2	Binomial dist.	n, p	$X \sim b(n, p)$ $f(x; n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0,1,\dots,n \\ 0 & \text{o.w.} \end{cases}$	$(1-p+pet)^n$	np	$np(1-p)$
3	Geometric dist.	p	$X \sim G(p)$ $f(x; p) = \begin{cases} p q^{x-1} & x=1,2,\dots \\ 0 & \text{o.w.} \end{cases}$	$\frac{pet}{1-qet}$ for $t < \ln(\frac{1}{q})$	$\frac{1}{p}$	$\frac{q}{p^2}$
4	Poisson dist.	m $m \in \mathbb{N}$	$X \sim P(m)$ $f(x; m) = \begin{cases} \frac{e^{-m} m^x}{x!} & x=0,1,\dots \\ 0 & \text{o.w.} \end{cases}$	$e^{m(e^t-1)}$ for $t < \infty$	m	m

Table of Continuous Probability Distributions

No.	Name of dist.	Parameters	Function $f(x)$	M.g.f. $M_x(t)$	Mean μ_x	Variance σ_x^2
1	Gamma dist.	α, β	$X \sim G(\alpha, \beta)$ $f(x; \alpha, \beta) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^\alpha} & x > 0 \\ 0 & \text{o.w.} \end{cases}$	$(1 - \beta t)^{-\alpha}$	$\alpha \beta$	$\alpha \beta^2$
2	Chi-Square dist.	r	$X \sim \chi^2(r)$ $f(x; r) = \begin{cases} \frac{x^{\frac{r}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{r}{2}} \Gamma(\frac{r}{2})} & x > 0 \\ 0 & \text{o.w.} \end{cases}$	$(1 - 2t)^{-\frac{r}{2}}$	r	$2r$
3	Exponential dist.	λ	$X \sim \text{exp.}(\lambda)$ $f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$	$(1 - \frac{t}{\lambda})^{-1}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
4	Normal dist.	μ_x, σ_x^2	$X \sim N(\mu_x, \sigma_x^2)$ $f(x; \mu_x, \sigma_x^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x - \mu_x)^2}{2\sigma_x^2}}$ $-\infty < x < \infty$	$e^{\mu_x t + \frac{\sigma_x^2 t^2}{2}}$ $-\infty < t < \infty$	μ_x	σ_x^2
5	Standard Normal dist. (S.N.d.)	0, 1	$X \sim N(0, 1)$ $f(x; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$	$e^{\frac{t^2}{2}}$ $-\infty < t < \infty$	0	1

Chapter Six

Some Special Probability Distributions

Discrete Probability Distributions

① Binomial Distribution (توزيع ذو الحدين)

Bernoulli Distribution

Bernoulli Experiment: is an experiment that has only two possible outcomes, event A (Success trial) and event A^c (failure trial).

For example:

Toss a coin once.

We get H or T , H, T are only two possible outcomes.

Let A : to get H

A^c : to get T

Suppose that $P(A) = p$, $0 < p < 1$

$$P(A^c) = 1 - p$$

Then, if we repeat the Bernoulli experiment n -times ($n \in \mathbb{I}^+$), and if the r.v. X ;

$X \equiv$ number of times that A happens.

For example:

Toss a coin 3-times.

A : to get H

$X \equiv$ number of H

$$X = 0, 1, 2, 3$$

Know, $X \equiv$ number of times that event A happens, then the function of X is:

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, 2, \dots, n \\ 0 & \text{o.w.} \end{cases}$$

Know, to show that $f(x)$ is a p.m.f. since X is d.r.v.

Cond. ① T.P $f(x) \geq 0 \quad \forall x \in R_x$

$n \in I^+, X \leq n, n-x \geq 0, x=0, 1, \dots, n$

$$0 < p < 1 \Rightarrow 0 < 1-p < 1$$

$$\binom{n}{x} > 0, p^x > 0 \Rightarrow (1-p)^{n-x} > 0$$

$$\text{Then } f(x) = \binom{n}{x} p^x (1-p)^{n-x} > 0 \text{ for } x=0, 1, 2, \dots, n \\ = 0 \text{ o.w.}$$

$$\text{Cond. ② T.P } \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} = 1$$

$$\text{i.e. } \sum_{x=0}^n f(x) = 1$$

$$\text{Let } a=1-p, b=p \\ (a+b)^n = \sum_{x=0}^n \binom{n}{x} b^x a^{n-x}$$

$$[(1-p)+p]^n = \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} \\ 1 = \sum_{x=0}^n f(x)$$

Note: The function $f(x)$ above is called the "Binomial p.f." We can use the function only if we have a Bernoulli experiment. Let $b(n, p)$ to denote of the "Binomial distribution with parameters n, p ".

If $X \sim b(n, p)$, then the p.m.f. $f(x)$ of X is Binomial function:

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x=0, 1, 2, \dots, n \\ 0 & \text{o.w.} \end{cases}$$

Example ①: If $X \sim b(5, \frac{2}{3})$. Find $f(x)$.

$$\text{Sol: } n=5, p=\frac{2}{3} \Rightarrow 1-p=\frac{1}{3}$$

$$f(x) = \begin{cases} \binom{5}{x} \left(\frac{2}{3}\right)^x \left(1-\frac{2}{3}\right)^{5-x} & \text{for } x=0, 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases} \\ = \begin{cases} \binom{5}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{5-x} & \text{for } x=0, 1, \dots, 5 \\ 0 & \text{o.w.} \end{cases}$$

Example ②: Toss dice 8-times

Let X be the number of times that no. (3) happens.
Find $f(x)$.

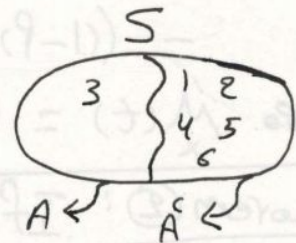
Sol: We must show this experiment is a Bernoulli experiment.

Toss a dice once:

$A = \{d=3\}$ = Success trial كاملة النجاح

$A^c = \{d; d \neq 3\}$ = failure trial كاملة الفشل

$S = \{d; 1 \leq d \leq 6\}$



then this experiment is Bernoulli experiment.

$$\therefore P(A) = \frac{1}{6} \text{ \& } P(A^c) = \frac{5}{6}, n = 8 \text{ (8-times)}$$

$$\therefore X \sim b(8, \frac{1}{6})$$

$$f(x) = \begin{cases} \binom{8}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{8-x} & \text{for } x = 0, 1, 2, \dots, 8 \\ 0 & \text{o.w.} \end{cases}$$

Now, Find $P(X \geq 1)$ احتمال الحصول على الرقم (3) على الأقل مرة واحدة

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - f(0)$$

$$= 1 - \binom{8}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^8 = 1 - \left(\frac{5}{6}\right)^8$$

$$\underline{\text{OR}} \quad P(X \geq 1) = \sum_{x=1}^8 f(x) = \sum_{x=1}^8 \binom{8}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{8-x}$$

$$\text{Find } P(X=5) = f(5) = \binom{8}{5} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^{8-5} = \left(\frac{1}{6}\right)^5$$

↑ احتمال ان يظهر الرقم (3) 5 مرات

Theorem ①: If $X \sim b(n, p)$, then the M.g.f. of X is as follows:

$$M_X(t) = (1 - p + pe^t)^n$$

Proof:

$$\therefore X \sim b(n, p)$$

$$\therefore f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, 2, \dots, n \\ 0 & \text{o.w.} \end{cases}$$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^n e^{tx} f(x)$$

$$= \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

Let $a = 1-p$, $b = pe^t$ & by $(a+b)^n = \sum_{x=0}^n \binom{n}{x} b^x a^{n-x}$

$$= (1-p + pe^t)^n$$

$$\therefore M_x(t) = (1-p + pe^t)^n$$

Theorem ②: If $X \sim b(n, p)$, then $\mu_x = E(x) = np$ and $\sigma_x^2 = \text{Var}(x) = np(1-p)$.

Proof: $\therefore X \sim b(n, p)$

$$\therefore M_x(t) = (1-p + pe^t)^n$$

$$M'_x(t) = n[1-p + pe^t]^{n-1} (pe^t)$$

$$M'_x(0) = n[1-p + pe^0]^{n-1} (pe^0) \quad \times (e^0 = 1)$$

$$= n(1-p + p)^{n-1} p = np$$

$$M''_x(t) = np[1-p + pe^t]^{n-1} (e^t) + pe^t(n-1)[1-p + pe^t]^{n-2} pe^t.$$

$$M''_x(0) = (1-p + pe^0)^{n-1} np e^0 + e^0(n-1)(1-p + pe^0)^{n-2} pe^0.$$

$$= np[1 + (n-1)p]$$

$$E(x^2) = np + n(n-1)p^2$$

$$= np + n^2p^2 - np^2$$

$$\therefore \text{Var}(x) = E(x^2) - (E(x))^2$$

$$= np + n^2p^2 - np^2 - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

Example ③: Given $M_x(t) = (\frac{1}{4} + \frac{3}{4}e^t)^6$, find $\mu_x, \sigma_x^2, f(x)$.

Sol. $1-p = \frac{1}{4}, p = \frac{3}{4}, n = 6$

$$\therefore X \sim b(6, \frac{3}{4})$$

$$f(x) = \begin{cases} \binom{6}{x} \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{6-x} & \text{for } x=0, 1, \dots, 6 \\ 0 & \text{o.w.} \end{cases}$$

$$= \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

Let $a = 1-p$, $b = pe^t$ & by $(a+b)^n = \sum_{x=0}^n \binom{n}{x} b^x a^{n-x}$

$$= (1-p + pe^t)^n$$

$$\therefore M_x(t) = (1-p + pe^t)^n$$

Theorem ②: If $X \sim b(n, p)$, then $\mu_x = E(x) = np$ and $\sigma_x^2 = \text{Var}(x) = np(1-p)$.

Proof: $\therefore X \sim b(n, p)$

$$\therefore M_x(t) = (1-p + pe^t)^n$$

$$M'_x(t) = n(1-p + pe^t)^{n-1} (pe^t)$$

$$M'_x(0) = n(1-p + pe^0)^{n-1} (pe^0) \quad \times (e^0 = 1)$$

$$= n(1-p + p)^{n-1} p = np$$

$$M''_x(t) = np(1-p + pe^t)^{n-1} (e^t) + pe^t(n-1)(1-p + pe^t)^{n-2} pe^t$$

$$M''_x(0) = (1-p + pe^0)^{n-1} np e^0 + e^0(n-1)(1-p + pe^0)^{n-2} pe^0$$

$$= np[1 + (n-1)p]$$

$$E(x^2) = np + n(n-1)p^2$$

$$= np + n^2p^2 - np^2$$

$$\therefore \text{Var}(x) = E(x^2) - (E(x))^2$$

$$= np + n^2p^2 - np^2 - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

Example ③: Given $M_x(t) = (\frac{1}{4} + \frac{3}{4}e^t)^6$, find $\mu_x, \sigma_x^2, f(x)$.

Sol. $1-p = \frac{1}{4}, p = \frac{3}{4}, n = 6$

$$\therefore X \sim b(6, \frac{3}{4})$$

$$f(x) = \begin{cases} \binom{6}{x} (\frac{3}{4})^x (\frac{1}{4})^{6-x} & \text{for } x=0, 1, \dots, 6 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \mu_x = np = 6\left(\frac{3}{4}\right) = \frac{18}{4} = \frac{9}{2}$$

$$V(X) = \sigma_x^2 = np(1-p) = 6\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{18}{16} = \frac{9}{8}$$

Example ④ Given $f(x) = \begin{cases} \binom{7}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{7-x} & \text{for } x=0,1,\dots,7 \\ 0 & \text{o.w.} \end{cases}$
Find $M_X(t)$.

sol. $\therefore X \sim b(7, \frac{2}{3})$

$$\therefore M_X(t) = (1-p+pe^t)^n \\ = \left(\frac{1}{3} + \frac{2}{3}e^t\right)^7$$

Example ⑤ Given $X \sim b(n, \frac{1}{3})$. Find the smallest value, s.t.

$$P(X \geq 1) > 0.8.$$

sol. $\therefore X \sim b(n, \frac{1}{3})$

$$\therefore f(x) = \begin{cases} \binom{n}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{n-x} & \text{for } x=0,1,2,\dots,n \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - f(0)$$

$$= 1 - \binom{n}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n$$

$$= 1 - \left(\frac{2}{3}\right)^n > 0.8 \quad (\text{since } P(X \geq 1) > 0.8)$$

$$-\left(\frac{2}{3}\right)^n > -(0.2)$$

$$\left(\frac{2}{3}\right)^n < (0.2)$$

$$\text{if } n=0 \Rightarrow \left(\frac{2}{3}\right)^0 = 1 \nless 0.2$$

$$\text{if } n=1 \Rightarrow \left(\frac{2}{3}\right)^1 = .67 \nless 0.2$$

$$\text{if } n=2 \Rightarrow \left(\frac{2}{3}\right)^2 = \frac{4}{9} = .44 \nless 0.2$$

$$\text{if } n=3 \Rightarrow \left(\frac{2}{3}\right)^3 = \frac{8}{27} = .29 \nless 0.2$$

$$\text{if } n=4 \Rightarrow \left(\frac{2}{3}\right)^4 = \frac{16}{81} = .19 < .2 \quad \checkmark$$

$$\text{if } n=5 \Rightarrow \left(\frac{2}{3}\right)^5 = \frac{32}{243} \approx .12 < .2 \quad \checkmark$$

then $n \geq 4$ s.t. $\left(\frac{2}{3}\right)^n < .2$

$\therefore$ the smallest value of (n) is equal to (4) .

Example (6) If X has $f(x) = \begin{cases} \binom{7}{x} (\frac{1}{2})^x (1-\frac{1}{2})^{7-x} & x=0,1,\dots,7 \\ 0 & \text{o.w.} \end{cases}$

(1) Show that $f(x)$ is a p.d.f.

(2) Find M.g.f. of X , μ_x , σ_x^2

(3) Find μ_x & σ_x^2 by two ways.

Sol. (1) H.w.

(2) $M_X(t) = (1-p+pe^t)^n = (\frac{1}{2} + \frac{1}{2}e^t)^7$ (since $X \sim b(7, \frac{1}{2})$)

$\mu_x = E(X) = np = 7(\frac{1}{2}) = \frac{7}{2}$

$\sigma_x^2 = E(X^2) - (E(X))^2 = np(1-p) = 7(\frac{1}{2})(\frac{1}{2}) = \frac{7}{4} > 0$

(3) $\mu_x = np = \frac{7}{2}$

or $\mu_x = M'_X(0) = np = \frac{7}{2}$ (Drive)

$\sigma_x^2 = np(1-p) = \frac{7}{4} > 0$

or $\sigma_x^2 = E(X^2) - (E(X))^2$
 $= M''_X(0) - (M'_X(0))^2$
 $= np + np^2 - np^2 - np^2$
 $= np - np^2 = np(1-p) = \frac{7}{4} > 0$

Example (7) If M.g.f. of X is $M_X(t) = (\frac{1}{3} + \frac{2}{3}e^t)^5$,

Find $P(0 \leq X \leq 4)$, $P(1 \leq X \leq 3)$, $P(X > 3)$, $P(X \leq 2)$,

$P(X \leq 1)$, $P(X = 2 \text{ or } 3)$.

Sol. $X \sim b(5, \frac{2}{3})$

$\therefore f(x) = \begin{cases} \binom{5}{x} (\frac{2}{3})^x (\frac{1}{3})^{5-x} & \text{for } x=0,1,\dots,5 \\ 0 & \text{o.w.} \end{cases}$

$P(0 \leq X \leq 4) = \sum_{x=1}^4 f(x) = f(1) + f(2) + f(3) + f(4)$ (continue ...)

$P(1 \leq X \leq 3) = \sum_{x=1}^3 f(x) = f(1) + f(2) + f(3)$ (continue ...)

$P(X \geq 2) = \sum_{x=2}^5 f(x)$ or $= 1 - P(X < 2) = 1 - [f(0) + f(1)]$

$P(X \leq 1) = \sum_{x=0}^1 f(x) = f(0) + f(1)$ (continue ...)

$P(X = 2 \text{ or } X = 3) = P(X = 2) + P(X = 3)$ (Continue the solution)
 $= f(2) + f(3)$

Example ⑧: If the M.g.f of X is

$$M_X(t) = \left(\frac{2}{3} + \frac{1}{3} e^t\right)^9$$

Show that $P(\mu_x - 2\sigma_x < X < \mu_x + 2\sigma_x) = \sum_{x=1}^5 \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$.

Sol. $\because X \sim b(9, \frac{1}{3})$

$$\because \mu_x = np = \frac{9}{3} = 3$$

$$\sigma_x^2 = np(1-p) = \frac{9 \cdot 2}{3 \cdot 3} = 2$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{2} \approx 1.4$$

$$P(3 - 2\sqrt{2} < X < 3 + 2\sqrt{2}) \approx P(1 \leq X \leq 5) = \sum_{x=1}^5 f(x) \\ = \sum_{x=1}^5 \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

Example ⑨: If X is $b(n, p)$ show that:

$$E\left(\frac{X}{n}\right) = p, \quad E\left(\left(\frac{X}{n} - p\right)^2\right) = \frac{p(1-p)}{n}$$

Sol. $\because X \sim b(n, p)$

$$\because E(X) = np \Rightarrow E\left(\frac{X}{n}\right) = \frac{1}{n} E(X) = \frac{1}{n} np = p$$

$$E\left[\left(\frac{X}{n} - p\right)^2\right] = E\left[\frac{X^2}{n^2} - 2\frac{Xp}{n} + p^2\right] \\ = \frac{1}{n^2} E(X^2) - \frac{2p}{n} E(X) + p^2 \quad (\text{since } n, p, 2 \text{ constants}) \\ = \frac{1}{n^2} (n^2 p^2 + np - np^2) - \frac{2p}{n} (np) + p^2 \\ = \cancel{p^2} + \frac{p}{n} - \frac{p^2}{n} - \cancel{2p^2} + p^2 = \frac{p(1-p)}{n}$$

Example ⑩: Let $X \sim b(2, p)$ & $Y \sim b(4, p)$. If $P(X \geq 1) = \frac{5}{9}$,

find $P(Y \geq 1)$.

Sol. $\because X \sim b(2, p)$

$$\because f(x, 2, p) = \begin{cases} \binom{2}{x} p^x (1-p)^{2-x} & \text{for } x=0, 1, 2 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \geq 1) = \frac{5}{9} \Rightarrow 1 - P(X < 1) = \frac{5}{9} \Rightarrow 1 - P(X=0) = \frac{5}{9}$$

$$f(0) = \frac{4}{9} \Rightarrow \binom{2}{0} p^0 (1-p)^2 = \frac{4}{9}$$

$$\Rightarrow (1-p)^2 = \frac{4}{9} \Rightarrow 1-p = \pm \frac{2}{3} \\ \Rightarrow p = 1 + \frac{2}{3} = \frac{5}{3} > 1 \text{ (discarded)} \\ p = 1 - \frac{2}{3} = \frac{1}{3} < 1$$

$$p = \frac{1}{3} \Rightarrow 0 < \frac{1}{3} < 1 \Rightarrow p = \frac{1}{3} \text{ only.}$$

$$\therefore Y \sim b(4, p)$$

$$f(y, 4, p) = \begin{cases} \binom{4}{y} p^y (1-p)^{4-y} & y = 0, 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \binom{4}{y} \left(\frac{1}{3}\right)^y \left(\frac{2}{3}\right)^{4-y} & y = 0, 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases}$$

$$P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y = 0) = 1 - f(0)$$

$$= 1 - \binom{4}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 = 1 - \left(\frac{2}{3}\right)^4 = 1 - \frac{16}{81} = \frac{65}{81} < 1$$

Note: If Bernoulli experiment have occurred once only (it have not repeated) is called "Bernoulli distribution"

$$X \sim b(1, p)$$

$$f(x, 1, p) = \begin{cases} p^x (1-p)^{1-x} & \text{for } x = 0, 1 \\ 0 & \text{o.w.} \end{cases}$$

$f(x)$ is called Bernoulli function.

Ex. 1 Tossing a coin once, Let X be the r.v. of head appears.

$P(X) = p = \frac{1}{2}$ for $x = 1$, i.e. when the head is appeared.

$P(X) = 1 - p = \frac{1}{2}$ for $x = 0$, i.e. when the tail is appeared.

The p.m.f. of each trail is

$$f(x) = \begin{cases} p^x (1-p)^{1-x} & x = 0, 1 \\ 0 & \text{o.w.} \end{cases}$$

Ex. 2 An Urn Contains 10 red and 20 white balls. Draw five balls at random from the urn, one at a time and with replacement. Let the draw of red ball be considered success.

Sol.

$P(X) = p = \frac{1}{3}$ when the red ball is success.

$1 - P(X) = \frac{2}{3}$ when the white ball is failure.

$$f(x) = \begin{cases} p^x (1-p)^{1-x} & \text{for } x=0,1 \\ 0 & \text{o.w.} \end{cases}$$
$$= \begin{cases} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{1-x} & \text{for } x=0,1 \\ 0 & \text{o.w.} \end{cases}$$

$\therefore X \sim \text{Bernoulli dist.}$
 $X \sim b(1, \frac{1}{3})$

Ex. 3 A die is tossed once, call the outcome a success if a six is rolled, and all other outcomes being considered failures.

Sol. We have a Bernoulli experiment with pr. $p = \frac{1}{6}$.

$P(X) = \frac{1}{6}$ if the six is rolled.

$1 - P(X) = \frac{5}{6}$ if the non six is rolled.

$\therefore X \sim \text{Bernoulli dist. (i.e. } X \sim b(1, \frac{1}{6})$

$$f(x) = \begin{cases} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{1-x} & \text{for } x=0,1 \\ 0 & \text{o.w.} \end{cases}$$

H.w. ① $Y \sim b(n, \frac{1}{4})$, Find the smallest (n) such that $P(1 \leq Y) \geq 0.7$.

② An Urn contains 7 red and 11 white balls. Draw one ball at random from the urn. Let $X=1$ if a red ball is drawn and let $X=0$ if a white ball is drawn. Find the p.m.f of X .



(2) Geometric Distribution

التوزيع الهندسي

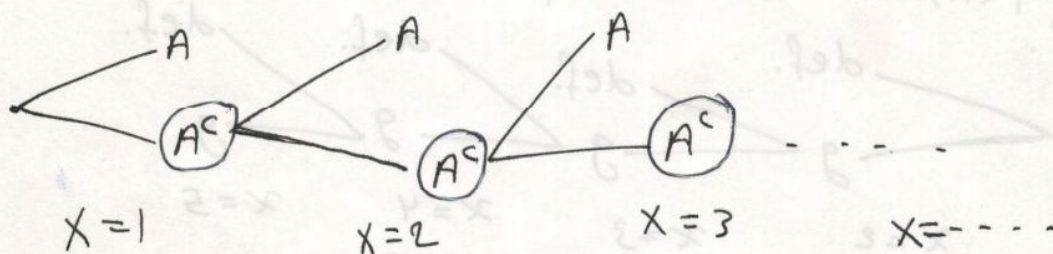
Given a Bernolli experiment that has only two events A & A^c .

$$\text{Let } P(A) = p \quad 0 < p < 1$$

$$P(A^c) = 1 - p = q \quad 0 < q < 1$$

Let the r.v. X be a number of experiment until event A happens for the first time.

For the first time.



$$\therefore X = 1, 2, 3, \dots \quad (\text{d.r.v.})$$

Then the function of X is:

$$f(x; p) = \begin{cases} pq^{x-1} & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

* Know to show that $f(x; p)$ is a p.m. f. of X ?

