By th. (10) $\Rightarrow M_y(t) = e^{bt} M_x(at) = e^{t} M_x(-2t)$

$\because M_x(t) = \dfrac{1}{1-3t}$ for $t < \dfrac{1}{3}$

$M_x(-2t) = \dfrac{1}{1-3(-2t)} = \dfrac{1}{1+6t}$ for $t > \dfrac{-1}{6}$

$M_y(t) = e^{t} \cdot \dfrac{1}{1+6t}$ for $t > \dfrac{-1}{6}$

## *The bounded of probability:*

حدود الاحتمالية

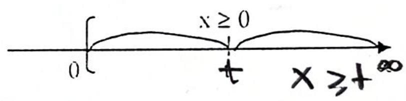
Theorem "11": ((Markov inquality))

If x is ar.v. and if $P(x \geq 0) = 1$ then $P(x \geq t) \leq \dfrac{E(x)}{t}$, for $t > 0$

Proof: case "1": if x is a c.r.v with a p.d.f f(x)

$$\because p(x \geq 0) = 1 \Rightarrow \int_0^{\infty} f(x)dx = 1$$

$$f(x) > 0 \quad \text{for