Condition ①

$$x = 1, 2, 3, \dots > 0$$

$$x-1 = 0, 1, 2, \dots \geq 0$$

$$0 < p < 1 \rightarrow p > 0$$

$$0 < q < 1 \rightarrow q^{x-1} > 0$$

$$\left. \begin{aligned} f(x) > 0 & \text{ for } x = 1, 2, \dots \\ = 0 & \text{ o.w.} \end{aligned} \right\} \Rightarrow \therefore f(x) \geq 0$$

$$\text{Condition ②} \quad \text{T.P.} \quad \sum_{x=1}^{\infty} f(x) = 1$$

$$\text{i.e.} \quad \sum_{x=1}^{\infty} pq^{x-1} = 1$$

$$\sum_{x=1}^{\infty} p q^{x-1} = p [1 + q + q^2 + q^3 + \dots] \rightarrow \text{Geometric Series is Conv.}$$

$$= p \cdot \left[\frac{1}{1-q} \right]$$

$$|q| < 1$$

$$= \cancel{p} \cdot \frac{1}{\cancel{p}} = 1 \quad (\text{since } 1-q=p)$$

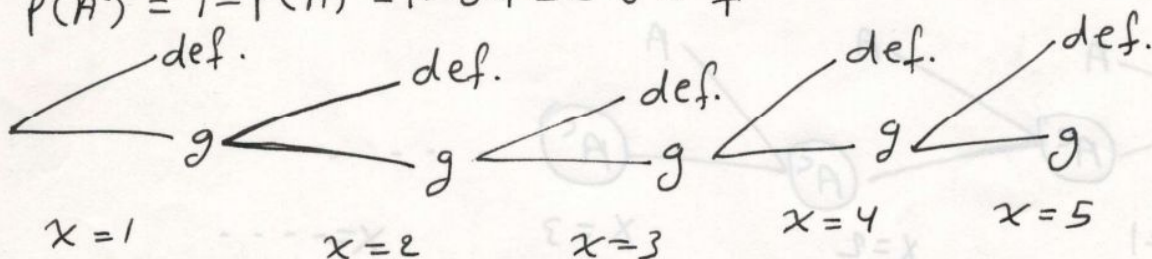
$\therefore f(x)$ is a p.m.f.

Example Suppose the pr. to choose one defective tube is (0.4). Find the pr. to get one defective tube after at most (4) good tubes are chosen.

sol. Let A: to get def. tube.

$$P(A) = 0.4 = p$$

$$P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6 = q$$



$$P(\text{ggggd}) = P(X=5) = f(X=5) = f(5)$$

$$\therefore X \sim G(p) \Rightarrow X \sim G(0.4)$$

$$P(X=5) = f(5; 0.4) = p q^{x-1} = (0.4)(0.6)^{5-1} = (0.4)(0.6)^4$$

Theorem ③ If X has a geometric dist. with parameter p , then the M.g.f. of X is

$$M_X(t) = \frac{pe^t}{1 - qe^t}, \text{ for } t < \ln\left(\frac{1}{q}\right)$$

Proof: $\therefore X \sim G(p)$

$$f(x; p) = \begin{cases} p q^{x-1} & \text{for } x=1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} p q^{x-1} = \frac{p}{q} \sum_{x=1}^{\infty} (q e^t)^x$$

$$M_X(t) = \frac{p}{q} [q e^t + (q e^t)^2 + (q e^t)^3 + \dots + (q e^t)^n + \dots]$$

$$= \frac{p}{q} (q e^t) [1 + (q e^t) + (q e^t)^2 + \dots + (q e^t)^{n-1} + \dots]$$

$r = q e^t$, then this series is convergent only where

$$|qe^t| < 1 \rightarrow qe^t < 1 \rightarrow e^t < \frac{1}{q} \rightarrow t < \ln\left(\frac{1}{q}\right)$$

$$\therefore \boxed{t < \ln\left(\frac{1}{q}\right)}$$

$$M_X(t) = pe^t [1 + qe^t + (qe^t)^2 + \dots + (qe^t)^{n-1} + \dots] \quad \text{--- (1)}$$

$$qe^t M_X(t) = pe^t [qe^t + (qe^t)^2 + \dots + (qe^t)^n + \dots] \quad \text{--- (2)}$$

$$M_X(t) - qe^t M_X(t) = \lim_{n \rightarrow \infty} pe^t [1 - (qe^t)^n]$$

$$(1 - qe^t) M_X(t) = pe^t \left[1 - \lim_{n \rightarrow \infty} q^n (e^t)^n \right]$$

$$= pe^t [1 - 0] = pe^t \quad \left(\text{since } \lim_{n \rightarrow \infty} q^n = 0, 0 < q < 1 \right)$$

$$(1 - qe^t) M_X(t) = pe^t$$

$$\therefore M_X(t) = \frac{pe^t}{1 - qe^t}; \quad t < \ln\left(\frac{1}{q}\right)$$

Theorem (4) If X has a geometric dist. then:

$$E(X) = M_X' = \frac{1}{p}; \quad \text{Var}(X) = \frac{q}{p^2} \quad 0 < p < 1$$

Proof: $X \sim G(p)$

$$M_X(t) = \frac{pe^t}{1 - qe^t} \quad t < \ln\left(\frac{1}{q}\right)$$

$$M_X'(t) = \frac{(1 - qe^t)pe^t + pe^t(qe^t)}{(1 - qe^t)^2} = \frac{pe^t}{(1 - qe^t)^2}$$

$$E(X) = M_X'(0) = \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$M_X''(t) = \frac{(1 - qe^t)^2 pe^t + 2pe^t(qe^t)(1 - qe^t)}{(1 - qe^t)^4}$$

$$M_X''(0) = E(X^2) = \frac{2 - p}{p^2}$$

$$\therefore \text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{2 - p}{p^2} - \frac{1}{p^2} = \frac{1 - p}{p^2} = \frac{q}{p^2}$$

Theorem (5) If X has a geometric distribution with parameter p , then $P(X > k+j | X > k) = P(X > j)$ where $k, j \in \mathbb{I}$.

Proof:

$(X > k+j)$: A happens after $(k+j)$ experiment.

A^c happens in all $(k+j)$ experiment.

$$P(X > k+j) = [P(A^c)]^{k+j} = q^{k+j}$$

$(X > k)$: A happens after (k) exp.

A^c happens in all (k) exp.

$$P(X > k) = [P(A^c)]^k = q^k$$

$$P(X > k+j | X > k) = \frac{P(AB)}{P(B)}$$

AB : $X > k+j$

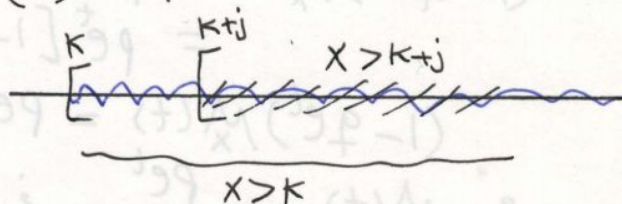
$$P(AB) = P(X > k+j) = q^{k+j}$$

B : $X > k$

$$P(B) = P(X > k) = q^k$$

$$P(X > k+j | X > k) = \frac{q^{k+j}}{q^k} = q^j = [P(A^c)]^j = P(X > j)$$

Memoryless Property
خاصية فقدان الذاكرة



Example (1): If X has a geometric with $p = \frac{1}{4}$,

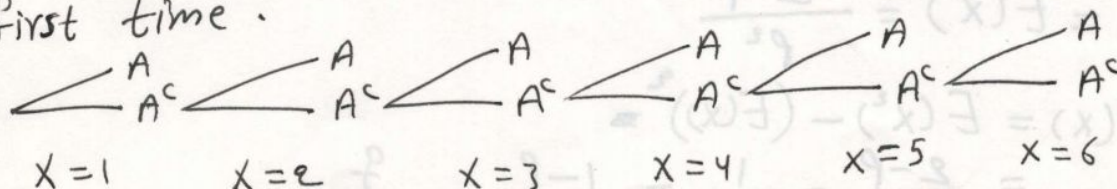
Find $P(X > 8 | X > 3)$

Sol. by Th. (5):

$$P(X > 8 | X > 3) = P(X > 5) = (q)^5 = \left(\frac{3}{4}\right)^5$$

Example (2): If X has a geometric distribution with parameter p , show that $P(X > 5 | X > 3) = P(X > 2)$

X = number of experiment until A happens for the first time.



$$P(A^c A^c A^c A^c A^c) = [P(A^c)]^5 = q^5$$

$(X > 5)$; A happens after (5) experiment; A^c happens in all (5) experiment.

$$P(X > 5) = [P(A^c)]^5 = q^5$$

$(X > 3)$: A happens after (3) experiment.

A^c happens in all (3) experiment.

$$P(X > 3) = [P(A^c)]^3 = q^3$$

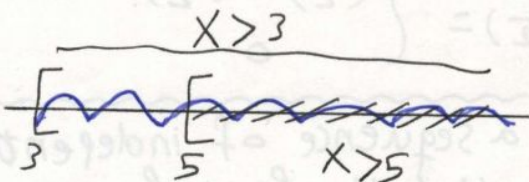
$$P(X > 5 | X > 3) = \frac{P(AB)}{P(B)}$$

AB : $(X > 5)$ & B : $(X > 3)$

$$P(AB) = P(X > 5) = q^5$$

$$P(B) = P(X > 3) = q^3$$

$$P(X > 5 | X > 3) = \frac{q^5}{q^3} = q^2 = [P(A^c)]^2 = P(X > 2)$$



Exercises about Geometric Distribution

Q1: If X has a M.g.f. as follows:

$$M_X(t) = \frac{2e^t}{5(1 - \frac{3}{5}e^t)} \quad \text{for } t < \ln(\frac{5}{3})$$

Then find $P(X > 7 | X > 3)$.

Sol. $M_X(t) = \frac{(\frac{2}{5})e^t}{(1 - \frac{3}{5}e^t)} = \frac{pe^t}{1 - qe^t}$

$$\therefore X \sim G(\frac{2}{5}) ; (p = \frac{2}{5})$$

$$\therefore P(X > 7 | X > 3) = P(X > 4) = q^4 = (\frac{3}{5})^4 = \frac{81}{625} < 1$$

Q2: If X has a p.m.f:

$$f(x) = \begin{cases} (\frac{2}{3})(\frac{1}{3})^{x-1} & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Find $P(X > 5 | X > 2)$

Sol. $\therefore X \sim G(\frac{2}{3}) ; p = \frac{2}{3}$

$$P(X > 5 | X > 2) = P(X > 3) = q^3 = (\frac{1}{3})^3 = \frac{1}{27}$$

Q3: Suppose we flip a fair coin until we get a head.

نقلب العملة المتعادلة

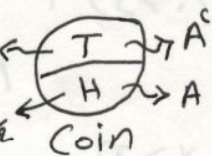
Let X be the number of trials to get a head. Find the probability mass function of X ?

Sol.

$$p(x) = f(x; p) = \begin{cases} q^{x-1} p & x = 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

parameter of distribution

$x = 1, 2, \dots$ $p = \frac{1}{2}$ $q = \frac{1}{2}$ Coin



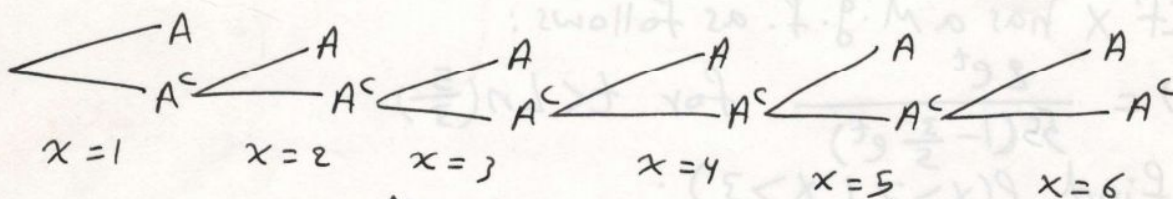
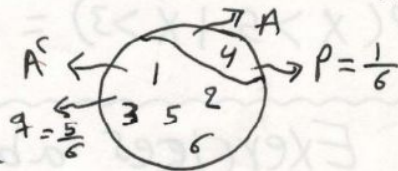
then

$$f(x; \frac{1}{2}) = \begin{cases} (\frac{1}{2})^{x-1} (\frac{1}{2}) & x = 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

Q4: In a sequence of independent rolls of a fair die, we find that the first four is observed on the sixth trial. Find the p.m.f. of the 6th trial.

Sol. $A = \{d = 4\} \Rightarrow P(A) = \frac{1}{6}$

$$A^c = \{d \neq 4 \mid d \leq 6\} \Rightarrow P(A^c) = \frac{5}{6}$$



p (the first four is observed in the sixth trial) = $P(X=6)$

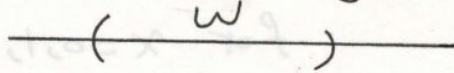
$$P(X=6) = f(6) = \begin{cases} (\frac{5}{6})^{6-1} (\frac{1}{6}) & \text{for } X=6 \\ 0 & \text{o.w.} \end{cases}$$

$$f(6) = (\frac{5}{6})^5 (\frac{1}{6}) = 0.067 \in [0, 1]$$



③ Poisson Distribution توزيع بواسون

Given an interval of length w , $w > 0$

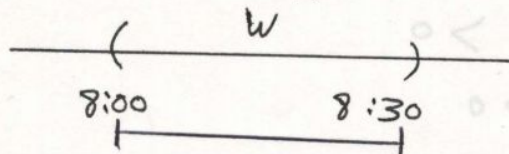


Let A be an event happen in w .

Note: w is interval of time or space or distance ... etc.

Example ①

A : Car accident that happens in interval $(8:00, 8:30)$



w : 30 minutes = interval of times.

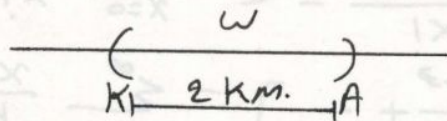
$X \equiv$ number of car accident in $w = 30$ minutes.

Example ②

A : Car accident between Kadimya and Adamia.

$X \equiv$ number of car accident in $w = 2$ Km.

$w =$ interval of distance = 2 Km.



Example ③

A : typing errors in one page.

$X \equiv$ number of typing errors in one page (w)

$w \equiv$ interval of space

Know:

If $X =$ number of events A that happens in interval of length, then the function of X is:

$$f(x) = \begin{cases} \frac{e^{-\lambda w} (\lambda w)^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

where $\lambda \equiv$ number of event A, when $w=1$ & $w>0, \lambda>0$

Let $m = \lambda \cdot w$; $m>0$, $w>0$

$$f(x; m) = \begin{cases} \frac{e^{-m} (m)^x}{x!} & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

where (m) is a parameter.

Know, to show that $f(x; m)$ is a p.m.f.

Cond. ① T.P $f(x; m) \geq 0 \quad \forall x \in \mathbb{R}_x$

$m>0$, $x \geq 0$, $x! > 0$

$(m)^x > 0$, $e^{-m} = \frac{1}{e^m} > 0$

$$\therefore f(x; m) = \begin{cases} \frac{e^{-m} (m)^x}{x!} > 0 & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

Cond. ② T.P $\sum_{x=0}^{\infty} f(x; m) = 1$

$$\sum_{x=0}^{\infty} f(x; m) = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!}$$

$$\therefore e^x = \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right] = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\therefore \sum_{x=0}^{\infty} f(x; m) = e^{-m} \cdot e^m = 1$$

Cond. ① & Cond. ② are satisfied; then $f(x; m)$ is p.m.f.

Note: $f(x; m)$ is called Poisson p.m.f.

Theorem (6): If X has a poisson dist. with parameter (m) , then the M.g.f. of X is:

$$M_X(t) = e^{m(e^t - 1)} \quad ; \quad -\infty < t < \infty$$

Proof: $\therefore X \sim P(m)$

$$\therefore f(x; m) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} M_X(t) &= E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f(x|m) \\ &= \sum_{x=0}^{\infty} e^{tx} \cdot e^{-m} \frac{m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!} \end{aligned}$$

$$\text{By } e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$M_X(t) = e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!} = e^{-m} \cdot e^{met} = e^{m(e^t-1)} \quad -\infty < t < \infty$$

Theorem (7): If X has a poisson dist. with parameter (m); then $\mu_X = E(X) = m$ & $\sigma_X^2 = \text{Var}(X) = m$.

Proof: $\circ \circ X \sim p(m)$

$$\circ \circ M_X(t) = e^{m(e^t-1)} \quad ; \quad -\infty < t < \infty$$

$$M_X'(t) = e^{m(e^t-1)} \cdot me^t$$

$$M_X'(0) = E(X) = 1 \cdot m \cdot 1 = m \Rightarrow \circ \circ E(X) = \mu_X = m$$

$$M_X''(t) = e^{m(e^t-1)} \cdot (me^t) + (me^t) e^{m(e^t-1)} \cdot (me^t)$$

$$M_X''(0) = E(X^2) = m + m^2$$

$$\begin{aligned} \sigma_X^2 &= \text{Var}(X) = E(X^2) - (E(X))^2 \\ &= m + m^2 - m^2 = m \Rightarrow \sigma_X^2 = m \end{aligned}$$

H.w.

① If $\mu_X = \sigma_X^2 = 2$; Find $P(X \geq 1)$, $M_X(t)$, $P(X=0)$

② Given $M_X(t) = e^{4(e^t-1)}$ for $-\infty < t < \infty$

Show that:

$$P(\mu_X - 2\sigma_X < X < \mu_X + 2\sigma_X) = 0.93$$

Examples

→
about Poisson distribution

Example ① The average number of homicides per day in (Los Angeles) Country is (2). Use the Poisson dist. to determine the pr. that on a given days:

- a. There will be three or less homicides in (Los Angeles) Country.
 b. There will be exactly three homicides.

Sol. $m = 2$; $w = 1$ day ; $X \equiv$ no. of homicides in a day. عدد القتل في كل يوم

$$\therefore X \sim P(2)$$

$$f(x; 2) = \begin{cases} \frac{e^{-2} 2^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$a. P(X \leq 3) = \sum_{x=0}^3 f(x) = f(0) + f(1) + f(2) + f(3)$$

$$b. P(X=3) = f(3) = \frac{e^{-2} 2^3}{3!} = \frac{8 e^{-2}}{6} = \frac{4}{3} e^{-2}$$

Example ② The average number of customers entering a store is (4) per minute. Assuming the number of customers has a Poisson dist. Find the pr. that at least (2) customers enter the store in ($\frac{1}{2}$) minute. معدل الزبائن الداخلين

Sol. $X \equiv$ number of customers enter the store. عدد الزبائن الذين يدخلون المحل

Given $m = 4$, $w = 1$ in one minute.

$$m = \lambda \cdot w$$

$$4 = \lambda \cdot 1 \Rightarrow \lambda = 4$$

if $w = \frac{1}{2}$; find $P(X \geq 2)$?

$$m = \lambda \cdot w$$

$$= 4 \cdot (\frac{1}{2}) = 2 \Rightarrow m = 2$$

$$X \sim P(2)$$

$$\therefore f(x; 2) = \begin{cases} \frac{e^{-2} 2^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) = 1 - P[(X=0) \cup (X=1)] \\ &= 1 - [P(X=0) + P(X=1)] \\ &= 1 - [f(0) + f(1)] = 1 - [e^{-2} + 2e^{-2}] \\ &= 1 - 3e^{-2} \end{aligned}$$

Example ③ The average number of defects on a tape is (4) per. (1000) feet.

Assuming that the number of defects has a poisson dist.

a. Find the pr. that a tape of long (2400) ft. has (2) defects.

b. Find the pr. that a tape of long (1200) ft. has no defects.

Sol. X = number of defects on a tape.

Given $m = 4$, $w = 1000$

$$m = \lambda \cdot w$$

$$4 = \lambda \cdot (1000) \Rightarrow \lambda = \frac{1}{250} = .004$$

a. if $w = 2400$; find $P(X=2)$

$$m = \lambda \cdot w$$

$$= \frac{1}{250} \cdot (2400) = 9.6 \Rightarrow \therefore m = 9.6$$

$$\therefore X \sim P(9.6)$$

$$f(x; 9.6) = \begin{cases} \frac{e^{-9.6} (9.6)^x}{x!} & ; x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X=2) = f(2) = \frac{e^{-9.6} (9.6)^2}{2!}$$

b. if $w = 1200$; find $P(X=0)$

$$m = \lambda \cdot w$$

$$= \frac{1}{250} \cdot (1200) = 4.8 \Rightarrow \therefore m = 4.8$$

$$\therefore X \sim P(4.8)$$

$$f(x; 4.8) = \begin{cases} \frac{e^{-4.8} (4.8)^x}{x!} & ; x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X=0) = f(0) = \frac{e^{-4.8} (4.8)^0}{0!} = e^{-4.8} \quad (\text{since } 0! = 1)$$

Example ④ The average number of phone calls in (5) per minute, find the pr. that there are (3) calls (15) per seconds.

Sol. X = number of phone calls in one minute.

Given $m = 5$ و $w = 1$ minute.

$$m = \lambda \cdot w$$

$$5 = \lambda \cdot 1 = 5 \Rightarrow \lambda = 5$$

if $w = 15$ seconds ; find $P(X=3)$?

$$w = \frac{15}{60} = \frac{1}{4} \text{ minute.}$$

$$m = \lambda \cdot w$$

$$= 5 \cdot \left(\frac{1}{4}\right) = \frac{5}{4} = 1.25 \Rightarrow m = 1.25$$

$$\therefore X \sim P(1.25)$$

$$f(x; 1.25) = \begin{cases} \frac{e^{-1.25} (1.25)^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X=3) = f(3) = \frac{e^{-1.25} (1.25)^3}{3!}$$

Example ⑤ The average number of defects in a roll of a certain type of wall paper is 2.5. Use the poisson dist. to determine the pr. that a roll will have (4) or more defects.

Sol. $X =$ number of defects in a roll of a type of wall paper.

$$\text{Given } m = 2.5$$

$$\therefore X \sim P(2.5)$$

$$f(x; 2.5) = \begin{cases} \frac{(2.5)^x e^{-2.5}}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} P(X \geq 4) &= 1 - P(X < 4) \\ &= 1 - \sum_{x=0}^3 f(x) = 1 - \sum_{x=0}^3 \frac{(2.5)^x e^{-2.5}}{x!} \\ &= 1 - [f(0) + f(1) + f(2) + f(3)] \end{aligned}$$

Example ⑥ The average number of traffic accidents that take place on the Baghdad Highway on a week day between 7:00 A.M. and 8:00 A.M. is (7) accident per hour. Use poisson dist. to determine the pr. that at least (2) traffic accidents would the Baghdad high way on morning between 7:00 A.M. & 8:00 A.M.

Sol.

$X \equiv$ number of traffic accidents that take place on the Baghdad high way on a week day between (7:00 - 8:00) A.m.

Given $m = 0.7$

$\therefore X \sim P(0.7)$

$$f(x; 0.7) = \begin{cases} \frac{e^{-0.7} (0.7)^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \leq 2) = \sum_{x=0}^2 f(x) = \sum_{x=0}^2 \frac{e^{-0.7} (0.7)^x}{x!} = f(0) + f(1) + f(2)$$

Example ⑦ A machine produces a certain item with (0.05) defectives. We select a sample of (100) item. Let the r.v. X be the number of defective. What is the pr. that no defective?

sol. $X \equiv$ number of defective in w .

$$w = 0.05 \quad ; \quad n = 100$$

$$m = n \cdot w \\ = (100) \cdot (0.05) = 5$$

$\therefore X \sim P(5)$

$$f(x; 5) = \begin{cases} \frac{e^{-5} 5^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X=0) = f(0) = \frac{e^{-5} 5^0}{0!} = e^{-5}$$

Example ⑧ A Car Salesman sells, on the average (2.5) cars per day. Use Poisson dist. to determine the pr. that on a given day the salesman would sell:

a. at least (4) cars, b. exactly (4) cars.

sol. $X =$ number of cars the salesman would sell.

Given $m = 2.5$

$$\therefore X \sim P(2.5)$$

$$f(x; 2.5) = \begin{cases} \frac{e^{-2.5} (2.5)^x}{x!} & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \text{a. } P(X \geq 4) &= 1 - P(X < 4) = 1 - \sum_{x=0}^3 f(x) \\ &= 1 - \sum_{x=0}^3 \frac{e^{-2.5} (2.5)^x}{x!} \\ &= 1 - (f(0) + f(1) + f(2) + f(3)) \end{aligned}$$

$$\text{b. } P(X=4) = f(4) = \frac{e^{-2.5} (2.5)^4}{4!}$$

Example 9 The average number of major fires per month in a certain city is (1.5). Use the Poisson dist. to determine the pr. that there will be exactly one major fire in a period of two months.

Sol.

عدد الحرائق الكبيرة كل شهر

$X \equiv$ number of major fires per month.

Given $m=1.5$, $w=1$ month

$$m = \lambda \cdot w$$

$$1.5 = \lambda \cdot 1 \Rightarrow \lambda = 1.5$$

if $w=2$; find $P(X=1)$?

$$m = \lambda \cdot w$$

$$= (1.5)(2) = 3 \Rightarrow m=3$$

$$\therefore X \sim P(3)$$

$$f(x; 3) = \begin{cases} \frac{e^{-3} 3^x}{x!} & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X=1) = f(1) = 3e^{-3}$$



Appendix B

Tables

TABLE I
The Poisson Distribution

$$\Pr(X \leq x) = \sum_{w=0}^x \frac{\mu^w e^{-\mu}}{w!}$$

x	$\mu = E(X)$											
	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
0	0.607	0.368	0.223	0.135	0.050	0.018	0.007	0.002	0.001	0.000	0.000	0.000
1	0.910	0.736	0.558	0.406	0.199	0.092	0.040	0.017	0.007	0.003	0.001	0.000
2	0.986	0.920	0.809	0.677	0.423	0.238	0.125	0.062	0.030	0.014	0.006	0.002
3	0.998	0.981	0.934	0.857	0.647	0.433	0.265	0.151	0.082	0.042	0.021	0.010
4	1.000	0.996	0.981	0.947	0.815	0.629	0.440	0.285	0.173	0.100	0.055	0.027
5		0.999	0.996	0.983	0.916	0.785	0.616	0.446	0.301	0.191	0.116	0.066
6		1.000	0.999	0.995	0.966	0.889	0.762	0.606	0.450	0.313	0.207	0.130
7			1.000	0.999	0.988	0.949	0.867	0.744	0.599	0.453	0.324	0.220
8				1.000	0.996	0.979	0.932	0.847	0.729	0.593	0.456	0.333
9					0.999	0.992	0.968	0.916	0.830	0.717	0.587	0.458
10					1.000	0.997	0.986	0.957	0.901	0.816	0.706	0.583
11						0.999	0.995	0.980	0.947	0.888	0.803	0.697
12						1.000	0.998	0.991	0.973	0.936	0.876	0.792
13							0.999	0.996	0.987	0.966	0.926	0.864
14							1.000	0.999	0.994	0.983	0.959	0.917
15								0.999	0.998	0.992	0.978	0.951
16								1.000	0.999	0.996	0.989	0.973
17									1.000	0.998	0.995	0.986
18										0.999	0.998	0.993
19										1.000	0.999	0.997
20											1.000	0.998
21												0.999
22												1.000

Chapter Six

⑦ Uniform Distribution

When X is d.r.v.

$$X \sim \text{unif}(K)$$

$$X = 1, 2, \dots, K$$

$$f(x) = \begin{cases} \frac{1}{K} & \text{for } x = 1, 2, \dots, K \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \sum_{\forall x} x f(x)$$

$$= \sum_{x=1}^K x \left(\frac{1}{K} \right)$$

$$= \frac{1}{K} \left[\frac{K(K+1)}{2} \right] \quad \text{when } \left(\sum_{x=1}^K x = \frac{K(K+1)}{2} \right)$$

$$= \left(\frac{K+1}{2} \right)$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{x=1}^K x^2 f(x)$$

$$= \left(\sum_{x=1}^K x^2 \right) \left(\frac{1}{K} \right)$$

$$= \frac{1}{K} \left[\frac{K(K+1)(2K+1)}{6} \right] ; \left[\sum_{x=1}^K x^2 = \frac{K(K+1)(2K+1)}{6} \right]$$

$$= \frac{(K+1)(2K+1)}{6},$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 \geq 0$$

$$\begin{aligned}\text{Var}(X) &= \frac{(K+1)(2K+1)}{6} - \frac{(K+1)^2}{4} \\ &= \frac{(K+1)(K-1)}{3}\end{aligned}$$

Continuous Probability Distributions

① Gamma Distribution (توزیع گاما)

Def. « Gamma function Γ »

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy ; \alpha > 0$$

by use integration by parts, we get :-

$$\boxed{\Gamma(\alpha) = (\alpha-1)!}$$

$X \sim G(\alpha, \beta)$; then the p.d.f. for Gamma dist. is :

$$f(x; \alpha, \beta) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^{\alpha}} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Theorem(8): If $X \sim G(\alpha, \beta)$, then the M.g.f. of X is :
 $M_X(t) = (1-\beta t)^{-\alpha}$

proof: $M_X(t) = E(e^{tx})$

$$\begin{aligned} &= \int_0^{\infty} e^{tx} \left(\frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^{\alpha}} \right) dx \\ &= \int_0^{\infty} \frac{x^{\alpha-1} e^{-x} \left(\frac{1-\beta t}{\beta} \right)}{\Gamma(\alpha) \beta^{\alpha}} dx \end{aligned}$$

$$\text{let } y = \frac{x(1-\beta t)}{\beta} ; x = \frac{\beta}{1-\beta t} y$$

$$dy = \frac{1-\beta t}{\beta} dx ; dx = \frac{\beta}{1-\beta t} dy$$

$$\begin{aligned} \therefore M_X(t) &= \int_0^{\infty} \frac{\left(\frac{\beta y}{1-\beta t} \right)^{\alpha-1} e^{-y}}{\Gamma(\alpha) \beta^{\alpha}} \left(\frac{\beta}{1-\beta t} \right) dy \\ &= \underbrace{\int_0^{\infty} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy}_{= \Gamma(\alpha)} \cdot \frac{\beta^{\alpha}}{\beta^{\alpha} (1-\beta t)^{\alpha}} = (1-\beta t)^{-\alpha} \quad \text{for } t < \frac{1}{\beta} \end{aligned}$$

Theorem (9): If $X \sim G(\alpha, \beta)$, then $\mu_x = E(X) = \alpha\beta$,
 $\sigma_x^2 = \text{Var}(X) = \alpha\beta^2$.

proof:

$$\because X \sim G(\alpha, \beta)$$

$$\because M_X(t) = (1 - \beta t)^{-\alpha}$$

$$M'_X(t) = +\alpha\beta(1 - \beta t)^{-\alpha-1} = \alpha\beta(1 - \beta t)^{-(\alpha+1)}$$

$$M'_X(0) = \alpha\beta(1-0)^{-(\alpha+1)} = \alpha\beta = \mu_x$$

$$\therefore \mu_x = \alpha\beta$$

$$M''_X(t) = -\alpha\beta(\alpha+1) \cdot (1 - \beta t)^{-\alpha+2} \cdot (-\beta)$$

$$= \alpha\beta^2(\alpha+1)(1 - \beta t)^{-\alpha+2}$$

$$M''_X(0) = \alpha\beta^2(\alpha+1) \cdot 1 = \alpha^2\beta^2 + \alpha\beta^2$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

$$\therefore \sigma_x^2 = \alpha\beta^2$$

Example ① Find the p.d.f. of X whose M.g.f. is:

$$M_X(t) = (1 - 3t)^{-2}$$

Sol. $\because X \sim G(2, 3)$

$$\because f(x; 2, 3) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^\alpha} = \frac{x e^{-\frac{x}{3}}}{1 \cdot (3)^2} = \frac{1}{9} x e^{-\frac{x}{3}}$$

Example ② Find the M.g.f. of X has a p.d.f.

$$f(x) = \begin{cases} \frac{x e^{-\frac{x}{2}}}{4} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Sol. $\because X \sim G(2, 2)$

$$\because M_X(t) = (1 - \beta t)^{-\alpha} = (1 - 2t)^{-2}$$

$$\alpha - 1 = 1$$

$$\therefore \alpha = 2$$

$$\beta = 2$$

Example ③ Given a.p.d.f. of x :

$$f(x) = \begin{cases} \frac{x^2 e^{-\frac{x}{3}}}{\Gamma(3)(27)} & ; \quad 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

Find the M.g.f. of X ?

Sol.

$$\alpha - 1 = 2 \Rightarrow \alpha = 3, \beta = 3$$

$$\therefore X \sim G(3, 3)$$

$$\begin{aligned} \therefore M_X(t) &= (1 - \beta t)^{-\alpha} \\ &= (1 - 3t)^{-3} \end{aligned}$$

Example ④ If $X \sim G(2, 1)$; find $P(X=0)$

$$\therefore X \sim G(2, 1)$$

$$\therefore f(x; 2, 1) = \frac{x^{2-1} e^{-\frac{x}{1}}}{\Gamma(2)(1)^2} = x e^{-x} \quad \text{for } x > 0$$

$$P(X=0) \neq f(0) = 0$$

$$\text{i.e. } P(X=0) = 0$$

So; find M_X & σ_X^2 :

$$M_X = \alpha \beta = (2)(1) = 2$$

$$\sigma_X^2 = \alpha \beta^2 = (2)(1)^2 = 2$$

$$M_X(t) = (1 - \beta t)^{-\alpha} = (1 - t)^{-2}$$

H.w

① If $X \sim G(3, 4)$; Find p.d.f. of X , M_X , σ_X^2 .

② Given M.g.f. of X : $M_X(t) = (1 - 2t)^{-6}$; find $f(x)$?

TABLE II
The Chi-Square Distribution*

$$\Pr(X \leq x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2} dw$$

r	Pr(X ≤ x)					
	0.01	0.025	0.050	0.95	0.975	0.99
1	0.000	0.001	0.004	3.84	5.02	6.63
2	0.020	0.051	0.103	5.99	7.38	9.21
3	0.115	0.216	0.352	7.81	9.35	11.3
4	0.297	0.484	0.711	9.49	11.1	13.3
5	0.554	0.831	1.15	11.1	12.8	15.1
6	0.872	1.24	1.64	12.6	14.4	16.8
7	1.24	1.69	2.17	14.1	16.0	18.5
8	1.65	2.18	2.73	15.5	17.5	20.1
9	2.09	2.70	3.33	16.9	19.0	21.7
10	2.56	3.25	3.94	18.3	20.5	23.2
11	3.05	3.82	4.57	19.7	21.9	24.7
12	3.57	4.40	5.23	21.0	23.3	26.2
13	4.11	5.01	5.89	22.4	24.7	27.7
14	4.66	5.63	6.57	23.7	26.1	29.1
15	5.23	6.26	7.26	25.0	27.5	30.6
16	5.81	6.91	7.96	26.3	28.8	32.0
17	6.41	7.56	8.67	27.6	30.2	33.4
18	7.01	8.23	9.39	28.9	31.5	34.8
19	7.63	8.91	10.1	30.1	32.9	36.2
20	8.26	9.59	10.9	31.4	34.2	37.6
21	8.90	10.3	11.6	32.7	35.5	38.9
22	9.54	11.0	12.3	33.9	36.8	40.3
23	10.2	11.7	13.1	35.2	38.1	41.6
24	10.9	12.4	13.8	36.4	39.4	43.0
25	11.5	13.1	14.6	37.7	40.6	44.3
26	12.2	13.8	15.4	38.9	41.9	45.6
27	12.9	14.6	16.2	40.1	43.2	47.0
28	13.6	15.3	16.9	41.3	44.5	48.3
29	14.3	16.0	17.7	42.6	45.7	49.6
30	15.0	16.8	18.5	43.8	47.0	50.9

* This table is abridged and adapted from "Tables of Percentage Points of the Incomplete Beta Function and of the Chi-Square Distribution," *Biometrika*, 32 (1941). It is published here with the kind permission of Professor E. S. Pearson on behalf of the author, Catherine M. Thompson, and of the Biometrika Trustees.

② Chi-Square Distribution (توزیع کای)

Chi-Square dist. is a special case of Gamma dist. where $\alpha = \frac{r}{2}$, r is positive integer and $\beta = 2$.

The new dist. is called "Chi-Square dist." which is denoted by $\chi^2(r)$, where (r) is called "degree of freedom" (d.f.). Therefore; if $X \sim \chi^2(r)$, then the p.d.f. of X is as follows:

$$f(x; r) = \begin{cases} \frac{x^{\frac{r}{2}-1} e^{-\frac{x}{2}}}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Also; the M.g.f. of X is:

$$M_X(t) = (1-2t)^{-\frac{r}{2}} \quad ; \quad t < \frac{1}{2}$$

$$E(X) = M_X' = \alpha\beta = \left(\frac{r}{2}\right) \cdot 2 = r$$

$$\text{Var}(X) = \alpha^2\beta^2 = \left(\frac{r}{2}\right) \cdot 4 = 2r$$

Example ① If X is a.r.v. with p.d.f.:

$$f(x) = \begin{cases} \frac{x e^{-\frac{x}{2}}}{4} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find M.g.f. of X .

Sol. $\because X \sim G(2, 2)$

$$\therefore \frac{r}{2} = \alpha \Rightarrow 2 = \frac{r}{2} \rightarrow r = 4$$

$$\therefore M_X(t) = (1-2t)^{-\frac{r}{2}} = (1-2t)^{-\frac{4}{2}} = (1-2t)^{-2}$$

$$\therefore X \sim \chi^2(4)$$

$$\therefore M_X = r = 4, \quad \sigma_X^2 = 2r = 8$$

Example ② If X is a.r.v. with a m.g.f. :

$$M_X(t) = (1-2t)^{-7} \quad ; t < \frac{1}{2}$$

then find the p.f. of X .

Sol.

$$\beta = 2, \alpha = 7 ; \frac{r}{2} = 7 \rightarrow r = 14$$

$$\therefore X \sim \chi^2(14)$$

$$\therefore f(x; r) = \begin{cases} \frac{x^6 \cdot e^{-\frac{x}{2}}}{\Gamma(7) 2^7} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

H.w.

① $X \sim \chi^2(7)$; Find $M_X(t)$, $f(x)$, σ_x^2 , M_x .

② $X \sim \chi^2(5)$; Find $f(x)$, $M_X(t)$, σ_x^2 , M_x .

Table of Chi-Square Distribution

$$P(X \leq x) = \int_0^x \left(\frac{x^{\frac{r}{2}-1} e^{-\frac{x}{2}}}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} \right) dx$$

d.f.

(r)	.001	0.025	0.05	0.95	0.975	.99
1	0.000	0.001	0.004	3.84	5.02	6.63
2	0.020	0.051	0.103	5.99	7.38	9.21
3	0.115	0.216	0.352	7.81	9.35	11.3
4	0.297	0.484	0.711	9.49	11.1	13.3
5	0.554	0.831	1.15	11.1	12.8	15.1
6	0.872	1.24	1.64	12.6	14.4	16.8
7	1.24	1.69	2.17	14.1	16.0	18.5
8	1.65	2.18	2.73	15.5	17.5	20.1
9	2.09	2.70	3.33	16.9	19.0	21.7
10	2.56	3.25	3.94	18.3	20.5	23.2

∴ Note: $P(a \leq x \leq b) = P(X \leq b) - P(X \leq a)$

Example ③ If $X \sim \chi^2(10)$, then find:
 $P(3.25 \leq X \leq 20.5)$

Sol.

$$\begin{aligned} P(3.25 \leq X \leq 20.5) &= P(X \leq 20.5) - P(X \leq 3.25) \\ &= 0.975 - 0.025 \\ &= 0.950 \in [0,1] \end{aligned}$$

Example ④ Let $Y \sim \chi^2(3)$; find $P(0.352 \leq Y \leq 11.3)$.

Sol.

$$Y \sim \chi^2(3) \quad ; r=3$$

$$\begin{aligned} P(0.352 \leq Y \leq 11.3) &= P(Y \leq 11.3) - P(Y \leq 0.352) \\ &= 0.99 - 0.05 \\ &= 0.94 \in [0,1] \end{aligned}$$

Example ⑤ If $M_X(t) = (1-2t)^{-6}$ for $t < \frac{1}{2}$; find

① $P(X \leq 4.4)$

② $P(5.23 \leq X \leq 26.2)$

Sol. $M_X(t) = (1-2t)^{-6} \Rightarrow \beta = 2 \text{ \& } \alpha = 6 ; \frac{r}{2} = 6 \Rightarrow r=12$
 $\therefore X \sim \chi^2(12)$

① $P(X \leq 4.4) = 0.025$

$$\begin{aligned} \text{② } P(5.23 \leq X \leq 26.2) &= P(X \leq 26.2) - P(X \leq 5.23) \\ &= 0.99 - 0.050 \\ &= 0.94 \in [0,1] \end{aligned}$$

Example ⑥ If X has a p.d.f.:

$$f(x) = \begin{cases} \frac{x^2}{16} e^{-\frac{x}{2}} & 0 \leq x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find: ① $P(X \leq 0.872)$ ② $P(1.24 \leq X \leq 16.8)$

Sol. $\beta = 2, \alpha = 2 \Rightarrow \alpha = 3 = \frac{r}{2} \rightarrow r = 6$
 $\therefore X \sim \chi^2(6)$

① $P(X \leq 0.872) = 0.01$

$$\begin{aligned} \text{② } P(1.24 \leq X \leq 16.8) &= P(X \leq 16.8) - P(X \leq 1.24) \\ &= 0.99 - 0.025 = 0.965 \in [0,1] \end{aligned}$$

Lemma: If $X \sim \chi^2(\frac{\alpha}{2})$, $\alpha = \frac{Y}{2}$, $\beta \geq 0$, then the
r.v. $Y = \frac{2X}{\beta} \sim \chi^2(r)$.

Example: Find $P(3.28 < X < 25.2)$

$$\alpha = 3, \beta = 4$$

$$\text{let } Y = \frac{2X}{\beta}, \quad r = 2\alpha = 6$$

$$\therefore X = \frac{Y\beta}{2} = \frac{Y \cdot 4}{2} = 2Y$$

$$\begin{aligned} P(3.28 < X < 25.2) &= P(3.28 < 2Y < 25.2) \\ &= P(1.64 < Y < 12.6) \\ &= P(Y \leq 12.6) - P(Y \leq 1.64) \\ &= 0.95 - 0.05 \\ &= 0.90 \end{aligned}$$

Example ⑦ $X \sim \chi^2(5)$; Find:

$$\begin{aligned} P(1.145 \leq X \leq 12.83) &= P(X \leq 12.83) - P(X \leq 1.145) \\ &= 0.975 - 0.05 \\ &= 0.925 \end{aligned}$$

$$\begin{aligned} \forall P(X > 15.09) &= 1 - P(X \leq 15.09) \\ &= 1 - 0.99 \\ &= 0.01 \end{aligned}$$

Q: If $X_i (i=1, 2, 3, \dots, n)$ are independent r.v.s such that
 $X_i \sim G(a_i, b)$, then show that:

$$Y = \sum_{i=1}^n X_i \sim G\left(\sum_{i=1}^n a_i, b\right)$$

$$\begin{aligned} \text{proof: } \mu_Y(t) &= E(e^{ty}) = [E(e^{(\sum X_i)t})] = E(e^{X_1 t + X_2 t + \dots + X_n t}) \\ &= E(e^{X_1 t}) \cdot E(e^{X_2 t}) \cdot \dots \cdot E(e^{X_n t}) \quad (\text{since } X_i \text{ are indep.}) \\ &= \mu_{X_1}(t) \cdot \mu_{X_2}(t) \cdot \dots \cdot \mu_{X_n}(t) \end{aligned}$$

$$\therefore X_i \sim G(a_i, b) \Rightarrow \mu_{X_i}(t) = (1 - bt)^{-a_i}$$

→

$$\begin{aligned} \circ \circ \quad M_Y(t) &= M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t) \\ &= (1 - b\bar{t})^{a_1} (1 - b\bar{t})^{a_2} \cdots (1 - b\bar{t})^{a_n} \\ &= (1 - b\bar{t})^{\sum_{i=1}^n a_i} \end{aligned}$$

$$\therefore \beta = b, \alpha = \sum_{i=1}^n a_i$$

$$\circ \circ \quad Y \sim G\left(\underbrace{\sum_{i=1}^n a_i}_{\alpha}, b\right) \rightarrow \beta$$

③ Exponential Distribution

التوزيع الأسي

Exponential dist. is a special case of Gamma dist., when $\alpha = 1$, $\beta = \frac{1}{\lambda}$ ($\lambda > 0$); the new dist. is called "exponential dist." denoted by "exp. (λ)" and its pr. function is as follows:

$$X \sim \text{exp.}(\lambda)$$

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } 0 \leq x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Also; the m.g.f. of X is:

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$\mu_X = E(X) = (1) \left(\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

$$\sigma_X^2 = \text{Var}(X) = (1) \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Example ① If $X \sim \text{exp.}(\lambda)$, then find the p.d.f. & m.g.f. of X ; σ_X^2 ; μ_X .

Sol. $f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$\mu_X = \frac{1}{\lambda} = E(X)$$

$$\sigma_X^2 = \frac{1}{\lambda^2} = V(X)$$

Example ② If the M.g.f. of X is:

$$M_X(t) = \frac{4}{4-t} \quad ; \text{ then find:}$$

- ① p.d.f. of X ② μ_x & σ_x^2

Sol.

$$M_X(t) = \frac{4}{4-t} = (1 - \frac{t}{4})^{-1} = (1 - \frac{t}{\lambda})^{-1} \rightarrow \lambda = 4$$

$$\therefore X \sim \text{exp.}(4)$$

$$\textcircled{1} f(x; 4) = \begin{cases} 4e^{-4x} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$\textcircled{2} \mu_x = \frac{1}{\lambda} = \frac{1}{4} \quad \& \quad \sigma_x^2 = \frac{1}{\lambda^2} = \frac{1}{16}$$

Example ③ If $X \sim \text{exp.}(5)$, then find $M_X(t)$, σ_x^2 , μ_x .

Sol. $\therefore X \sim \text{exp.}(5)$

$$\therefore M_X(t) = (1 - \frac{t}{\lambda})^{-1} \quad ; \lambda = 5$$
$$\text{(M.g.f.)} = (1 - \frac{t}{5})^{-1}$$

$$\mu_x = E(X) = \frac{1}{\lambda} = \frac{1}{5}$$

$$\sigma_x^2 = \text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{25}$$

H.w. Given M.g.f. of X as follows:

$$M_X(t) = (1 - \frac{t}{3})^{-1}$$

- ① Find μ_x , σ_x^2 , $f(x)$
② $Y = 2 + 3X$; find $M_Y(t)$.

(4) Beta Distribution (توزیع بیتا)

A c.r.v. X have a Beta dist. its p.d.f. has the following form:

$$f(x; a, b) = \begin{cases} \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Where $a > 0$ & $b > 0$; denoted by $X \sim B(a, b)$

$$\text{And } B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

The M.g.f. of $X \sim B(a, b)$ is:

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \frac{B(a+n, b)}{B(a, b)}$$

$$E(X) = \frac{a}{a+b}, \quad \text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$$

In particular:

1. If $a=b=1$, then Beta dist. reduces to uniform dist. on $(0, 1)$:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Since } f(x, a=1, b=1) = \begin{cases} \frac{1}{B(1,1)} x^{1-1} (1-x)^{1-1} & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\& B(1,1) = \frac{\Gamma(1) \Gamma(1)}{\Gamma(2)} = 1 \Rightarrow X \sim B(1,1) \\ \Rightarrow X \sim \text{unif.}(0,1)$$

2. If $a=1$ & $b=2$ then Beta dist. reduces to dist. which is known as triangular dist.:

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Theorem: If $X \sim p(m)$ & if $x_0 = \text{mode of } X$, then

$$x_0 = \begin{cases} m, m-1 & \text{if } m \in \mathbb{I}^+ \\ [m] & \text{if } m \notin \mathbb{I}^+ \end{cases}$$

Ex. $X \sim p(5)$ then mode $= X = 5, 4$.

$X \sim p(2.6)$ then mode $= X = 2$.

Example The ^{نسبة}percentage of ^{مفوزين}bettors who make money of the race on a given day has Beta distribution with parameters $a=1$ and $b=5$. Find the pr. that fewer than ten ^{نسبة مئوية}percent comment ^{الفائزين}winners on given day?

Sol. $P(X < .1)$?

$$P(X < .1) = \int_0^1 \frac{\Gamma(1+5)}{\Gamma(1)\Gamma(5)} \cdot x^0 (1-x)^4 dx$$

$$= \frac{5!}{0! 4!} \int_0^1 (1-x)^4 dx$$

$$= -5 \frac{(1-x)^5}{5} \Big|_0^1 = (.9)^5$$



(5) Normal Distribution (التوزيع الطبيعي)

If X is a c.r.v. and

$$f(x; a, b) = \frac{1}{b\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2b^2}} \quad ; \quad -\infty < x < \infty$$

to show $f(x; a, b)$ is a p.d.f. of X .

Cond. ① T.P $f(x; a, b) \geq 0 \quad \forall x \in \mathbb{R}_x$

$$\frac{1}{b\sqrt{2\pi}} > 0 \quad \& \quad (x-a)^2 > 0, \quad \frac{1}{2b^2} > 0, \quad e^{-\frac{(x-a)^2}{2b^2}} > 0$$

$$\therefore f(x; a, b) \geq 0.$$

Cond ② T.P $\int_{-\infty}^{\infty} f(x; a, b) dx = 1$

$$\text{Let } y = \frac{x-a}{b} \rightarrow x = by + a$$

$$dx = b dy$$

$$\int_{-\infty}^{\infty} f(x; a, b) dx = \frac{1}{b\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \cdot b dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = 1$$

Note: Any c.r.v. X has a p.d.f. $f(x; a, b)$ above is said to be normal dist. with parameters a & b^2 denoted by $X \sim \mathcal{N}(a, b^2)$.

Theorem (10): If a c.r.v. X has a normal dist. with parameters a and b^2 , then the M.g.f. of X is:

$$M_X(t) = e^{at + \frac{b^2 t^2}{2}} \quad \text{for } -\infty < t < \infty$$

Proof: $\because X \sim \mathcal{N}(a, b^2)$

$$\therefore f(x; a, b) = \frac{1}{b\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2b^2}}$$

$$\begin{aligned} \therefore M_x(t) &= E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{b\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2b^2}} dx \\ &= \frac{1}{b\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{[2b^2tx - (x-a)^2]}{2b^2}} dx \end{aligned}$$

Let $X = by + b^2t + a$, $dx = bdy$

$$\begin{aligned} \therefore M_x(t) &= \frac{1}{b\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{[2b^2t(by+b^2t+a) - (by+b^2t)^2]}{2b^2}} \cdot bdy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{(2b^3ty + 2b^4t^2 + 2b^2at - b^2y^2 - 2b^3yt - b^4t^2)}{2b^2}} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{at + \frac{b^2t^2}{2} - \frac{y^2}{2}} dy \\ &= \frac{e^{at + \frac{b^2t^2}{2}}}{\sqrt{2\pi}} \cdot \left\{ \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \right\} = \sqrt{2\pi} \end{aligned}$$

$$M_x(t) = e^{at + \frac{b^2t^2}{2}} \quad \text{for } -\infty < t < \infty$$

Theorem (11): If $X \sim N(a, b^2)$, then:

$$M_x = a \quad \& \quad \sigma_x^2 = b^2$$

proof: $\therefore X \sim N(a, b^2)$

$$\therefore M_x(t) = e^{(at + \frac{b^2t^2}{2})}$$

$$M'_x(t) = e^{(at + \frac{b^2t^2}{2})} \cdot (a + b^2t)$$

$$M'_x(0) = e^0 (a + 0) = a = E(x)$$

$$M''_x(t) = e^{at + \frac{b^2t^2}{2}} (b^2) + (a + b^2t) e^{at + \frac{b^2t^2}{2}} (a + b^2t)$$

$$M''_x(0) = e^0 (b^2) + (a) e^0 (a) = a^2 + b^2 = E(x^2)$$

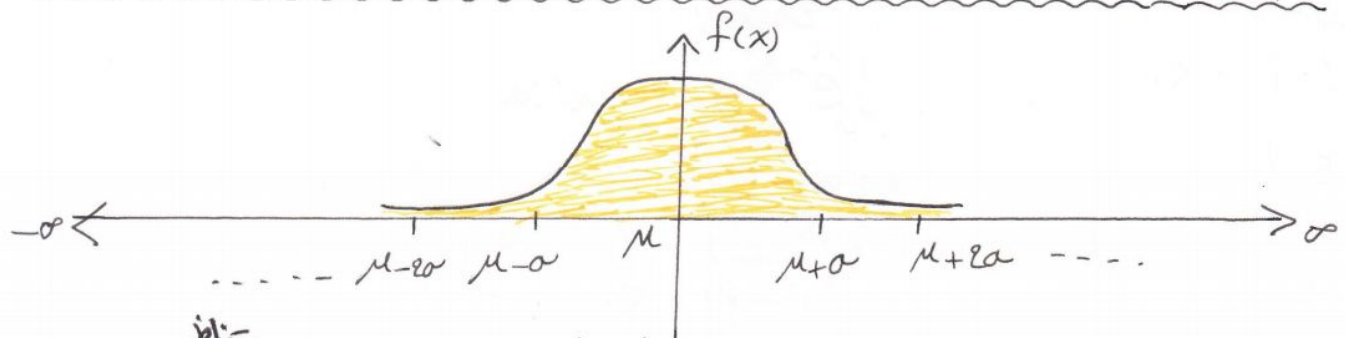
$$\text{Var}(X) = \sigma_x^2 = E(X^2) - (E(X))^2$$

Note: If $X \sim N(\mu_x, \sigma_x^2)$

$$f(x; \mu_x, \sigma_x^2) = \frac{1}{\sigma \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}; -\infty < x < \infty$$

$$M_X(t) = e^{\mu_x t + \frac{\sigma_x^2 t^2}{2}}$$

Some Properties of graph of Normal distribution

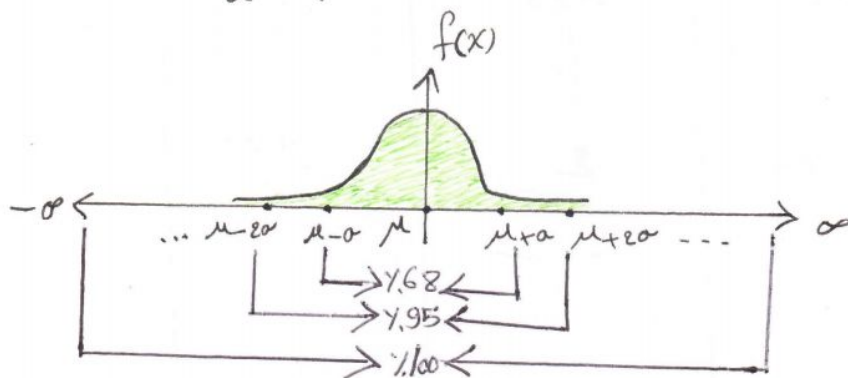


- ① Symmetric about Vertical axis through $x = \mu_x$ (متناظر المحور العمودي)
- ② There are points of inflection at $\mu_x \pm \sigma_x$ (انقلاب)
- ③ The max. point of $f(x)$ is at $x = \mu_x$
- ④ The x -axis is asymptote line. (المماسية)

Note: $X \sim N(\mu_x, \sigma_x^2)$

$$P(X \leq a) = \int_{-\infty}^a \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx$$

$$P(a \leq X \leq b) = \int_a^b \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx$$



⑥ Standard Normal Distribution (S.N.d.) التوزيع الطبيعي القياسي

If X be a r.v. distributed as $N(\mu, \sigma^2)$ and if $\mu=0$ and $\sigma^2=1$, then the new dist. is called "standard normal dist." denoted by $N(0,1)$, therefore; the p.d.f. of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ for } -\infty < x < \infty$$

And the M.g.f. of X is $M_X(t) = e^{t^2/2}$

Note: If $X \sim N(0,1)$ then:

$$1. P(X \leq a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = N(a)$$

$$2. P(X > a) = 1 - P(X \leq a) = 1 - N(a)$$

$$3. P(X \leq -a) = 1 - P(X \leq a) = 1 - N(a) = N(-a)$$

$$4. P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = N(b) - N(a)$$

Table of S.N.d.

$$N(x) = P(X \leq x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

x	$N(x)$	x	$N(x)$	x	$N(x)$
0.0	.50	1.1	.864	2.05	.98
.1	.54	1.2	.885	2.15	.98
.25	.549	1.3	.903	2.30	.989
⋮	⋮	⋮	⋮	⋮	⋮
.85	.802	1.85	.968	2.8	.997
⋮	⋮	⋮	⋮	⋮	⋮
1.05	.853	2	.977	3	.999

Example ① If $X \sim N(0,1)$, then find:

(i) $P(X < .85) = N(.85) = .802$

(ii) $P(X < -1.85) = 1 - P(X < 1.85) = 1 - N(1.85) = 1 - .968$

(iii) $P(1.05 < X < 2.15) = N(2.15) - N(1.05) = .989 - .853$

(iv) $P(-2.15 < X < 3) = N(3) - N(-2.15) = N(3) - [1 - N(2.15)]$
 $= .999 - 1 + .984 = .983$

Theorem (12): If $X \sim N(\mu_x, \sigma_x^2)$ and $Y = aX + b$; a, b are constant. then:

① $Y \sim N(a\mu_x + b, a^2\sigma_x^2)$

② The r.v. $Z = \frac{X - \mu_x}{\sigma_x}$; $\sigma_x > 0$ is $N(0,1)$

i.e. $Z \sim N(0,1)$

③ If $X_1, X_2, \dots, X_n$ are independent r.v.'s s.t. $X_i \sim N(\mu_i, \sigma_i^2)$ then: $Y = \sum_{i=1}^n X_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$

Proof:

① $Y = aX + b$, $\mu_y = E(Y) = E(aX + b) = aE(X) + b = a\mu_x + b$

$\text{Var}(Y) = \sigma_y^2 = V(aX + b) = a^2 \text{Var}(X) = a^2 \sigma_x^2$

$M_y(t) = E(e^{yt}) = E[e^{(ax+b)t}] = e^{bt} E(e^{axt})$ (by Theorem)

$= e^{bt} M_x(at)$

$= e^{bt} \left[e^{a\mu_x t + \frac{a^2 \sigma_x^2 t^2}{2}} \right]$ (by M.G.F. of $N(\mu_x, \sigma_x^2)$)

$= e^{(a\mu_x + b)t + \frac{a^2 \sigma_x^2 t^2}{2}}$

② $Z = \frac{X - \mu_x}{\sigma_x} \rightarrow Z = \frac{X}{\sigma_x} - \frac{\mu_x}{\sigma_x}$

$M_z(t) = E(e^{tz}) = E[e^{(\frac{X}{\sigma_x} - \frac{\mu_x}{\sigma_x})t}] = e^{-\frac{\mu_x}{\sigma_x} t} E(e^{\frac{X}{\sigma_x} t})$ (by $E(ax) = aE(x)$)

$= e^{-\frac{\mu_x}{\sigma_x} t} M_x\left(\frac{t}{\sigma_x}\right)$ (by def. of M.G.F.)

$= e^{-\frac{\mu_x}{\sigma_x} t} e^{\frac{\mu_x}{\sigma_x} t + \frac{\sigma_x^2 t^2}{2\sigma_x^2}}$ ($X \sim N(\mu_x, \sigma_x^2)$)

$\therefore M_z(t) = e^{\frac{t^2}{2}} \Rightarrow \boxed{Z \sim N(0,1)}$
 (M.G.F. of $N(0,1)$)

$$\therefore X \sim N(0, 1)$$

$$(3) Y = \sum_{i=1}^n X_i$$

$$\begin{aligned} M_Y(t) &= E(e^{ty}) = E(e^{t(\sum_{i=1}^n X_i)}) \\ &= E(e^{tX_1 + tX_2 + \dots + tX_n}) \\ &= E(e^{tX_1}) \cdot E(e^{tX_2}) \dots E(e^{tX_n}) \\ &= M_{X_1}(t) \cdot M_{X_2}(t) \dots M_{X_n}(t) \\ &= \left(e^{M_1 t + \frac{\sigma_1^2 t^2}{2}} \right) \left(e^{M_2 t + \frac{\sigma_2^2 t^2}{2}} \right) \dots \left(e^{M_n t + \frac{\sigma_n^2 t^2}{2}} \right) \\ &= e^{\sum_{i=1}^n M_i t + \sum_{i=1}^n \frac{\sigma_i^2 t^2}{2}} \end{aligned}$$

$$\therefore Y \sim N\left(\sum_{i=1}^n M_i, \sum_{i=1}^n \sigma_i^2\right)$$

Example (2) If $X \sim N(5, 16)$, find $P(X \leq 13)$.

$$\begin{aligned} \text{sol. } P(X \leq 13) &= P\left(\frac{X - \mu}{\sigma} \leq \frac{13 - 5}{4}\right) \\ &= P(Z \leq 2) = .977 \end{aligned}$$

Example (3) If $X \sim N(5, 4)$; find $P(1 < X < 8)$.

$$\begin{aligned} \text{sol. } P(1 < X < 8) &= P\left(\frac{1-5}{2} < \frac{X - \mu}{\sigma} < \frac{8-5}{2}\right) \\ &= P\left(-2 < Z < \frac{3}{2}\right) \\ &= P(Z < 1.5) - P(Z < -2) \\ &= P(Z < 1.5) - [1 - P(Z < 2)] \\ &= .933 - (1 - .977) \\ &= .910 \end{aligned}$$

Example (4)

(1) If $X \sim N(2, 25)$, then find $P(0 < X < 16)$

(2) If $X \sim N(75, 100)$, then find $P(70 < X < 100)$ & $P(X < 60)$

Sol.

(1) $X \sim N(2, 25)$

$$\begin{aligned} P(0 < X < 10) &= P\left(\frac{0-2}{5} < \frac{X-2}{5} < \frac{10-2}{5}\right) \\ &= P\left(\frac{-2}{5} < Z < \frac{8}{5}\right) \\ &= P\left(Z < \frac{8}{5}\right) - P\left(Z < \frac{-2}{5}\right) \\ &= N\left(\frac{8}{5}\right) - N\left(\frac{-2}{5}\right) \\ &= N(1.6) - N(-0.4) \\ &= 0.945 - [1 - N(0.4)] \\ &= 0.945 - [1 - 0.655] \\ &= 0.945 + 0.655 - 1 = 0.6 \end{aligned}$$

(2) $X \sim N(75, 100)$

$$\begin{aligned} P(X < 60) &= P\left(\frac{X-75}{10} < \frac{60-75}{10}\right) \\ &= P\left(Z < \frac{-15}{10}\right) = P(Z < -1.5) = N(-1.5) \\ &= 1 - N(1.5) = 1 - 0.933 = 0.067 \end{aligned}$$

$$\begin{aligned} P(70 < X < 100) &= P\left(\frac{70-75}{10} < \frac{X-75}{10} < \frac{100-75}{10}\right) \\ &= P(-0.5 < Z < 2.5) \\ &= P(Z < 2.5) - P(Z < -0.5) \\ &= N(2.5) - N(-0.5) \\ &= N(2.5) - [1 - N(0.5)] \\ &= N(2.5) + N(0.5) - 1 \\ &= 0.994 + 0.691 - 1 = 0.685 \end{aligned}$$

Example (5):

(1) $X \sim N(3, 4)$; find $P(X \leq 4)$

(2) $X \sim N(3, 9)$; find $P(X > 0)$

Sol.

(1) $X \sim N(3, 4)$

Z

$$P(X \leq 4) = P\left(\frac{X-3}{2} \leq \frac{4-3}{2}\right) \\ = P\left(Z \leq \frac{1}{2}\right) = P(Z \leq 0.5) = \mathcal{N}(0.5) = .6915$$

$$(2) X \sim \mathcal{N}(3, 9)$$

$$P(X > 0) = P\left(\frac{X-3}{3} > \frac{0-3}{3}\right) \\ = P(Z > -1) \\ = 1 - P(Z \leq -1) \\ = 1 - \mathcal{N}(-1) = 1 + \mathcal{N}(1) = \mathcal{N}(1) = .8413$$

Example(6) If $X \sim \mathcal{N}(\mu, \sigma^2)$, show that:

$$P(\mu - \sigma < X < \mu + \sigma) = .682$$

$$\text{proof: } P(\mu - \sigma < X < \mu + \sigma) = P\left(\frac{\mu - \sigma - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + \sigma - \mu}{\sigma}\right)$$

$$= P\left(\frac{\mu - \sigma - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + \sigma - \mu}{\sigma}\right)$$

$$= P(-1 < Z < 1) = P(Z < 1) - P(Z < -1)$$

$$= \mathcal{N}(1) - \mathcal{N}(-1)$$

$$= \mathcal{N}(1) - [1 - \mathcal{N}(1)]$$

$$= 2\mathcal{N}(1) - 1$$

$$= 2(.841) - 1 = .682$$

Example(7) If $X \sim \mathcal{N}(3, 9)$, then find $P(|X-3| > 6)$.

$$\text{sol. } P(|X-3| > 6) = 1 - P(|X-3| \leq 6)$$

$$= 1 - P(-6 \leq X-3 \leq 6)$$

$$= 1 - P\left(\frac{-6}{3} \leq \frac{X-3}{3} \leq \frac{6}{3}\right) \quad \left(\text{since } \mu=3\right)$$

$$= 1 - P(-2 \leq Z \leq 2)$$

$$= 1 - [P(Z \leq 2) - P(Z \leq -2)]$$

$$= 1 - [\mathcal{N}(2) - \mathcal{N}(-2)]$$

$$= 1 - \mathcal{N}(2) + [1 - \mathcal{N}(2)]$$

$$= 2 - 2\mathcal{N}(2)$$

$$= 2 - 2(.9772)$$

$$= .0456 \in [0, 1]$$

Example (8) If $X \sim N(25, 36)$, find the constant (C) from the $P(|X-25| \leq C) = .9544$

Sol. $P(|X-25| \leq C) = .9544$
 $= P(-C \leq X-25 \leq C)$

$$= P\left(-\frac{C}{6} \leq \underbrace{\frac{X-25}{6}}_{Z} \leq \frac{C}{6}\right)$$

$$= P\left(-\frac{C}{6} \leq Z \leq \frac{C}{6}\right)$$

$$= P\left(Z \leq \frac{C}{6}\right) - P\left(Z \leq -\frac{C}{6}\right)$$

$$= N\left(\frac{C}{6}\right) - N\left(-\frac{C}{6}\right)$$

$$= N\left(\frac{C}{6}\right) - [1 - N\left(\frac{C}{6}\right)]$$

$$.9544 = 2N\left(\frac{C}{6}\right) - 1$$

$$\therefore N\left(\frac{C}{6}\right) = (.9544 + 1)/2 = .9772$$

$$\therefore \frac{C}{6} = 2 \Rightarrow \boxed{C=12}$$

Example (9) If $X \sim N(\mu_x, \sigma_x^2)$ & $P(X < 60) = .10$;
 $P(X > 90) = .05$, then find σ_x^2, μ_x .

Sol. $P(X < 60) = P\left(\frac{X - \mu_x}{\sigma_x} < \frac{60 - \mu_x}{\sigma_x}\right) = .10$
 $= P\left(Z < \frac{60 - \mu_x}{\sigma_x}\right) = .10$
 $= N\left(\frac{60 - \mu_x}{\sigma_x}\right) = .10 \dots \textcircled{1}$

by table $\rightarrow -1.282 = \frac{60 - \mu_x}{\sigma_x} \dots \textcircled{1'}$

$$\begin{aligned} P(X > 90) &= 1 - P(X \leq 90) \\ .05 &= 1 - P\left(\frac{X - \mu_x}{\sigma_x} \leq \frac{90 - \mu_x}{\sigma_x}\right) \\ &= 1 - P\left(Z \leq \frac{90 - \mu_x}{\sigma_x}\right) = 1 - N\left(\frac{90 - \mu_x}{\sigma_x}\right) = .05 \end{aligned}$$

$$N\left(\frac{90 - \mu_x}{\sigma_x}\right) = 1.005 = 0.95 \quad (2)$$

by table $\frac{90 - \mu_x}{\sigma_x} = 1.645 \quad \dots (2')$

By (1) & (2') we get:

$$90 - \mu_x = 1.645 \sigma_x \quad \dots (2')$$

$$60 - \mu_x = -1.282 \sigma_x \quad \dots (1)$$

$$\Rightarrow \mu_x = 73.1 \text{ \& } \sigma = 10.2$$

Theorem (13)

$$Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2(1)$$

$$\sum_{i=1}^n Z_i^2 \sim \chi^2(n)$$

↑
لا علاقة بين التوزيعات

Theorem (13) $X \sim N(\mu, \sigma^2)$, then $y = \left(\frac{X - \mu}{\sigma}\right)^2$ is $\chi^2(1)$

ملاحظة:
توزيع بيتا التوزيعات
(مطلوبه هذا)
(الاطلاعه)

(1) $X \sim N(0,1)$, then $y = x^2$ is $\chi^2(1)$

(2) $X_1, X_2, \dots, X_n$ r.v.s $\sim N(0,1)$ &

$$y = \sum_{i=1}^n x_i^2, \text{ then } y \text{ is } \chi^2(n)$$

ملاحظة:
توزيع بيتا التوزيعات
(مطلوبه هذا)

Example (10): If M.g.f. $M_x(t) = e^{3t + 8t^2}$, find $P(-1 < X < 9)$.

Sol.

$$\mu_x \text{ Coefficient of } t : 3t = \mu t \rightarrow \mu = 3$$

$$\sigma_x^2 \text{ Coefficient of } t^2 : 8t^2 = \frac{\sigma^2 t^2}{2} \rightarrow \sigma^2 = 16 \rightarrow \sigma = 4$$

$$\therefore X \sim N(3, 16)$$

$$\begin{aligned} P(-1 < X < 9) &= P(X < 9) - P(X < -1) \\ &= P\left(\frac{X - 3}{4} < \frac{9 - 3}{4}\right) - P\left(\frac{X - 3}{4} < \frac{-1 - 3}{4}\right) \\ &= P(Z < 1.5) - P(Z < -1) \\ &= \Phi(1.5) - \Phi(-1) \\ &= \Phi(1.5) - [1 - \Phi(1)] \\ &= 0.933 - 1 + 0.341 \\ &= 0.274 \end{aligned}$$

Example (11) If $X \sim N(5, 10)$, find

$P(0.04 < (X-5)^2 < 38.4)$?

Sol. $P(0.04 < (X-5)^2 < 38.4) = P\left(\frac{0.04}{10} < \frac{(X-5)^2}{10} < \frac{38.4}{10}\right)$
 $= P(0.004 < \left(\frac{X-5}{\sqrt{10}}\right)^2 < 3.84)$

by Theorem: $W = \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2(1)$

$\therefore W \sim \chi^2(1)$

by Table of $\chi^2(1) \rightarrow$

$= P(0.004 < W < 3.84)$

$= P(W < 3.84) - P(W < 0.004)$

$= 0.95 - 0.050$

$= .90$

Example (12) If $X \sim N(1, 4)$, find $P(1 < X^2 < 9)$.

Sol. $(1 < X^2 < 9) = (1 < X < 3) \cup (-3 < X < -1)$

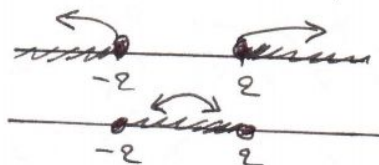
$P(1 < X^2 < 9) = P(1 < X < 3) + P(-3 < X < -1)$
 $= P(X < 3) - P(X < 1) + P(X < -1) - P(X < -3)$
 $= P\left(\frac{X-1}{2} < \frac{3-1}{2}\right) - P\left(\frac{X-1}{2} < \frac{1-1}{2}\right) + P\left(\frac{X-1}{2} < \frac{-1-1}{2}\right) - P\left(\frac{X-1}{2} < \frac{-3-1}{2}\right)$
 $= P(Z < 1) - P(Z < 0) + P(Z < -1) - P(Z < -2)$
 $= N(1) - N(0) + N(-1) - N(-2)$
 $= N(1) - N(0) + (1 - N(1)) - (1 - N(2))$
 $= N(1) - N(1) - N(0) + N(2) + 1 - 1$
 $= N(2) - N(0)$
 $= 0.977 - 0.5 = 0.477$

Example (13) If $X \sim N(1, 4)$; find ① $P(|X| > 2)$

② $P(|X| \leq 2)$.

① $|X| > 2 \rightarrow -2 > X > 2$

② $|X| \leq 2 \rightarrow -2 \leq X \leq 2$



Sol.

② $P(|X| \leq 2) = P(-2 \leq X \leq 2)$
 $= P(X \leq 2) - P(X \leq -2)$
 $= P\left(\frac{X-1}{2} \leq \frac{2-1}{2}\right) - P\left(\frac{X-1}{2} \leq \frac{-2-1}{2}\right)$

||

$$\begin{aligned}
 &= P(Z \leq 0.5) - P(Z \leq -1.5) \\
 &= \mathcal{N}(0.5) - 1 + \mathcal{N}(1.5) \\
 &= .691 - 1 + .933 = .624
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{1} \quad P(|X| \geq 2) &= 1 - P(|X| < 2) \\
 &= 1 - P(-2 < X < 2) \\
 &= 1 - [P(X < 2) - P(X < -2)] \\
 &= 1 - P\left(\frac{X-1}{2} < \frac{2-1}{2}\right) + P\left(\frac{X-1}{2} < \frac{-2-1}{2}\right) \\
 &= 1 - P(Z < 0.5) + P(Z < -1.5) \\
 &= 1 - \mathcal{N}(0.5) + \mathcal{N}(-1.5) \\
 &= 1 - \mathcal{N}(0.5) + 1 - \mathcal{N}(1.5) \\
 &= 2 - \mathcal{N}(0.5) - \mathcal{N}(1.5) \\
 &= 2 - .691 - .933 = .376
 \end{aligned}$$

H.W.

① If $X \sim \mathcal{N}(\mu, \sigma^2)$ & $P(X < 89) = 0.9$, $P(X < 94) = .95$, then find μ & σ^2 .

② When $X \sim \mathcal{N}(\mu, \sigma^2)$ show that:

- (i) $P(\mu - 2\sigma < X < \mu + 2\sigma) = .95$
- (ii) $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = .99$
- (iii) $P(X > \mu + \sigma) = .1587$

③ If $X \sim \chi^2(5)$, determine the constants (c) & (d). show that $P(c < X < d) = .95$ and $P(X < c) = .025$.

④ If $X \sim \chi^2(17)$; find the following:

- (i) $P(X < 7.564)$
- (ii) $P(X > 27.59)$
- (iii) $P(8.408 \leq X \leq 27.59)$

⑤ If $X \sim \chi^2(23)$, find the following:

a. $P(10.20 < X < 35.17)$

b. If $P(a < X < b) = .95$ & $P(X < a) = .025$; find the constants (a) & (b).

c. The p.d.f. of X ($f(x)$).



ثبوت التوزيع الطبيعي القياسي $X^2(1)$

Th. $X \sim N(0, 1)$

$$Y = X^2 \sim \chi^2(1)$$

(Using Moment Technique)

Proof. $X \sim N(0, 1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$M(t) = E(e^{tx})$$

$$= E(e^{tx^2})$$

$$= \int_{-\infty}^{\infty} e^{tx^2} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2(1-2t)}{2}} dx$$

$$\text{Let } w = \sqrt{1-2t} \ X$$

$$dw = \sqrt{1-2t} \ dx$$

$$\circ\circ \quad M_y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} \frac{dw}{\sqrt{1-2t}}$$

$$= \frac{1}{\sqrt{1-2t}}$$

$$\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} dw \right)$$

Since $(w \sim N(0,1))$

$\Rightarrow$

$$M_y(t) = (1-2t)^{-\frac{1}{2}}$$

$$\circ\circ \quad Y \sim \chi^2(1)$$

~~XXXX~~

Note

$$X \sim N(\mu, \sigma^2)$$

Then $\Rightarrow$ ① $Z \sim N(0,1)$

② $Z^2 \sim \chi^2(1)$

③ $\sum_{i=1}^n Z^2 \sim \chi^2(n)$

TABLE III
The Normal Distribution

$$\Pr(X \leq x) = N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$$

$$[N(-x) = 1 - N(x)]$$

x	N(x)	x	N(x)	x	N(x)
0.00	0.500	1.10	0.864	2.05	0.980
0.05	0.520	1.15	0.875	2.10	0.982
0.10	0.540	1.20	0.885	2.15	0.984
0.15	0.560	1.25	0.894	2.20	0.986
0.20	0.579	1.282	0.900	2.25	0.988
0.25	0.599	1.30	0.903	2.30	0.989
0.30	0.618	1.35	0.911	2.326	0.990
0.35	0.637	1.40	0.919	2.35	0.991
0.40	0.655	1.45	0.926	2.40	0.992
0.45	0.674	1.50	0.933	2.45	0.993
0.50	0.691	1.55	0.939	2.50	0.994
0.55	0.709	1.60	0.945	2.55	0.995
0.60	0.726	1.645	0.950	2.576	0.995
0.65	0.742	1.65	0.951	2.60	0.995
0.70	0.758	1.70	0.955	2.65	0.996
0.75	0.773	1.75	0.960	2.70	0.997
0.80	0.788	1.80	0.964	2.75	0.997
0.85	0.802	1.85	0.968	2.80	0.997
0.90	0.816	1.90	0.971	2.85	0.998
0.95	0.829	1.95	0.974	2.90	0.998
1.00	0.841	1.960	0.975	2.95	0.998
1.05	0.853	2.00	0.977	3.00	0.999

تذكير ①

① $X \sim N(0, 1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

$$M_x(t) = e^{t^2/2} \quad -\infty < t < \infty$$

② $X \sim N(\mu, \sigma^2)$

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad x, \mu \in (-\infty, \infty), \sigma^2 > 0$$

$$M_x(t) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

③ $X \sim G(\alpha, \beta)$

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} \quad x \in (0, \infty)$$

$$M_x(t) = (1 - \beta t)^{-\alpha} \quad t < \frac{1}{\beta}$$

④ $X \sim \chi^2(r)$ $[r = 2\alpha, \beta = 2]$

$$f(x; r) = \frac{x^{r/2-1} e^{-x/2}}{\Gamma(r/2) 2^{r/2}} \quad -\infty < x < \infty$$

$$M_x(t) = (1 - 2t)^{-r/2} \quad t < \frac{1}{2}$$

Note ① $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

② $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$

$X \sim \chi^2(1) \rightarrow f(x) = \frac{x^{-1/2} e^{-x/2}}{\Gamma(1/2) 2^{1/2}}, M(t) = (1-2t)^{-1/2}$

② $X \sim N(\mu, \sigma^2) \rightarrow Z = \frac{X - \mu}{\sigma} \sim N(0, 1) \leftrightarrow$ نورمالي

④ $Z \sim N(0, 1) \rightarrow Y = Z^2 \sim \chi^2(1)$

⑤ $X_i \text{ (i.i.d.)} \sim N(0, 1) \text{ indep.}, Y = \sum X_i^2 \sim \chi^2(n)$

$r=1$
 $\alpha=r/2$
 $\alpha=1/2$

Proof (4)

by transformation

by M.g.f. $[M_y(t)]$

$$Z \sim N(0, 1) \rightarrow f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$y = z^2 \rightarrow z = \pm \sqrt{y} \rightarrow \begin{matrix} z = \sqrt{y} \\ z = -\sqrt{y} \end{matrix}$$

$$|dz/dy| = |J| = \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2\sqrt{y}}$$

$$g(y) = \sum_{\pm} f(z(y)) |J| = \sum_{\pm} f(z) \cdot |J|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{y^{-1/2}}{\sqrt{\pi}} e^{-y/2} \quad \leftarrow \Gamma(1/2) = \sqrt{\pi}$$

$$= \frac{y^{1/2-1} e^{-y/2}}{\Gamma(1/2) (2)^{1/2}}$$

$\therefore Y \sim \chi^2(1)$

$$Z \sim N(0, 1) \rightarrow f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$y = z^2$$

$$M_y(t) = E[e^{ty}] = E[e^{t z^2}]$$

$$= \int_{-\infty}^{\infty} e^{t z^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}(1-2t)} dz$$

put $w = \sqrt{1-2t} z \rightarrow z = \frac{1}{\sqrt{1-2t}} w$

$$dz = \frac{1}{\sqrt{1-2t}} dw$$

$$M_y(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-w^2/2} \cdot \frac{1}{\sqrt{1-2t}} dw$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} e^{-w^2/2} dw = \sqrt{2\pi}$$

$$= (1-2t)^{-1/2}$$

$\therefore Y \sim \chi^2(1)$

or $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-w^2/2} dw = 1$
S.N.D

(2)

proof (s) $X_i \sim \chi^2(1) \rightarrow M_{X_i}(t) = E(e^{X_i t}) = (1-2t)^{-1/2}$
 $i=1, \dots, n$ indep.

$$Y = \sum_{i=1}^n X_i \sim ? \chi^2(n)$$

by M.g.f. of Y

ex 2 *

If $X_1, X_2 \sim \chi^2(1)$ are indep. $Y = X_1 + X_2 \sim \chi^2(2)$
--

$$M_Y(t) = E(e^{tY})$$

$$= E[e^{t \sum_{i=1}^n X_i}] = E[e^{t(X_1 + X_2 + \dots + X_n)}]$$

$$= E[e^{tX_1 + tX_2 + \dots + tX_n}]$$

$$= E[e^{tX_1} \cdot e^{tX_2} \cdot \dots \cdot e^{tX_n}] \quad \text{if } X_i \text{ indep.}$$

$$= E[e^{tX_1}] \cdot E[e^{tX_2}] \cdot \dots \cdot E[e^{tX_n}]$$

$$= (1-2t)^{-1/2} \cdot (1-2t)^{-1/2} \cdot \dots \cdot (1-2t)^{-1/2}$$

$$= (1-2t)^{-n/2}$$

$\beta=2$, $r=n$
 $\alpha = n/2$

$\therefore Y \sim \chi^2(n)$

(3)

Thm If $X \sim N(\mu, \sigma^2)$, then the m.g.f. of X is

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \quad -\infty < t < \infty$$

proof

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\text{let } y = \frac{x-\mu}{\sigma} \rightarrow x = \sigma y + \mu \rightarrow dx = \sigma dy$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(\mu + \sigma y)} \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2} \sigma dy$$

$$= \frac{e^{\mu t}}{e} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma t y - \frac{y^2}{2}} dy$$

$$= \frac{e^{\mu t}}{e} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{[y^2 - 2\sigma t y]}{2}} dy$$

$$= \frac{e^{\mu t}}{e} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{[y^2 - 2\sigma t y + \sigma^2 t^2 + \sigma^2 t^2 - \sigma^2 t^2]}{2}} dy$$

$$= \frac{e^{\mu t}}{e} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y - \sigma t)^2}{2} + \frac{\sigma^2 t^2}{2}} dy$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{e} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y - \sigma t)^2}{2}} dy$$

$$\text{let } z = y - \sigma t \rightarrow y = z + \sigma t \rightarrow dy = dz$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{e} \cdot \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \right) = 1$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{e}$$

Thm If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$ & $\text{Var}(X) = \sigma^2$

proof

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$M_X'(t) = \left(\frac{\mu t + \frac{\sigma^2 t^2}{2}}{e} \right) (\mu + \sigma^2 t) = M_X(t) [\mu + \sigma^2 t]$$

$$E(X) = M_X'(0) = M_X(0) [\mu + \sigma^2 \cdot 0] = \mu$$

$M_X(0) = 1$

(4)

$$\begin{aligned}\dot{M}_x(t) &= M_x(t) [H + \sigma^2 t] \\ &= H M_x(t) + \sigma^2 [t \cdot M_x(t)]\end{aligned}$$

$$\ddot{M}_x(t) = M \cdot \dot{M}_x(t) + \sigma^2 [t \cdot \dot{M}_x(t) + M_x(t) \cdot 1]$$

$$\ddot{M}_x(0) = M \cdot \dot{M}_x(0) + \sigma^2 [0 + M_x(0)]$$

$$= M \cdot M + \sigma^2 \cdot 1$$

$$= M^2 + \sigma^2$$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= (M^2 + \sigma^2) - (M)^2$$

$$= \sigma^2$$

$$\binom{n}{n-1} = \frac{n!}{(n-1)!}$$

1 Binomial dist.

$$f(x, n, p) = \binom{n}{x} p^x (1-p)^{n-x} \quad x=0, 1, 2, \dots, n$$

$$F_X(x) = P(X \leq x) = \sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} p^k (1-p)^{n-k} \quad x=0, 1, \dots, n$$

[No simple closed form, evaluate numerically]

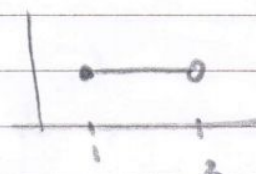
Interval	int.	$f(x)$	$F(x) = \sum_{x \leq r} f(r)$
$x < 0$	No int.	0	0
$0 \leq x < 1$	0	$(1-p)^n$	$(1-p)^n$
$1 \leq x < 2$	1	$np(1-p)^{n-1}$	$(1-p)^n + np(1-p)^{n-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 \leq x < n$	$(n-1)$	$n p^{n-1} (1-p)$	$(1-p)^n + np(1-p)^{n-1} + \dots + n p^{n-1} (1-p)$
$x \geq n$	n	p^n	$(1-p)^n + np(1-p)^{n-1} + \dots + n p^{n-1} (1-p) + p^n$ $= \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}$

Remember

$f(x) = \lfloor x \rfloor$ The greatest integer Function
 Greatest integer less than x

ex.

$$1 \leq x < 2 \rightarrow f(x) = \lfloor x \rfloor = 1$$

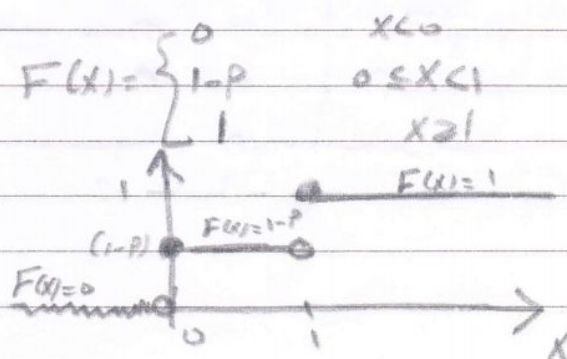


2 Bernoulli dist.

$$f(x; p) = p^x (1-p)^{1-x} \quad x=0, 1$$

$$F_X(x) = P(X \leq x) = \sum_{k=0}^{\lfloor x \rfloor} p^k (1-p)^{1-k} \quad x=0, 1$$

int.	int.	$f(x; p)$	$F(x) = \sum f(x)$
$x < 0$	No	0	0
$0 \leq x < 1$	0	$(1-p)$	$(1-p)$
$x \geq 1$	1	p	$(1-p) + p = 1$



Q2 ex If X_i ($i=1, \dots, m$) are independent r.v.
 where $X_i \sim \text{Exp}(\lambda)$, then prove that
 $T = \frac{\sum X_i}{m} \sim G(m, \frac{1}{m\lambda})$

Sol. $X_i \sim \text{Exp}(\lambda) \rightarrow f(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$
 $M_{X_i}(t) = (1 - \frac{1}{\lambda} t)^{-1}$

T.P $T = \frac{\sum X_i}{m} \sim G(m, \frac{1}{m\lambda})$

Proof: $M_T(t) = E[e^{tT}] = E[e^{\frac{\sum X_i}{m} t}]$

$= E[e^{(X_1 + X_2 + \dots + X_m) \frac{t}{m}}]$

$= E[e^{X_1 \frac{t}{m}} \cdot e^{X_2 \frac{t}{m}} \cdot \dots \cdot e^{X_m \frac{t}{m}}]$

X_i are indep.
 $= E[e^{X_1 \frac{t}{m}}] \cdot E[e^{X_2 \frac{t}{m}}] \cdot \dots \cdot E[e^{X_m \frac{t}{m}}]$

$= M_{X_1}(\frac{t}{m}) \cdot M_{X_2}(\frac{t}{m}) \cdot \dots \cdot M_{X_m}(\frac{t}{m})$

Answer

$= (1 - \frac{1}{\lambda} \frac{t}{m})^{-1} \cdot (1 - \frac{1}{\lambda} \frac{t}{m})^{-1} \cdot \dots \cdot (1 - \frac{1}{\lambda} \frac{t}{m})^{-1}$

$= (1 - (\frac{1}{m\lambda}) t)^{-m}$

$\alpha = m, \beta = \frac{1}{m\lambda}$

Answer

$\therefore T \sim G(m, \frac{1}{m\lambda})$ *

الموزع الغاوسي
 Lab من توزيع
 $\beta = \frac{1}{\lambda}, \alpha = 1$ مع

$X \sim G(\alpha, \beta)$

$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1}$ $0 < x < \infty$

$M_X(t) = (1 - \beta t)^{-\alpha}$

2

[3] Geometric Dist.

سوي اديف

$$f(x, p) = \begin{cases} p q^{x-1} & \text{for } x = 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

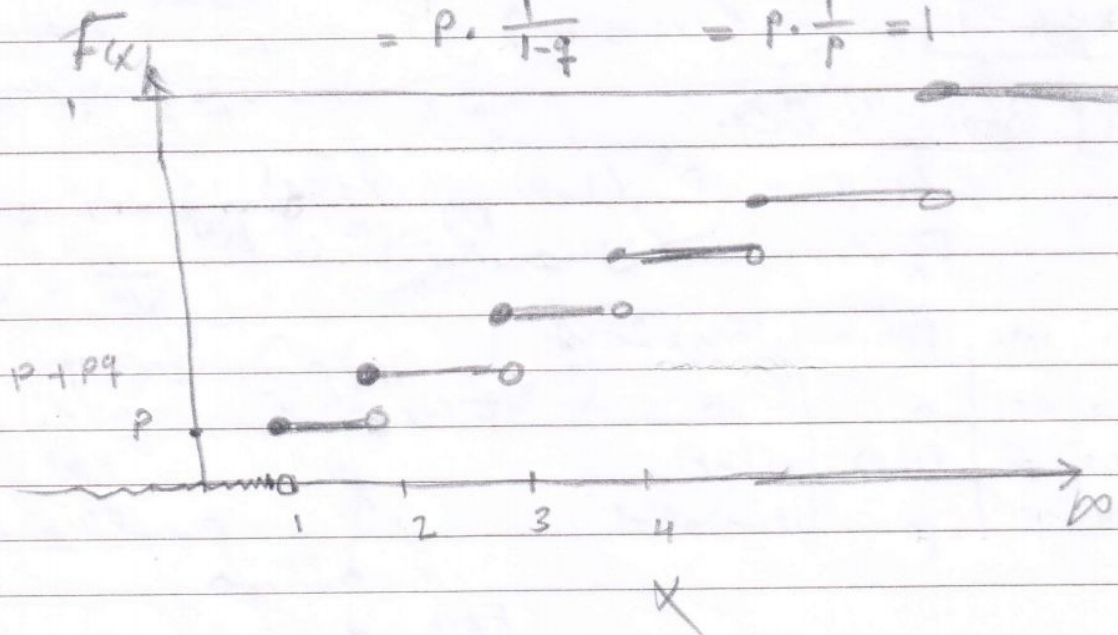
$$f(x, p) = \begin{cases} p q^x & x = 0, 1, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$F(x) = P(X \leq x) = 1 - P(X > x) = 1 - q^x = \sum_{k=1}^x p q^{k-1}$$

interval	intk	f(x)	F(x) = $\sum_{k=1}^x p q^{k-1}$
$x < 1$	no.	0	0
$1 \leq x < 2$	1	p	p
$2 \leq x < 3$	2	p q	p + p q
$3 \leq x < 4$	3	p q ²	p + p q + p q ²
$n \leq x < n+1$	n	p q ⁿ	$\sum_{k=1}^n p q^{k-1}$

$$\begin{aligned} \lim_{x \rightarrow \infty} F(x) &= \lim_{x \rightarrow \infty} [1 - q^x] \\ &= 1 - \lim_{x \rightarrow \infty} q^x \\ &= 1 - 0 \\ &= 1 \end{aligned} \quad \text{as } q < 1$$

$$\begin{aligned} n \rightarrow \infty \rightarrow F(x) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n p q^{k-1} = \lim_{n \rightarrow \infty} p \sum_{k=1}^n q^{k-1} \\ &= \lim_{n \rightarrow \infty} p [1 + q + \dots + q^{n-1}] \\ &= p [1 + q + \dots + q^{n-1} + \dots] \\ &= p \cdot \frac{1}{1-q} = p \cdot \frac{1}{p} = 1 \end{aligned}$$



4 Poisson Dist.

$$f(x) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

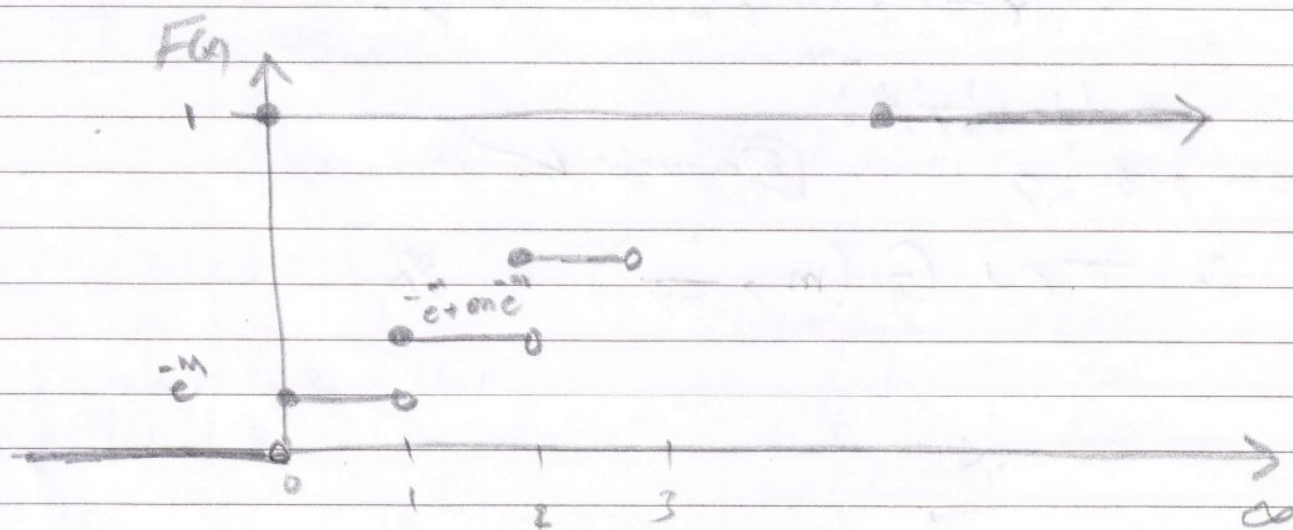
$$F(x) = P(X \leq x) = \sum_{k=0}^x \frac{e^{-m} m^k}{k!}$$

if $x \rightarrow \infty \rightarrow \lim_{x \rightarrow \infty} F(x) = P(X \leq x; x \rightarrow \infty)$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \sum_{k=0}^x \frac{e^{-m} m^k}{k!} \\ &= e^{-m} \sum_{k=0}^{\infty} \frac{m^k}{k!} = e^{-m} \cdot e^m = 1 \end{aligned}$$

interval	integer	$f(x)$	$F(x) = \sum f(x)$
$x < 0$	no	0	0
$0 \leq x < 1$	0	e^{-m}	e^{-m}
$1 \leq x < 2$	1	$e^{-m} \frac{m}{1!}$	$e^{-m} + e^{-m} \frac{m}{1!}$
$2 \leq x < 3$	2	$e^{-m} \frac{m^2}{2!}$	$e^{-m} + e^{-m} \frac{m}{1!} + e^{-m} \frac{m^2}{2!}$
$\vdots$			
$x \rightarrow \infty$			1

$e^{-m} > 0$



If X_i ($i=1,2,3,\dots,n$) are independent random variables, such that $X_i \sim \exp(\theta)$, then show that:

$$Y = \sum_{i=1}^n X_i \sim G(n, \frac{1}{\theta}).$$

Hint: Start with Moment generating function ($M(t)$).

Sol.

$$\begin{aligned} M_Y(t) &= E(e^{ty}) \\ &= E(e^{t(X_1 + X_2 + \dots + X_n)}) \\ &= E(e^{X_1 t + X_2 t + \dots + X_n t}) \\ &= E(e^{X_1 t} \cdot e^{X_2 t} \dots e^{X_n t}) \\ &= E(e^{X_1 t}) \cdot E(e^{X_2 t}) \dots E(e^{X_n t}) \quad \left(\begin{array}{l} X_i \text{ are} \\ \text{indep.} \\ \text{r.v.s} \end{array} \right) \\ &= M_{X_1}(t) \cdot M_{X_2}(t) \dots M_{X_n}(t) \quad \left(\begin{array}{l} \text{def.} \\ \text{of} \\ \text{m.g.f.} \end{array} \right) \\ &\stackrel{\left(\begin{array}{l} \text{Since} \\ X_i \sim \exp(\theta) \end{array} \right)}{\rightarrow} = \left(1 - \frac{t}{\theta}\right)^{-1} \left(1 - \frac{t}{\theta}\right)^{-1} \dots \left(1 - \frac{t}{\theta}\right)^{-1} \\ &= \left(1 - \frac{t}{\theta}\right)^{-n} \\ &= \text{M.g.f. of } G(n, \frac{1}{\theta}) \end{aligned}$$

$\nwarrow \quad \nwarrow$
 $\alpha \quad \beta$

$$\therefore Y = \sum_{i=1}^n X_i \sim G(n, \frac{1}{\theta})$$

#

①

If $X_i (i=1, 2, 3, \dots, m)$ are independent random variables, such that $X_i \sim \exp(\phi)$, then show that:

$$Z = \sum_{i=1}^m \frac{X_i}{m} \sim G\left(m, \frac{1}{m\phi}\right).$$

Hint: Start with m.g.f. ($M_X(t)$).

Sol.

$$M_Z(t) = E(e^{tZ})$$

$$= E\left(e^{t\left(\frac{\sum X_i}{m}\right)}\right)$$

$$= E\left(e^{\frac{t}{m}(X_1 + X_2 + \dots + X_m)}\right)$$

$$= E\left(e^{\frac{t}{m}X_1} \cdot e^{\frac{t}{m}X_2} \cdot \dots \cdot e^{\frac{t}{m}X_m}\right)$$

(Since X_i are indep. r.v.s)

$$= E\left(e^{\frac{t}{m}X_1}\right) \cdot E\left(e^{\frac{t}{m}X_2}\right) \cdot \dots \cdot E\left(e^{\frac{t}{m}X_m}\right)$$

$$= M_{X_1}\left(\frac{t}{m}\right) \cdot M_{X_2}\left(\frac{t}{m}\right) \cdot \dots \cdot M_{X_m}\left(\frac{t}{m}\right)$$

(def. of m.g.f.)

(Since $X_i \sim \exp(\phi)$)

$$= \left(1 - \frac{t}{m\phi}\right)^{-1} \cdot \left(1 - \frac{t}{m\phi}\right)^{-1} \cdot \dots \cdot \left(1 - \frac{t}{m\phi}\right)^{-1}$$

(m-times)

$$= \left(1 - \frac{t}{m\phi}\right)^{-m}$$

$$= \text{m.g.f. of } G\left(m, \frac{1}{m\phi}\right)$$

$$\therefore Z \sim G\left(m, \frac{1}{m\phi}\right).$$

#

If $X_i (i=1, 2, 3, \dots, k)$ are independent random variables, such that $X_i \sim \text{EXP}(\omega)$, then show that:

$$Y = \sum_{i=1}^k X_i \sim G(k, \frac{1}{\omega}).$$

Hint: Start with m.g.f. $M_X(t)$.

Sol.

$$M(t) = E(e^{ty})$$

$$= E(e^{t(\sum_{i=1}^k X_i)})$$

$$= E(e^{X_1 t + X_2 t + \dots + X_k t})$$

$$= E(e^{X_1 t} \cdot e^{X_2 t} \dots e^{X_k t})$$

$$= E(e^{X_1 t}) \cdot E(e^{X_2 t}) \dots E(e^{X_k t})$$

$$= M_{X_1}(t) \cdot M_{X_2}(t) \dots M_{X_k}(t) \quad \left(\begin{array}{l} \text{def.} \\ \text{of m.g.f.} \end{array} \right)$$

$$\left(\begin{array}{l} \text{Since} \\ X_i \sim \text{exp}(\omega) \end{array} \right) \rightarrow = \underbrace{\left(1 - \frac{t}{\omega}\right)^{-1} \cdot \left(1 - \frac{t}{\omega}\right)^{-1} \dots \left(1 - \frac{t}{\omega}\right)^{-1}}_{(k \text{ - times})}$$

$$= \left(1 - \frac{t}{\omega}\right)^{-k}$$

$$= \text{m.g.f. of } G(k, \frac{1}{\omega})$$

$$Y = \sum_{i=1}^k X_i \sim G(k, \frac{1}{\omega})$$

#

3

If $X_i (i=1, 2, 3, \dots, n)$ are independent random Variables, such that $X_i \sim \exp(\gamma)$, then show that:

$$T = \sum_{i=1}^n \frac{X_i}{n} \sim G(n, \frac{1}{n\gamma})$$

Hint: Start with m.g.f. ($M(t)$).

Sol.

$$M_T(t) = E(e^{tT})$$

$$= E(e^{t(\sum_{i=1}^n \frac{X_i}{n})})$$

$$= E(e^{\frac{t}{n}X_1 + \frac{t}{n}X_2 + \dots + \frac{t}{n}X_n})$$

(Since X_i are indep. r.v.s)

$$= E(e^{\frac{t}{n}X_1} \cdot e^{\frac{t}{n}X_2} \cdot e^{\frac{t}{n}X_3} \dots e^{\frac{t}{n}X_n})$$

$$= E(e^{\frac{t}{n}X_1}) \cdot E(e^{\frac{t}{n}X_2}) \cdot E(e^{\frac{t}{n}X_3}) \dots E(e^{\frac{t}{n}X_n})$$

(def of m.g.f.)

$$= M_{X_1}(\frac{t}{n}) \cdot M_{X_2}(\frac{t}{n}) \cdot M_{X_3}(\frac{t}{n}) \dots M_{X_n}(\frac{t}{n})$$

(Since $X_i \sim \exp(\gamma)$)

$$= (1 - \frac{t}{n\gamma})^{-1} \cdot (1 - \frac{t}{n\gamma})^{-1} \cdot (1 - \frac{t}{n\gamma})^{-1} \dots (1 - \frac{t}{n\gamma})^{-1}$$

$$= (1 - \frac{t}{n\gamma})^{-n} \text{ (n-times)}$$

$$= \text{m.g.f. of } G(n, \frac{1}{n\gamma})$$

$\downarrow \alpha \quad \downarrow \beta$

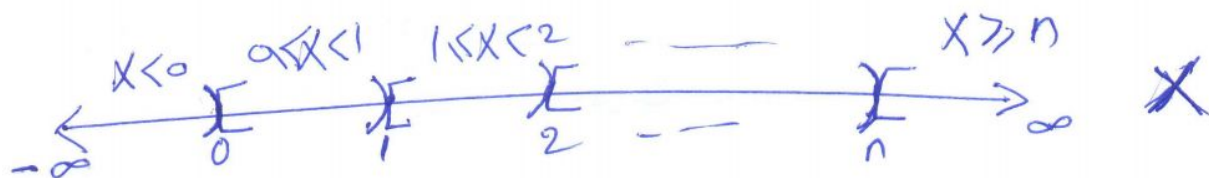
$$\therefore T = \sum_{i=1}^n \frac{X_i}{n} \sim G(n, \frac{1}{n\gamma})$$

#

(4)

F(x) of Binomial dist.

$$f(x) = \begin{cases} C_x^n p^x (1-p)^{n-x} & \text{for } x=0,1,2,\dots,n \\ 0 & \text{o.w.} \end{cases}$$



interval of x	$x_i \in I$	$f(x_i) \uparrow$	$F(x) = \sum_{x_i \leq x} f(x_i)$
$x < 0$	no intg.	0	0
$0 \leq x < 1$	0	$f(0) = C_0^n p^0 (1-p)^{n-0}$	$F(0) = 0 + f(0)$
$1 \leq x < 2$	1	$f(1) = C_1^n p^1 (1-p)^{n-1}$	$F(1) = 0 + f(0) + f(1)$
$\vdots$			
$x \geq n$	n	$f(n) = C_n^n p^n (1-p)^0$	$F(n) = 0 + f(0) + f(1) + \dots + f(n)$ $= 1 \quad \left(\sum_{x_i=0}^n f(x_i) \right)$

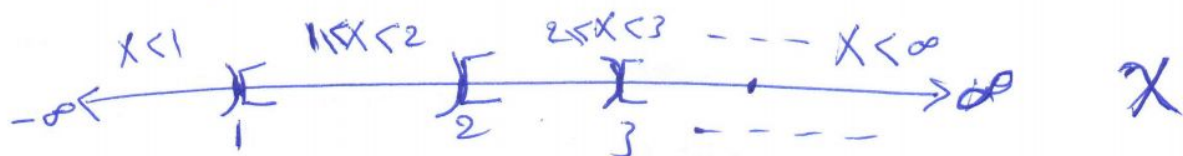
$$F(x) = \begin{cases} 0 & x < 0 \\ F(0) & 0 \leq x < 1 \\ F(1) & 1 \leq x < 2 \\ \vdots & \\ F(n) = 1 & x \geq n \end{cases}$$

#

(5)

$F(x)$ of Geometric dist.

$$f(x) = \begin{cases} p q^{x-1} & \text{for } x=1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$



interval of x	$x_j \in I$	$f(x) = \uparrow$	$F(x) = \sum_{x_j \in I} f(x_j)$
$x < 1$	No intg.	0	0
$1 \leq x < 2$	1	$f(1) = p q^0$	$F(1) = 0 + f(1)$
$2 \leq x < 3$	2	$f(2) = p q^1$	$F(2) = 0 + f(1) + f(2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x < \infty$		$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} p q^{x-1}$	$\lim_{x \rightarrow \infty} F(x) = 1$ (by Th.)

$$\begin{aligned}
 F(\infty) &= 1 \quad (\text{by Th.}) \\
 &= \sum_{x=1}^{\infty} p q^{x-1} \\
 &= \frac{p}{q} \sum_{x=1}^{\infty} q^x \\
 &= \frac{p}{q} \left[\frac{1}{1-q} \right] = p \\
 &= \frac{p}{q} \left[\frac{1}{p} \right] \\
 &= 1
 \end{aligned}$$

Geometric series (sum of infinite terms)

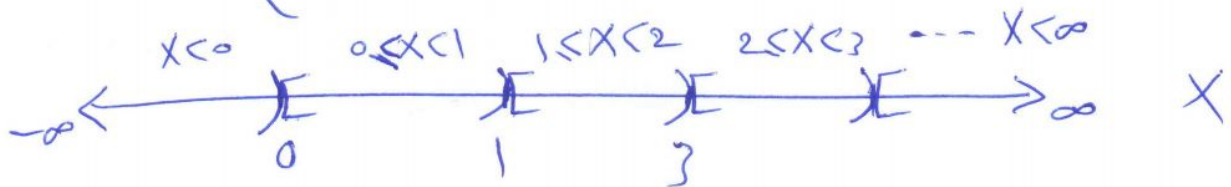
$$\begin{aligned}
 \sum_{x_j \in I} f(x_j) &= \begin{cases} 0 & \text{for } x < 1 \\ F(1) & \text{for } 1 \leq x < 2 \\ F(2) & \text{for } 2 \leq x < 3 \\ \vdots & \vdots \\ \vdots & \text{for } x < \infty \end{cases} \\
 \therefore F(x) &= \begin{cases} 0 & \text{for } x < 1 \\ F(1) & \text{for } 1 \leq x < 2 \\ F(2) & \text{for } 2 \leq x < 3 \\ \vdots & \vdots \\ \vdots & \text{for } x < \infty \end{cases}
 \end{aligned}$$

#

6

F(x) of Poisson dist

$$f(x) = \begin{cases} \frac{e^{-m} m^x}{x!} & \text{for } x=0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$



intervals	no. of integ.	$f(x) = \uparrow$	$F(x) = \sum_{0 \leq x_j \leq x} f(x_j)$
$x < 0$	no integ.	0	0
$0 \leq x < 1$	0	$f(0) = \frac{e^{-m} m^0}{0!}$	$F(0) = 0 + f(0)$
$1 \leq x < 2$	1	$f(1) = \frac{e^{-m} m^1}{1!}$	$F(1) = 0 + f(0) + f(1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x < \infty$	$\vdots$	$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{e^{-m} m^x}{x!}$	$F(\infty) = 1$ (by Theorem). $= \sum_{x=0}^{\infty} e^{-m} \frac{m^x}{x!}$ $= e^{-m} \left(\sum_{x=0}^{\infty} \frac{m^x}{x!} \right)$ $= e^{-m} e^m$ $= e^0$ $= 1$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ F(0) & \text{for } 0 \leq x < 1 \\ F(1) & \text{for } 1 \leq x < 2 \\ \vdots & \vdots \\ 1 & x < \infty \end{cases}$$

#

حلول مختصرة لبعض الواجبات
 للفصل الخامس / توزيع كاي
 والتوزيع الطبيعي والعلاقة بينها

① If $X \sim N(\mu, \sigma^2)$ & $P(X < 89) = 0.9$,
 $P(X < 94) = 0.95$, then
 Find μ & σ^2 (للمهولة نكتب μ, σ^2)

Sol.

$$P(X < 89) = 0.90$$

$$= P\left(\frac{X - \mu}{\sigma} < \frac{89 - \mu}{\sigma}\right) = 0.90$$

$$= P\left(Z < \frac{89 - \mu}{\sigma}\right) = 0.90$$

← $\left(\begin{array}{l} Z \sim N(0,1) \\ Z = \frac{X - \mu}{\sigma} \end{array}\right)$
 للتحويل الى توزيع Z

$$= N\left(\frac{89 - \mu}{\sigma}\right) = 0.90$$

$$\Rightarrow \frac{89 - \mu}{\sigma} = 1.282 \quad (\text{by } Z\text{-table})$$

$$\Rightarrow 89 - \mu = 1.282 \sigma \quad \text{--- ①} \quad \left(\begin{array}{l} \text{تكون معادله} \\ \text{أكمل الحد ...} \end{array}\right)$$

by the same way: بنفس الطريقة

$$P(X < 94) = 0.95$$

$$= P\left(\frac{X - \mu}{\sigma} < \frac{94 - \mu}{\sigma}\right) = 0.95$$

$$= P\left(Z < \frac{94 - \mu}{\sigma}\right) = 0.95$$

فكرة الحل
 من معادله ① و ② وبعد حلها
 نجد قيم كل من μ و σ المطلوبة

$$= N\left(\frac{94 - \mu}{\sigma}\right) = 0.95$$

$$= N\left(\frac{94 - \mu}{\sigma}\right) = 0.95$$

$$\Rightarrow \frac{94 - \mu}{\sigma} = 1.645 \quad (\text{by } Z\text{-table})$$

$$\Rightarrow 94 - \mu = 1.645 \sigma \quad \text{--- ②} \quad \left(\begin{array}{l} \text{تكون معادله} \\ \text{ونكمل الحد ...} \end{array}\right)$$

تحل معادله ① ومعادله ② أيضاً لإيجاد μ و σ

①

② When $X \sim N(\mu, \sigma^2)$, show that:

(i) $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95$

sol. First, use Z-table ($X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$)

$$= P\left(\frac{\mu - 2\sigma - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + 2\sigma - \mu}{\sigma}\right)$$

$$= P(-2 < Z < 2)$$

$$= P(Z < 2) - P(Z < -2)$$

$$= N(2) - N(-2)$$

$$= N(2) - [1 - N(2)]$$

$$= 2N(2) - 1$$

$$= 2(0.977) - 1 \quad (\text{by Z-table } (N(2) = 0.977))$$

$$= 0.954 \quad \#$$

(ii) $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 0.99$

$$= P\left(\frac{\mu - 3\sigma - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{\mu + 3\sigma - \mu}{\sigma}\right)$$

$$= P(-3 \leq Z \leq 3)$$

$$= P(Z \leq 3) - P(Z \leq -3)$$

$$= N(3) - N(-3)$$

$$= N(3) - [1 - N(3)]$$

$$= 2N(3) - 1$$

$$= 2(0.999) - 1 \quad (\text{by Z-table } (N(3) = 0.999))$$

$$= 0.998$$

حل بعض تمارين الفصل الخامس / توزيع χ^2 و $N(\mu, \sigma^2)$

② (iii) $P(X > \mu + \sigma) = 0.1587$

$$P(X > \mu + \sigma) = P\left(\frac{X - \mu}{\sigma} > \frac{\mu + \sigma - \mu}{\sigma}\right)$$

$$= P(Z > 1)$$

$$= 1 - P(Z \leq 1)$$

$$= 1 - N(1)$$

$$= 1 - 0.841$$

$$= 0.1590$$

(نقطة التقاطع مع المحور)

(Note: $P(Z \leq z) = P(Z < z)$ since Z is continuous r.v.)

(by Z-table)

#

③ If $X \sim \chi^2(r)$, determine the constants c & d such that:

$$P(c < X < d) = 0.015 \quad \& \quad P(X < c) = 0.975$$

Sol.

$$P(c < X < d) = P(X < d) - P(X < c)$$

$$0.015 = P(X < d) - 0.975$$

$$\therefore P(X < d) = 0.015 + 0.975 = 0.990$$

(given)

(by table of $\chi^2(r)$ & $r=5$)

(الاحتمال التراكمي للمثلث الزائد)

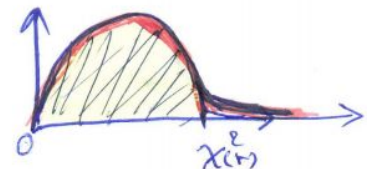
$$\Rightarrow P(X < d) = 0.990 \text{ & } r=5$$

$$\Rightarrow d = 15.09$$

$$\& \quad P(X < c) = 0.975 \text{ & } r=5$$

$$\Rightarrow c = 12.83$$

by $\chi^2(r)$ table



(المجموع التراكمي لتوزيع كاي)

(cumulative chi-square dist. table)

③

حل بعض مسائل الفحص الخامس / توزيع $N(1, 0.2)$ و $\chi^2(r)$

④ If $X \sim \chi^2(17)$; find the following probabilities:

(i) $P(X < 7.564) = 0.025$

(ii) $P(X > 27.59) = 1 - P(X \leq 27.59)$
 $= 1 - 0.950$
 $= 0.05$

} by
Cumulative
Chi-Square
dist. Table

(with $r=17$)
↓

(iii) $P(8.67 \leq X < 27.59)$
 $= P(X < 27.59) - P(X \leq 8.67)$
 $= 0.950 - 0.050$
 $= 0.90$

} (=)

⑤ If $X \sim \chi^2(23)$, then find the following:

a. $P(10.20 < X < 35.17)$

$= P(X < 35.17) - P(X < 10.20)$

$= 0.950 - 0.01$

$= 0.940 \in [0, 1]$

} → (by Cumulative
Chi-Square
dist. Table
with $r=23$)

b. If $P(a < X < b) = 0.950$ & $P(X < a) = 0.025$.
 Then find the constants a & b .

sol. $P(a < X < b) = P(X < b) - P(X < a)$
 $0.950 = P(X < b) - 0.025$ (given)

$\therefore P(X < b) = 0.950 + 0.025$
 $= 0.975$

$\Rightarrow b = 38.08$

& $P(X < a) = 0.025 \Rightarrow a = 11.69$

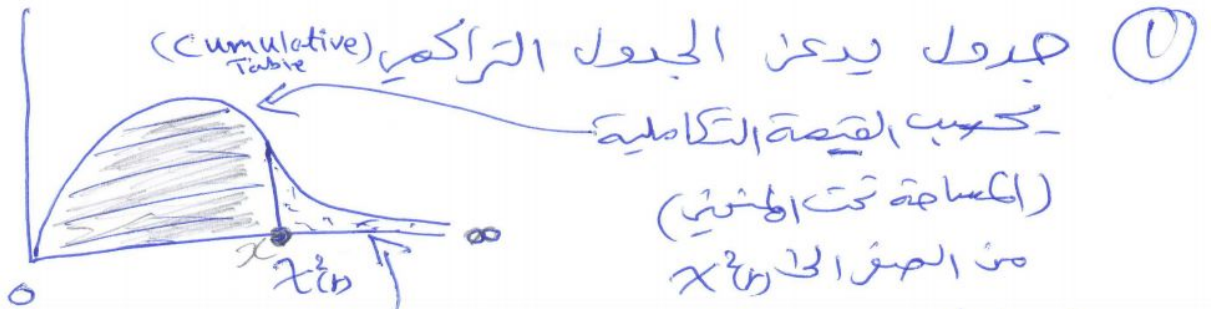
} (=)
جدول مربع كاي
الحركي
 $r=23$

④

#

ملاحظات حول جدول التوزيع مربع كاي $\chi^2(r)$

هناك جدولان لـ $\chi^2(r)$



(كلما زادت قيمة x زادت الاحتمالية (أي زادت المساحة تحت المنحنى))

$$P(X \leq x) = F(x)$$

② وهناك جدول آخر لـ $\chi^2(r)$ يدعى الجدول المكمل (right tail)

حسب المساحة تحت المنحنى المتبقية صاغة الى ∞

(كلما زادت القيمة x كلما قلت المساحة تحت المنحنى)

أي قلت نسبة الاحتمالية -

(المساحة الصغيرة على اليمين وتدعى الذيل الأيمن)

$$P(X > x)$$

$\chi^2(r)$ Tables

Cumulative Chi-Square Table

الجدول التراكمي لتوزيع مربع كاي

$$P(X \leq x) = \int_0^x f(t) dt$$

(المساحة الأكبر)

(نستخدمه في معظم الحالات)

Right Tail

Chi-Square Table

الجدول ذوالذيل الأيمن لتوزيع مربع كاي

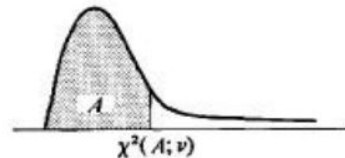
$$P(X > x) = \int_x^{\infty} f(t) dt$$

(المساحة الأصغر)

Chi-Square Distribution

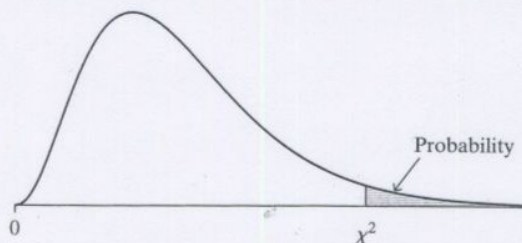
Table C-2. Percentiles of the χ^2 Distribution

Entry is $\chi^2(A; \nu)$ where $P\{\chi^2(\nu) \leq \chi^2(A; \nu)\} = A$



ν	A									
	.005	.010	.025	.050	.100	.900	.950	.975	.990	.995
1	0.004393	0.00157	0.00982	0.02393	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.60
3	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84
4	0.207	0.297	0.484	0.711	1.064	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.145	1.61	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.4	104.2
80	51.17	53.54	57.15	60.39	64.28	96.58	101.9	106.6	112.3	116.3
90	59.20	61.75	65.65	69.13	73.29	107.6	113.1	118.1	124.1	128.3
100	67.33	70.06	74.22	77.93	82.36	118.5	124.3	129.6	135.8	140.2

TABLE C: Chi-Squared Distribution Values for Various Right-Tail Probabilities



df	Right-Tail Probability						
	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.32	2.71	3.84	5.02	6.63	7.88	10.83
2	2.77	4.61	5.99	7.38	9.21	10.60	13.82
3	4.11	6.25	7.81	9.35	11.34	12.84	16.27
4	5.39	7.78	9.49	11.14	13.28	14.86	18.47
5	6.63	9.24	11.07	12.83	15.09	16.75	20.52
6	7.84	10.64	12.59	14.45	16.81	18.55	22.46
7	9.04	12.02	14.07	16.01	18.48	20.28	24.32
8	10.22	13.36	15.51	17.53	20.09	21.96	26.12
9	11.39	14.68	16.92	19.02	21.67	23.59	27.88
10	12.55	15.99	18.31	20.48	23.21	25.19	29.59
11	13.70	17.28	19.68	21.92	24.72	26.76	31.26
12	14.85	18.55	21.03	23.34	26.22	28.30	32.91
13	15.98	19.81	22.36	24.74	27.69	29.82	34.53
14	17.12	21.06	23.68	26.12	29.14	31.32	36.12
15	18.25	22.31	25.00	27.49	30.58	32.80	37.70
16	19.37	23.54	26.30	28.85	32.00	34.27	39.25
17	20.49	24.77	27.59	30.19	33.41	35.72	40.79
18	21.60	25.99	28.87	31.53	34.81	37.16	42.31
19	22.72	27.20	30.14	32.85	36.19	38.58	43.82
20	23.83	28.41	31.41	34.17	37.57	40.00	45.32
25	29.34	34.38	37.65	40.65	44.31	46.93	52.62
30	34.80	40.26	43.77	46.98	50.89	53.67	59.70
40	45.62	51.80	55.76	59.34	63.69	66.77	73.40
50	56.33	63.17	67.50	71.42	76.15	79.49	86.66
60	66.98	74.40	79.08	83.30	88.38	91.95	99.61
70	77.58	85.53	90.53	95.02	100.4	104.2	112.3
80	88.13	96.58	101.8	106.6	112.3	116.3	124.8
90	98.65	107.6	113.1	118.1	124.1	128.3	137.2
100	109.1	118.5	124.3	129.6	135.8	140.2	149.5

Source: Calculated using *StaTable*, software from Cytel Software, Cambridge, MA.

TABLE II
The Chi-Square Distribution*

$$\Pr(X \leq x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2} dw$$

r	Pr(X ≤ x)					
	0.01	0.025	0.050	0.95	0.975	0.99
1	0.000	0.001	0.004	3.84	5.02	6.63
2	0.020	0.051	0.103	5.99	7.38	9.21
3	0.115	0.216	0.352	7.81	9.35	11.3
4	0.297	0.484	0.711	9.49	11.1	13.3
5	0.554	0.831	1.15	11.1	12.8	15.1
6	0.872	1.24	1.64	12.6	14.4	16.8
7	1.24	1.69	2.17	14.1	16.0	18.5
8	1.65	2.18	2.73	15.5	17.5	20.1
9	2.09	2.70	3.33	16.9	19.0	21.7
10	2.56	3.25	3.94	18.3	20.5	23.2
11	3.05	3.82	4.57	19.7	21.9	24.7
12	3.57	4.40	5.23	21.0	23.3	26.2
13	4.11	5.01	5.89	22.4	24.7	27.7
14	4.66	5.63	6.57	23.7	26.1	29.1
15	5.23	6.26	7.26	25.0	27.5	30.6
16	5.81	6.91	7.96	26.3	28.8	32.0
17	6.41	7.56	8.67	27.6	30.2	33.4
18	7.01	8.23	9.39	28.9	31.5	34.8
19	7.63	8.91	10.1	30.1	32.9	36.2
20	8.26	9.59	10.9	31.4	34.2	37.6
21	8.90	10.3	11.6	32.7	35.5	38.9
22	9.54	11.0	12.3	33.9	36.8	40.3
23	10.2	11.7	13.1	35.2	38.1	41.6
24	10.9	12.4	13.8	36.4	39.4	43.0
25	11.5	13.1	14.6	37.7	40.6	44.3
26	12.2	13.8	15.4	38.9	41.9	45.6
27	12.9	14.6	16.2	40.1	43.2	47.0
28	13.6	15.3	16.9	41.3	44.5	48.3
29	14.3	16.0	17.7	42.6	45.7	49.6
30	15.0	16.8	18.5	43.8	47.0	50.9

* This table is abridged and adapted from "Tables of Percentage Points of the Incomplete Beta Function and of the Chi-Square Distribution," *Biometrika*, 32 (1941). It is published here with the kind permission of Professor E. S. Pearson on behalf of the author, Catherine M. Thompson, and of the Biometrika Trustees.

Some Estimation Methods

In this section we introduce some estimation methods to estimate the parameters of distribution as below

- ① Maximum Likelihood estimation Method. (MLE)
- ② Moments estimator Method (MOM).

① Maximum Likelihood Estimator Method (MLE)

The concept of this method to find and estimate the value of parameter θ of the distribution (population) using $\uparrow$ random Sample $(x_1, \dots, x_n)$ and this estimator denoted by

$\hat{\theta} = T(x)$ which called Maximum Likelihood estimator (MLE)

however, the Likelihood Function $L(x_1, x_2, \dots, x_n, \theta)$ is a joint probability function of random Sample $(x_1, \dots, x_n)$ and the value of $\hat{\theta}$ will maximize the Likelihood Function $L(x_1, \dots, x_n, \theta)$.

The steps of Finding Maximum likelihood Estimator (MLE) as below

- ① Find the Likelihood function $L(x_1, \dots, x_n, \theta)$ as follows

$$\text{i.e. } L(x_1, x_2, \dots, x_n, \theta) = \prod_{i=1}^n f(x_i, \theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdot \dots \cdot f(x_n, \theta) \\ = f(x_1, x_2, \dots, x_n, \theta) \quad (\text{J.P.F})$$

- ② Take the natural logarithm of $L(x_1, \dots, x_n, \theta)$

$$\text{i.e., } \ln L(x_1, \dots, x_n, \theta) = \ln L$$

- ③ Take the derivative (differentiate) of $\ln L(x_1, \dots, x_n, \theta)$ in ②

$$\text{i.e., } \frac{\partial \ln L(x_1, \dots, x_n, \theta)}{\partial \theta} \quad \text{or} \quad \frac{\partial \ln L}{\partial \theta}$$

- ④ Equate the result in step (3) to zero

$$\text{i.e., } \frac{\partial \ln L(x_1, \dots, x_n, \theta)}{\partial \theta} = 0 \quad \text{and the result is } \hat{\theta} \quad \left(\begin{array}{l} \text{Estimator} \\ \text{of } \theta \\ \text{using MLE} \end{array} \right)$$

- ⑤ To be sure that $\hat{\theta}$ maximize the likelihood function

$$\frac{\partial^2 \ln L(x_1, \dots, x_n, \theta)}{\partial \theta^2} < 0 \quad \text{or} \quad \frac{\partial^2 \ln L}{\partial \theta^2} < 0 \quad \#$$

ex.1 let $x_1, \dots, x_n$ be a.r.s. from Bernoulli distribution with parameter p . Estimate (p) using MLE Method.

Sol. $f(x; p) = \begin{cases} p^x (1-p)^{1-x} & \text{for } x=0,1 \\ 0 & \text{o.w} \end{cases}$

$$\begin{aligned} L = L(x_1, x_2, \dots, x_n; p) &= \prod_{i=1}^n f(x_i; p) \\ &= [p^{x_1} (1-p)^{1-x_1}] \cdot [p^{x_2} (1-p)^{1-x_2}] \cdots [p^{x_n} (1-p)^{1-x_n}] \\ &= p^{\sum x_i} (1-p)^{n-\sum x_i} \end{aligned}$$

$$\ln L = (\sum x_i) \ln p + (n - \sum x_i) \ln (1-p)$$

$$\frac{\partial \ln L}{\partial p} = (\sum x_i) \frac{1}{p} + (n - \sum x_i) \cdot \frac{1}{(1-p)} (-1) = 0$$

$$= \frac{\sum x_i}{\hat{p}} - \frac{n - \sum x_i}{1 - \hat{p}} = 0$$

$$\frac{\sum x_i (1 - \hat{p}) - (n - \sum x_i) \hat{p}}{\hat{p} (1 - \hat{p})} = 0$$

$$\sum x_i - \hat{p} (\sum x_i) - n \hat{p} + (\sum x_i) \hat{p} = 0$$

$$\sum x_i = n \hat{p} \rightarrow \hat{p} = \frac{\sum x_i}{n} = \bar{x} \quad [\hat{p}_{MLE} = \bar{x}]$$

ex.2 let $x_1, x_2, \dots, x_n$ be a.r.s. from exponential distribution with parameter θ . Estimate (θ) using MLE method.

Sol. $\because X \sim \text{Exp}(\theta) \rightarrow f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{o.w} \end{cases}$

$$\begin{aligned} L = L(x_1, x_2, \dots, x_n; \theta) &= \prod_{i=1}^n f(x_i; \theta) = \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} \\ &= \left[\frac{1}{\theta} e^{-x_1/\theta} \right] \times \left[\frac{1}{\theta} e^{-x_2/\theta} \right] \cdots \left[\frac{1}{\theta} e^{-x_n/\theta} \right] \\ &= \frac{1}{\theta^n} \cdot e^{-\frac{\sum x_i}{\theta}} \end{aligned}$$

$$\begin{aligned} \ln L &= \ln \left(\frac{1}{\theta^n} \cdot e^{-\frac{\sum x_i}{\theta}} \right) = \ln \frac{1}{\theta^n} + \ln \left[e^{-\frac{\sum x_i}{\theta}} \right] \\ &= -n \ln \theta - \frac{\sum x_i}{\theta} \end{aligned}$$

$$\frac{\partial \ln L}{\partial \theta} = -n \frac{1}{\theta} + \frac{1}{\theta^2} \sum x_i = 0 \quad * (\theta^2)$$

$$\Rightarrow -n \hat{\theta} + \sum x_i = 0 \rightarrow n \hat{\theta} = \sum x_i \Rightarrow \hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$$

ex.3 Let $x_1, \dots, x_n$ be a.r.s. from gamma dist. with parameter α and $(\frac{1}{\beta})$, where α is known and (β) is unknown. Estimate β using Mle.

Sol. $f(x; \beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$

$$L = L(x_1, x_2, \dots, x_n, \beta) = \prod_{i=1}^n f(x_i; \beta) = \left[\frac{\beta^\alpha}{\Gamma(\alpha)} x_1^{\alpha-1} e^{-\beta x_1} \right] \dots \left[\frac{\beta^\alpha}{\Gamma(\alpha)} x_n^{\alpha-1} e^{-\beta x_n} \right]$$

$$= (\beta^\alpha)^n [\Gamma(\alpha)]^{-n} \left(\prod_{i=1}^n x_i^{\alpha-1} \right) (e^{-\beta \sum x_i})$$

$$\ln L = (n\alpha) \ln \beta - n \ln \Gamma(\alpha) + \sum_{i=1}^n \ln x_i^{\alpha-1} - \beta \sum x_i$$

$$= n\alpha \ln \beta - n \ln \Gamma(\alpha) + (\alpha-1) \sum_{i=1}^n \ln x_i - \beta \sum x_i$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n\alpha}{\beta} - 0 + 0 - \sum x_i = 0$$

$$\Rightarrow n\alpha = \hat{\beta}(\sum x_i) \Rightarrow \hat{\beta} = \frac{n\alpha}{\sum x_i} = \frac{\alpha}{\frac{\sum x_i}{n}} = \frac{\alpha}{\bar{x}}$$

$$\therefore \hat{\beta}_{\text{MLE}} = \frac{\alpha}{\bar{x}} \quad \left(\left(\frac{1}{\beta} = \bar{x} \right) \right)$$

ex.4 If $x_1, \dots, x_n$ be a.r.s. from poisson dist. with λ Estimate λ using Mle.

Sol. $x \sim \text{po}(\lambda) \rightarrow f(x; \lambda) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & ; x \geq 0 \\ 0 & \text{o.w.} \end{cases}$

$$L = L(x_1, \dots, x_n; \lambda) = \prod_{i=1}^n f(x_i; \lambda) = \left(\frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \right) \dots \left(\frac{e^{-\lambda} \lambda^{x_n}}{x_n!} \right)$$

$$= \frac{e^{-n\lambda} \lambda^{\sum x_i}}{\prod_{i=1}^n x_i!}$$

$$\ln L = \ln e^{-n\lambda} + \ln \lambda^{\sum x_i} - \ln \prod_{i=1}^n x_i!$$

$$\frac{\partial \ln L}{\partial \lambda} = -n + \frac{\sum x_i}{\lambda} = 0$$

$$\Rightarrow n\lambda = \sum x_i \Rightarrow \hat{\lambda} = \frac{\sum x_i}{n} = \bar{x}$$

ex-5 let $x_1, x_2, \dots, x_n$ be a.r.s. from Geometric dist with parameter (p) . Estimate (p) using MLE Method.

Sol. $X \sim \text{Geo}(p) \rightarrow f(x, p) = \begin{cases} p(1-p)^{x-1} & \text{for } x=1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$

$$L = L(x_1, x_2, \dots, x_n, p) = \prod_{i=1}^n p(1-p)^{x_i-1}$$

$$= [p(1-p)]^n \cdot [p(1-p)^{x_i-1}]$$

$$= p^n (1-p)^{\sum x_i - n}$$

$$= \left[\frac{p}{1-p} \right]^n (1-p)^{\sum x_i}$$

$$\ln L = n \ln p + \sum x_i \ln(1-p)$$

$$\frac{\partial \ln L}{\partial p} = n \left(\frac{1-p}{p} \right) \left[\frac{-p \cdot 1 - (1-p)}{(1-p)^2} \right] + \sum x_i \cdot \frac{1}{1-p} (-1) = 0$$

$$= n \frac{(1-p)}{p} \cdot \frac{1}{(1-p)^2} - \frac{\sum x_i}{1-p} = 0$$

$$= \frac{n}{p(1-p)} - \frac{\sum x_i}{(1-p)} = 0$$

$$= \frac{n - p \sum x_i}{p(1-p)} = 0$$

$$\Rightarrow n - \hat{p} \sum x_i = 0$$

$$n = \hat{p} \sum x_i \Rightarrow \hat{p} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}} \quad \times$$

H.w. ① Suppose $x_1, \dots, x_n$ be a.r.s. have a p.d.f below

$$f(x; \theta) = \begin{cases} (\theta+1)x^{\theta+1} & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \quad \leftarrow \text{Check pdf.}$$

Estimate θ using MLE Method.

H.w. ② Given a distribution function as below

$$F(x; \theta_1, \theta_2) = \begin{cases} 1 - \left(\frac{\theta_1}{x}\right)^{\theta_2} & x > \theta_1 \\ 0 & \text{o.w.} \end{cases}$$

where θ_1 is known and θ_2 is unknown.

Estimate θ_2 using mle method.

Hint $f(x; \theta_1, \theta_2) = F(x; \theta_1, \theta_2) - F(x; \theta_1, \theta_2) = \theta_2 \theta_1^{-\theta_2} x^{-\theta_2-1}$

$$L = L(x_1, \dots, x_n; \theta_1, \theta_2) = \theta_2^n \theta_1^{-n\theta_2} \prod_{i=1}^n x_i^{-(\theta_2+1)}$$

$$\ln L = n \ln \theta_2 + n \theta_2 \ln \theta_1 - (\theta_2+1) \sum_{i=1}^n \ln x_i$$

$$0 = \frac{\partial \ln L}{\partial \theta_2} = n \cdot \frac{1}{\theta_2} + n \ln \theta_1 - \sum \ln x_i \Rightarrow \hat{\theta}_2 = \frac{n}{\sum \ln x_i - n \ln \theta_1}$$

[2] Moments Estimator Method (MoM)

let $x_1, x_2, \dots, x_n$ be a.r.s. from a population which has p.f. $f(x; \theta)$, where θ is unknown parameter that we have to estimate. The idea of this method (MoM) is to find the moments of population and the moments of sample and then equal them to find θ as follows:

- ① Find the population moments as a number of parameters ^{steps}
i.e. $\mu_r = E(X^r)$; $\mu_1 = E(X)$, $\mu_2 = E(X^2)$, ---
- ② Find the Sample moments as a number of parameters
i.e. $\mu'_r = \frac{\sum x_i^r}{n}$ $\rightarrow \mu'_1 = \frac{\sum x_i}{n}$, $\mu'_2 = \frac{\sum x_i^2}{n}$, ---
- ③ population moments = Sample moments (as No. of parameters)
i.e. $\mu_r = \mu'_r \Rightarrow \mu_1 = \mu'_1, \mu_2 = \mu'_2, \dots, \mu_k = \mu'_k$ [if No. of par. is k]

ex.1 let $x_1, x_2, \dots, x_n$ be a.r.s. from Binomial dist. with parameter (p) . Estimate p using MoM ($\hat{p}_{mom}$).

$\because f(x; p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0, 1, \dots, n \\ 0 & \text{o.w.} \end{cases}$

- ① $\mu_1 = E(X) = np$ (population moment) ^{1st} one parameter p
- ② $\mu'_1 = \frac{\sum x_i}{n} = \bar{X}$ (1st sample moment) substituted 1st eq.
- ③ $\mu_1 = \mu'_1 \rightarrow np = \bar{X} \Rightarrow \hat{p}_{mom} = \frac{\bar{X}}{n}$

ex.2 let $x_1, x_2, \dots, x_n$ be a.r.s. from exponential distribution with parameter (θ) . Find $\hat{\theta}_{mom}$.

sol. $X \sim \text{Exp}(\theta) \rightarrow E(X) = \frac{1}{\theta}$

$f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x > 0 \\ 0 & \text{o.w.} \end{cases}$

- ① $\mu_1 = E(X) = \frac{1}{\theta}$ ② $\mu'_1 = \frac{\sum x_i}{n} = \bar{X}$ one parameter
- ③ $\hat{\theta}_{mom} = \frac{\sum x_i}{n} = \bar{X}$ need only 1st moment of pop. & sample

ex 3 let $x_1, x_2, \dots, x_n$ be a r.s from gamma dist. with parameters α and β . Find $\hat{\alpha}_{mom}$ & $\hat{\beta}_{mom}$.

Sol $X \sim G(\alpha, \beta) \Rightarrow f(x; \alpha, \beta) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^\alpha} & x > 0 \\ 0 & o.w \end{cases}$

① $M_1 = E(X) = \alpha\beta$

② $\hat{M}_1 = \frac{\sum x_i}{n} = \bar{x}$

③ $M_1 = \hat{M}_1 \Rightarrow \hat{\alpha}\hat{\beta} = \bar{x} \Rightarrow \boxed{\hat{\beta} = \frac{\bar{x}}{\hat{\alpha}}} \quad \text{--- (1)}$

① $M_2 = E(X^2) = Var(X) + [E(X)]^2$
 $= \alpha\beta^2 + \alpha^2\beta^2$

② $\hat{M}_2 = \frac{\sum x_i^2}{n}$

③ $\hat{M}_2 = \hat{M}_2 \Rightarrow \boxed{\frac{\sum x_i^2}{n} = \hat{\alpha}^2 \hat{\beta}^2 + \hat{\alpha}^2 \hat{\beta}^2} \quad \text{--- (2)}$

Substitute equation (1) in equation (2), we get

$\frac{\sum x_i^2}{n} = \cancel{\hat{\alpha}} \cdot \frac{\bar{x}^2}{\hat{\alpha}^2} + \hat{\alpha}^2 \cdot \frac{\bar{x}^2}{\cancel{\hat{\alpha}}^2}$

$\frac{\sum x_i^2}{n} - \bar{x}^2 = \frac{\bar{x}^2}{\hat{\alpha}}$
 $s^2 = \frac{\bar{x}^2}{\hat{\alpha}} \Rightarrow \boxed{\hat{\alpha} = \frac{\bar{x}^2}{s^2}}$

substitute in eqn (1)

$\hat{\beta} = \frac{\bar{x}}{\frac{\bar{x}^2}{s^2}} = \frac{s^2}{\bar{x}}$

$\cancel{s^2} / s^2 = \frac{\sum (x_i - \bar{x})^2}{n} = \frac{\sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2)}{n}$
 $= \frac{\sum x_i^2 - 2n\bar{x} + n\bar{x}^2}{n}$
 $= \frac{\sum x_i^2}{n} - \bar{x}$

H.W. (1) $f(x, \theta) = \begin{cases} \frac{1}{\theta} & x \in [0, \theta] \\ 0 & o.w \end{cases}$

Find $\hat{\theta}_{mom}$

(2) $X \sim \text{Geom}(p)$
Find $\hat{p}_{mom}$

(3) $X \sim \text{po}(m)$
Find $\hat{m}_{mom}$

(4) $X \sim \text{Ber}(\theta)$
Find $\hat{\theta}_{mom}$

Some Properties of good estimator

I) Unbiased estimator بیاضی استیمایٹر

Let $x_1, x_2, \dots, x_n$ be a r.s. from population having p.f $f(x; \theta)$ with unknown parameter θ . The estimator $\hat{\theta}$ is said to be unbiased estimator of θ iff

$$E(\hat{\theta}) - \theta = \text{Zero} \longrightarrow E(\hat{\theta}) = \theta \quad [E(\hat{\theta} - \theta) = 0]$$

While the bias of an estimator $\hat{\theta}$ can be defined as below

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta \neq 0 \longrightarrow \text{if } E(\hat{\theta}) \neq \theta \quad [\hat{\theta} \text{ biased est.}]$$

$$\text{If } \text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta = 0 \longrightarrow \text{if } E(\hat{\theta}) = \theta \quad [\hat{\theta} \text{ unbiased est.}]$$

ex.1 let $x_1, x_2, \dots, x_n$ be a r.s. from exponential dist. with parameter θ . Show that $\bar{x}$ is unbiased estimator for θ

Sol. $\because X \sim \text{Exp}(\theta) \longrightarrow f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x > 0, \theta > 0 \\ 0 & \text{o.w} \end{cases}$

$$E(\bar{x}) = E\left[\frac{\sum x_i}{n}\right] = \frac{1}{n} [E(x_1) + \dots + E(x_n)] \quad [E(x_i) = \theta, \forall x_i = \theta] \text{ (ides)}$$

$$= \frac{1}{n} [\theta + \theta + \dots + \theta] = \frac{n\theta}{n} = \theta$$

$$E(\bar{x}) = \theta \longrightarrow \bar{x} \text{ is unbiased estimator}$$

or $\text{bias}(\bar{x}) = E(\bar{x}) - \theta = 0 \longrightarrow \bar{x} \text{ is unbiased estimator}$

Tip 1 $\text{Var}(\bar{x}) = \text{Var}\left[\frac{\sum x_i}{n}\right] = \frac{1}{n^2} \text{Var}[x_1 + \dots + x_n] = \frac{1}{n^2} [\text{Var}(x_1) + \dots + \text{Var}(x_n)]$

$$= \frac{1}{n^2} [\theta^2 + \dots + \theta^2] = \frac{n\theta^2}{n^2} = \frac{\theta^2}{n} \quad \times$$

ex.2 let $x_1, \dots, x_n$ be a r.s. from geometric dist. with pars θ . Show that the sample mean $(\bar{x})$ is unbiased estimator for $\frac{1}{\theta}$.

Sol. $X \sim \text{Geo}(\theta) \longrightarrow f(x; \theta) = \begin{cases} \theta(1-\theta)^{x-1} & \text{for } x=1, 2, \dots, \infty \\ 0 & \text{o.w} \end{cases}$

$$E(\bar{x}) = E\left(\frac{\sum x_i}{n}\right) = \frac{1}{n} \cdot E(x_1 + \dots + x_n) \quad [E(x_i) = \frac{1}{\theta}, \forall x_i = \frac{1}{\theta}]$$

$$= \frac{1}{n} [E(x_1) + \dots + E(x_n)]$$

$$= \frac{1}{n} \left[\frac{1}{\theta} + \frac{1}{\theta} + \dots + \frac{1}{\theta}\right]$$

$$= \frac{n}{n} \cdot \frac{1}{\theta} = \frac{1}{\theta}$$

$\therefore E(\bar{x}) = \frac{1}{\theta} \longrightarrow \bar{x} \text{ is unbiased estimator of } \frac{1}{\theta}$

H.W. If $X_1, \dots, X_n$ be a.r.s. from binomial dist. with parameters (n, p) where p is unknown. Find $\hat{p}_{MLE}$ and show that $\hat{p}_{MLE}$ is unbiased estimator for p .

Hint $f(x; p) = \binom{n}{x} p^x (1-p)^{n-x} \quad x=0, 1, \dots, n$ $E(X) = np$
 $Var(X) = np(1-p)$

$$L = L(X_1, X_2, \dots, X_n; p) = \prod_{i=1}^n f(x_i; p) = \prod_{i=1}^n \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$$

$$\ln L = \sum_{i=1}^n \ln \binom{n}{x_i} + \sum_{i=1}^n x_i \ln p + n \ln(1-p) - \sum_{i=1}^n x_i \ln(1-p)$$

$$\frac{\ln L}{n} = 0 \quad \longrightarrow \quad \hat{p}_{MLE} = \frac{\sum x_i}{n}$$

$$E(\hat{p}_{MLE}) = E\left(\frac{\sum x_i}{n}\right) = \frac{1}{n} \cdot n E(X_i) = \frac{1}{n} [np] = \frac{n}{n} p = p$$

H.W. If $X_1, \dots, X_n$ from poisson dist. with parameter λ . Find $\hat{\lambda}_{MLE}$ & $\hat{\lambda}_{mom}$ and Test the unbiasedness of them.

[2] Consistent estimator

Let $X_1, \dots, X_n$ be a.r.s. from population with p.f. $f(x; \theta)$ with unknown parameter θ , the estimator (statistic) $\hat{\theta}$ is said to be consistent estimator for θ iff

① $\lim_{n \rightarrow \infty} E(\hat{\theta}_n) = \theta$

② $\lim_{n \rightarrow \infty} Var(\hat{\theta}_n) = 0$

ex-1 let $X_1, X_2, \dots, X_n$ be a.r.s. from poisson dist with (λ) .

Find the $(\hat{\lambda}_{MLE})$ and test consistent

Sol $f(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \quad x=0, 1, 2, \dots, \infty$ $E(X) = Var(X) = \lambda$

We derive $\hat{\lambda}_{MLE} = \bar{X}$ $\rightarrow E(\bar{X}) = \frac{1}{n} E(X_i) = E(X_i) = \lambda$
 $\rightarrow Var(\bar{X}) = \frac{1}{n^2} Var(X_i) = \frac{1}{n} (\lambda) = \frac{\lambda}{n}$

① $\lim_{n \rightarrow \infty} E(\bar{X}) = \lim_{n \rightarrow \infty} \lambda = \lambda$

② $\lim_{n \rightarrow \infty} Var(\bar{X}) = \lim_{n \rightarrow \infty} \frac{\lambda}{n} = \lambda(0) = 0$ $\Rightarrow \hat{\lambda}_{MLE}$ consistent est.

ex.2 if $x_1, \dots, x_n$ be a.r.s. from geometric dist. with par. θ .

Find $\hat{\theta}_{MLE}$ and test unbiasedness and consistent for $\frac{1}{\theta}$

Sol. $f(x; \theta) = \theta(1-\theta)^{x-1}$ for $x=1, 2, \dots$; $E(X) = \frac{1}{\theta}$, $Var(X) = \frac{(1-\theta)}{\theta^2}$

as we know, $\bar{X}$ is unbiased estimator for $\frac{1}{\theta}$ [$E(\bar{X}) = \frac{1}{\theta}$]

Know, (1) $\lim_{n \rightarrow \infty} E(\bar{X}) = \lim_{n \rightarrow \infty} \frac{1}{\theta} = \frac{1}{\theta}$ [$Var(\bar{X}) = \frac{\theta}{n^2} \cdot \frac{(1-\theta)}{\theta^2} = \frac{(1-\theta)}{n\theta^2}$]

$$(2) \lim_{n \rightarrow \infty} Var(\bar{X}) = \lim_{n \rightarrow \infty} \frac{(1-\theta)}{n\theta^2} = \frac{(1-\theta)}{\theta^2} \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$\therefore \bar{X}$ is consistent estimator for $\frac{1}{\theta}$

ex.3 If $x_1, \dots, x_n$ be a.r.s. from exponential dist. with parameter α . Find $\hat{\alpha}_{MLE}$ and test unbiasedness and consistent of $\hat{\alpha}_{MLE}$ w.r.t α

Sol. $f(x; \alpha) = \begin{cases} \frac{1}{\alpha} e^{-x/\alpha} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$ $E(X) = \alpha$
 $Var(X) = \alpha^2$

as we know $\hat{\alpha}_{MLE} = \bar{X}$ & $E(\bar{X}) = \alpha$ [unbiased estimator]

$$(1) \lim_{n \rightarrow \infty} E(\bar{X}) = \lim_{n \rightarrow \infty} \alpha = \alpha \quad Var(\bar{X}) = \frac{\alpha^2}{n}$$

$$(2) \lim_{n \rightarrow \infty} Var(\bar{X}) = \lim_{n \rightarrow \infty} \frac{\alpha^2}{n} = 0$$

$\therefore \hat{\alpha}_{MLE} = \bar{X}$ is consistent estimator.

Properties of Estimation

Review

Methods of Estimation

Mle

طريقة
المحتمل الأعظم

$$\textcircled{1} L(x_i; \theta) = L(x_1, x_2, \dots, x_n; \theta) \\ = \prod_{i=1}^n f(x_i; \theta)$$

$$\textcircled{2} \ln L(x_i; \theta)$$

$$\textcircled{3} \frac{\partial \ln L(x_i; \theta)}{\partial \theta} = 0$$

$\Rightarrow \hat{\theta}_{MLE}$ estimator of θ

طريقة العزم

Mom

Population moment = Sample moment

$$E(x^r) = \sum_{i=1}^n \frac{x_i^r}{n}$$

$r = 1, 2, \dots$

$r = \text{no. of parameters}$

Properties of Estimation

صفة التقدير
الجيد

Unbiasedness

عدم التحيز

$$B(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$$= 0$$

$\Rightarrow E(\hat{\theta}) = \theta$
 $\Rightarrow \hat{\theta}$ is unbiased
estimator of θ

(الصفة المرغوبة)
للتقدير

$$\neq 0$$

$\Rightarrow E(\hat{\theta}) \neq \theta$
 $\Rightarrow \hat{\theta}$ is biased
estimator of θ

Consistency

الاتساق

$$\textcircled{1} \lim_{n \rightarrow \infty} E(\hat{\theta}) = \theta$$

$$\textcircled{2} \lim_{n \rightarrow \infty} V(\hat{\theta}) = 0$$

$\Rightarrow \hat{\theta}$ is consistent
estimator of θ

(صفة مرغوبة للتقدير)

Ex. 1 Let $X_1, X_2, \dots, X_n$ be a.r.s. follows $\exp(\theta)$.

① Show that $\hat{\theta} = \bar{X}$ is biased for θ .

Sol.

$$\textcircled{1} f(x; \theta) = \begin{cases} \theta e^{-\theta x} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \frac{1}{\theta}, \quad V(X) = \frac{1}{\theta^2}$$

$$\begin{aligned} E(\hat{\theta}) &= E(\bar{X}) = E\left(\frac{\sum_{i=1}^n X_i}{n}\right) \\ &= \frac{1}{n} E\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n} \cdot n \cdot E(X_i) \\ &= \frac{1}{\theta} \quad \left(\begin{array}{l} (X_i \sim \exp(\theta)) \\ E(X_i) = \frac{1}{\theta} \end{array} \right) \\ &\neq \theta \end{aligned}$$

$\therefore \hat{\theta} = \bar{X}$ is biased estimator of θ .

② Show that $\hat{\theta} = \bar{X}$ is unbiased estimator of θ .

$$\textcircled{2} f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \theta, \quad V(X) = \theta^2$$

$$\begin{aligned} E(\hat{\theta}) &= E(\bar{X}) = E\left(\frac{\sum_{i=1}^n X_i}{n}\right) \\ &= \frac{1}{n} E(\sum X_i) \\ &= \frac{1}{n} \cdot n \cdot E(X_i) \\ &= \theta \quad \left(\begin{array}{l} (X_i \sim \exp(\theta)) \\ E(X_i) = \theta \end{array} \right) \end{aligned}$$

$\therefore \hat{\theta} = \bar{X}$ is unbiased estimator of θ .

Ex. 2 Let X_i be a r.v.s. follows $\text{ber}(\theta)$, then $\hat{\theta}_{\text{mom}}$ estimator is equal $\bar{X}$; test the unbiasedness of $\hat{\theta}$.

Sol. $X_i \sim \text{ber}(\theta) \rightarrow E(X_i) = \theta \quad \forall \quad V(X_i) = \theta(1-\theta)$

$$f(x; \theta) = \begin{cases} \theta^x (1-\theta)^{1-x} & \text{for } x=0, 1, 2, \dots, n \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} E(\hat{\theta}_{\text{mom}}) &= E(\bar{X}) = E\left(\frac{\sum_{i=1}^n X_i}{n}\right) \\ &= \frac{1}{n} E(\sum X_i) \\ &= \frac{1}{n} \cdot n \cdot E(X_i) \\ &= \theta \end{aligned}$$

$\therefore \hat{\theta}_{\text{mom}}$ is unbiased estimator of θ .

Ex. 3 If $X_i \sim \text{ber}(\theta)$, then $\hat{\theta}_{\text{mom}} = \bar{X}$ is consistent estimator of θ .

Sol.

$$\begin{aligned} \textcircled{1} \lim_{n \rightarrow \infty} E(\hat{\theta}) &= \lim_{n \rightarrow \infty} E(\bar{X}) \\ &= \lim_{n \rightarrow \infty} \theta \\ &= \theta \quad (\text{Condition } \textcircled{1} \text{ is satisfied}). \end{aligned}$$

$$\begin{aligned} \textcircled{2} \lim_{n \rightarrow \infty} V(\hat{\theta}) &= \lim_{n \rightarrow \infty} V(\bar{X}) \\ &= \lim_{n \rightarrow \infty} V\left(\frac{\sum X_i}{n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2} V(\sum X_i) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \cdot n \cdot V(X_i) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} [\theta(1-\theta)] \\ &= \theta(1-\theta) \left(\lim_{n \rightarrow \infty} \frac{1}{n} \right) = \text{Zero} \quad \left(\frac{1}{n} \geq 0 \right) \\ &= \text{Zero} \quad (\text{Condition } \textcircled{2} \text{ is satisfied}). \end{aligned}$$

$\therefore$ by $\textcircled{1}$ & $\textcircled{2}$; $\hat{\theta}$ is consistent estimator of θ .

Ex.3 Let X_i ($i=1, 2, \dots, n$) be a r.s. follows $N(\mu, \sigma^2)$;
 (μ is unknown & σ^2 is known parameters) .

(H.W.) Estimate μ using (MLE & MOM). Then test the properties of estimation for $\hat{\mu}$ (Estimator of μ).
 (اختبر خواص التقدير للـ $\hat{\mu}$)

Sol. (Properties of $\hat{\mu}$)

I

Unbiasedness

خواص التقدير

We know $\hat{\mu}_{mle} = \bar{X}$

$$E(\hat{\mu}_{mle}) = E(\bar{X}) = \frac{1}{n} E(X_i) = E(X_i) = \mu$$

$$\therefore \boxed{E(\hat{\mu}_{mle}) = \mu} \Rightarrow \hat{\mu}_{mle} \text{ is unbiased estimator of } \mu.$$

II

Consistency

خواص التقدير

Note $Var(\hat{\mu}_{mle}) = Var(\bar{X}) = \frac{1}{n^2} \cdot n Var(X_i)$

$$= \frac{1}{n} Var(X_i) \quad \left(\begin{array}{l} X_i \sim N(\mu, \sigma^2) \\ Var(X_i) = \sigma^2 \end{array} \right)$$

$$\therefore \boxed{Var(\hat{\mu}_{mle}) = \frac{\sigma^2}{n}} = \frac{\sigma^2}{n}$$

لتحقيق الاتي

$$\textcircled{1} \lim_{n \rightarrow \infty} E(\hat{\mu}_{mle}) = \lim_{n \rightarrow \infty} E(\bar{X}) = \lim_{n \rightarrow \infty} \mu = \mu$$

$$\begin{aligned} \textcircled{2} \lim_{n \rightarrow \infty} V(\hat{\mu}_{mle}) &= \lim_{n \rightarrow \infty} V(\bar{X}) = \lim_{n \rightarrow \infty} \frac{\sigma^2}{n} \\ &= \sigma^2 \lim_{n \rightarrow \infty} \frac{1}{n} \\ &= \sigma^2 \cdot 0 \\ &= 0 \end{aligned}$$

$\therefore$ Condition $\textcircled{1}$ & $\textcircled{2}$ are Satisfied.

$\therefore \hat{\mu}_{mle} = \bar{X}$ is consistent estimator for μ . #

Ex.1 If $X_1, X_2, \dots, X_n$ are a.r.s. from $G(\alpha, \beta)$.
Estimate α and β using MoM.

Sol. Number of unknown parameter is (2)
 $\therefore$ We need $E(X) = \frac{\sum X_i}{n}$ — (1)
 $E(X^2) = \frac{\sum X_i^2}{n}$ — (2)

Note $\because X_i \sim G(\alpha, \beta) \rightarrow E(X) = \alpha\beta, V(X) = \alpha\beta^2$
 $E(X^2) = V(X) + (E(X))^2$
 $= \alpha\beta^2 + \alpha^2\beta^2$

From eq. (1); $E(X) = \alpha\beta = \frac{\sum X_i}{n} = \bar{X}$

$$\Rightarrow \hat{\alpha} = \frac{\bar{X}}{\hat{\beta}} \quad \text{--- (3)}$$

From eq. (2); $E(X^2) = \hat{\alpha}^2\hat{\beta}^2 + \hat{\alpha}^2\hat{\beta}^2 = \frac{\sum X_i^2}{n}$

$$\therefore \left(\frac{\bar{X}}{\hat{\beta}}\right)^2 \hat{\beta}^2 + \frac{\bar{X}^2}{\hat{\beta}^2} \hat{\beta}^2 = \frac{\sum X_i^2}{n}$$

$$\bar{X} \hat{\beta} + \bar{X}^2 = \frac{\sum X_i^2}{n}$$

$$\bar{X} \hat{\beta} = \frac{\sum X_i^2}{n} - \bar{X}^2 = S^2$$

$$\therefore \hat{\beta}_{\text{MoM}} = \frac{S^2}{\bar{X}} \quad \text{--- (4)}$$

جاء (3) و (4) في (3) و (4) و (4) و (4)

$$\hat{\alpha}_{\text{MoM}} = \frac{\bar{X}}{\frac{S^2}{\bar{X}}} = \frac{\bar{X}^2}{S^2}$$

Remark $S^2 = \frac{\sum (X_i - \bar{X})^2}{n} = \frac{\sum (X_i^2 - 2\bar{X}X_i + \bar{X}^2)}{n} = \frac{\sum X_i^2 - 2\bar{X}(\sum X_i) + n\bar{X}^2}{n}$
 $= \frac{\sum X_i^2}{n} - \bar{X}^2$

Examples of MLE & MoM

ex.1 If $X_1, X_2, \dots, X_n$ be a random sample from $G(\alpha, \beta)$ such that (α) known. Estimate $\hat{\beta}$ using MLE

Sol.

$$X_i \sim G(\alpha, \beta) \xrightarrow[\text{known}]{\alpha} f(x_i; \beta) = \begin{cases} \frac{x_i^{\alpha-1} e^{-x_i/\beta}}{\Gamma(\alpha) \beta^\alpha} & \alpha, \beta > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$L(\cdot) = L(X_1, \dots, X_n; \beta) = \prod_{i=1}^n f(x_i; \beta) = \frac{\prod_{i=1}^n x_i^{\alpha-1} e^{-\frac{\sum x_i}{\beta}}}{(\Gamma(\alpha))^n (\beta^\alpha)^n}$$

$$\ln L(\cdot) = (\alpha-1) \sum_{i=1}^n \ln x_i - \frac{\sum x_i}{\beta} - n \ln \Gamma(\alpha) - n\alpha \ln \beta$$

$$\frac{\partial \ln L(\cdot)}{\partial \beta} = + \frac{\sum x_i}{\beta^2} - n\alpha \cdot \frac{1}{\beta} = 0 \quad * \beta^2$$

$$\sum x_i = \hat{\beta} (n\alpha) \rightarrow \hat{\beta}_{MLE} = \frac{\sum x_i}{n\alpha} = \frac{\bar{x}}{\alpha} \quad (2)$$

ملفد في حاله (α) مجهول ايضاً يتطلب ان نشتق من دالة الاحتمال بالنسبة لـ α ما يؤدي الى اشتقاق دالة لوج ليكلير التي تتطلب حلاً عددياً

$$\frac{\partial \ln L(\cdot)}{\partial \alpha} = \sum \ln x_i - n \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \right) - n \ln \beta \quad \psi(\alpha) \quad (3)$$

✓ ex.3 If $X_1, X_2, \dots, X_n$ are i.i.d. from $N(\mu, \sigma^2)$,
Estimate μ, σ^2 using MLE Method.

Sol.

$$X_i \sim N(\mu, \sigma^2) \rightarrow f(X_i; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(X_i - \mu)^2}{2\sigma^2}}$$

$\rightarrow \sigma^2 > 0$

$$L(\cdot) = L(X_1, X_2, \dots, X_n; \mu, \sigma^2) = \prod_{i=1}^n f(X_i; \mu, \sigma^2)$$

$$= (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum_{i=1}^n (X_i - \mu)^2}{2\sigma^2}}$$

$$\ln L(\cdot) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{\sum_{i=1}^n (X_i - \mu)^2}{2\sigma^2}$$

$$\frac{\partial \ln L(\cdot)}{\partial \mu} = + \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu) = 0$$

$$\therefore \sum_{i=1}^n (X_i - \mu) = 0 \Rightarrow \sum_{i=1}^n X_i - n\mu = 0$$

$$\text{so } \boxed{\hat{\mu} = \frac{\sum_{i=1}^n X_i}{n} = \bar{X}} \quad (1)$$

$$\frac{\partial \ln L(\cdot)}{\partial \sigma^2} = -\frac{n}{2} \frac{1}{\sigma^2} + \frac{\sum_{i=1}^n (X_i - \mu)^2}{2\sigma^4} = 0 \quad \left(\left(\sigma^2 \right)^{-1} \right)' = (-1)(\sigma^2)^{-2}$$

ضرب في $(2\sigma^4)$ $\rightarrow$ ضرب في $2\sigma^4$

$$-n\sigma^2 + \sum_{i=1}^n (X_i - \mu)^2 = 0$$

$$\text{so } \hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \mu)^2}{n} = S^2 \quad \#$$

ex.4 MOM method for finding μ, σ^2

$$E(X) = \mu, \quad V(X) = \sigma^2$$

$$E(X^2) = \sigma^2 + \mu^2$$

$$(1) \quad E(X) = \hat{\mu} = \frac{\sum_{i=1}^n X_i}{n} = \bar{X}$$

$$(2) \quad E(X^2) = \hat{\sigma}^2 + \left[\frac{\sum_{i=1}^n X_i}{n} \right]^2 = \frac{\sum_{i=1}^n X_i^2}{n}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n X_i^2}{n} - \bar{X}^2 = S^2$$

Q1/ Suppose that $X_1, \dots, X_n$ are independent and identically random variables distributed *Bernoulli* with parameter (p) . Find the maximum likelihood (ML) estimator, and test the properties for this estimator of the parameter p .

Q2/ Suppose that $X_1, \dots, X_n$ are independent and identically random variables distributed *Geometric distribution* with parameter (p) . Find the Moment Method (MOM) estimator.

Q3/ Suppose that $X_1, \dots, X_n$ are independent and identically random variables distributed *Gamma distribution* with parameters $(4, \beta)$. Find the maximum likelihood (ML) estimator, and test the properties for this estimator of the parameter β .

Q4/ Suppose that $X_1, \dots, X_n$ are independent and identically random variables distributed *Gamma distribution* with parameters (α, β) . Find Moment Method (MOM) estimator, and test the properties for this estimator of the parameters α and β .

Q5/ Suppose that $X_1, \dots, X_n$ are independent and identically random variables distributed *Normal distribution* with parameters (μ, σ^2) . Find Moment Method (MOM) estimator, and test the properties for this estimator of the parameters μ and σ^2 .

Q6/ Suppose that $X_1, \dots, X_n$ are independent and identically random variables distributed *Normal distribution* with parameters $(\mu, 9)$. Find Maximum likelihood (ML) estimator, and test the properties for this estimator of the parameters μ .

Q7/ Suppose that $X_1, \dots, X_n$ are independent and identically distributed *Poisson* (λ) random variables. Find the maximum likelihood (ML) estimator, and test the properties for this estimator of the parameter λ .

Q8/ Suppose that $X_1, \dots, X_n$ are independent and identically distributed *Exponential* (λ) random variables. Find the maximum likelihood (ML) estimator, and test the properties for this estimator of the parameter λ .

Q9/ Suppose that $X_1, \dots, X_n$ are independent and identically distributed *Poisson* (λ) random variables. Find the Find Moment Method (MOM) estimator, and test the properties for this estimator of the parameter λ .

Q3 / $X_i (i=1, \dots, n)$ are i.i.d r.v.s follows $G(4, \beta)$

Find $\hat{\beta}_{MLE}$ and test its properties

Sol: $X \sim G(4, \beta) \rightarrow f(x) = \frac{3}{\Gamma(4)} \frac{e^{-x/\beta}}{\beta^4} \quad 0 < x < \infty$

$$L(\cdot) \rightarrow L(x_1, \dots, x_n, \beta) = \frac{\prod_{i=1}^n f(x_i)}{[\Gamma(4)]^n [\beta]^{4n}} \quad \begin{aligned} E(X) &= 4\beta \\ \text{Var}(X) &= 4\beta^2 \end{aligned}$$

$$\ln L(\cdot) = 3 \sum \ln x_i - \frac{\sum x_i}{\beta} - n \ln \Gamma(4) - 4n \ln \beta$$

$$\frac{\partial \ln L(\cdot)}{\partial \beta} = 0 + \frac{\sum x_i}{\beta^2} - 0 - \frac{4n}{\beta} = 0 \quad \times \beta^2$$

$$\sum x_i = (4n) \hat{\beta} \rightarrow \hat{\beta}_{MLE} = \frac{\sum x_i}{4n} = \frac{\bar{x}}{4}$$

(I) $E(\hat{\beta}_{MLE}) = \frac{1}{4n} E[\sum x_i] = \frac{n}{4n} E(X) = \frac{1}{4} [4\beta] = \beta$
 so $\hat{\beta}$ is unbiased estimator for β

(II) (i) $\lim_{n \rightarrow \infty} E[\hat{\beta}_{MLE}] = \lim_{n \rightarrow \infty} \beta = \beta \quad \checkmark$

(ii) $\lim_{n \rightarrow \infty} \text{Var}[\hat{\beta}_{MLE}] = \lim_{n \rightarrow \infty} \text{Var}\left[\frac{\sum x_i}{4n}\right]$

$$= \lim_{n \rightarrow \infty} \frac{1}{(4n)^2} \times \text{Var}(\sum x_i)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{16n} (4\beta^2)$$

$$= \frac{\beta^2}{4} \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$= 0 \quad \checkmark$$

so $\hat{\beta}_{MLE}$ is consistent estimator

Q₂ $X_1, \dots, X_n$ are iid r.v.s distributed $G(4, \beta)$, with unknown (β) . Find the (MoM) estimator and test the properties of this estimator of β . (2)

Sol. $X_0 \sim G(4, \beta)$

$$E(X) = 4\beta \quad \& \quad \text{Var}(X) = 4\beta^2$$

Using MoM estimation

$$E(X) = \frac{\sum X_i}{n} = \bar{X} \Rightarrow 4\hat{\beta} = \bar{X}$$

$$\text{so } \hat{\beta}_{\text{mom}} = \frac{\bar{X}}{4}$$

by the same method of Q₃, we prove the properties

Q₃ $X_i \sim \text{poisson}(\lambda)$, $i=1, \dots, n$

Find $\hat{\lambda}_{\text{mom}}$ and test its properties

Sol. $f(x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $E(X) = \lambda$, $\text{Var}(X) = \lambda$

by using MoM estimation, put $E(X) = \frac{\sum X_i}{n} = \bar{X}$

so $\hat{\lambda}_{\text{mom}} = \bar{X} \rightarrow$ (i) $E(\hat{\lambda}_{\text{mom}}) = E(\bar{X}) = \frac{n}{n} E(X) = \lambda$
 so $\hat{\lambda}$ is unbiased for λ

(2) (i) $\lim_{n \rightarrow \infty} E(\hat{\lambda}_{\text{mom}}) = \lim_{n \rightarrow \infty} E(\bar{X}) = \lim_{n \rightarrow \infty} \lambda = \lambda \checkmark$

(ii) $\lim_{n \rightarrow \infty} \text{Var}(\hat{\lambda}_{\text{mom}}) = \lim_{n \rightarrow \infty} \text{Var}(\bar{X}) = \lim_{n \rightarrow \infty} \frac{n}{n^2} \text{Var}(X)$
 $= \lim_{n \rightarrow \infty} \frac{1}{n} (\lambda) = 0 \checkmark$

so from (i) & (ii) $\rightarrow \hat{\lambda}_{\text{mom}}$ is consistent.

(3)

Q2/ $X_i \sim \text{Geo}(p), i=1, \dots, n$
Find $\hat{p}_{\text{mom}}$ and $\hat{p}_{\text{MLE}}$ for the parameter p .

Sol. $f(x) = p q^{x-1}$ $x=1, 2, \dots$, $q=1-p$
o.w.

$$E(X) = \frac{1}{p}, \quad \text{Var}(X) = \frac{q}{p^2} = \frac{1-p}{p^2}$$

$$E(X) = \frac{\sum X_i}{n} = \bar{x} \rightarrow \frac{1}{\hat{p}_{\text{mom}}} = \frac{\sum X_i}{n} = \bar{x} \rightarrow \hat{p}_{\text{mom}} = \frac{1}{\bar{x}}$$

$$L(p) = p^n q^{\sum X_i - n} = p^n (1-p)^{\sum X_i - n}$$

$$\ln L(p) = n \ln p + (\sum X_i - n) \ln(1-p)$$

$$\frac{\partial \ln L(p)}{\partial p} = \frac{n}{p} - \frac{(\sum X_i - n)}{1-p} = 0 \quad (p \neq 1-p)$$

$$\Rightarrow n(1-p) - (\sum X_i - n)p = 0$$

$$n - np - \sum X_i p + np = 0$$

$$n = \sum_{i=1}^n X_i \hat{p} \rightarrow \hat{p}_{\text{MLE}} = \frac{n}{\sum X_i} = \frac{1}{\bar{x}}$$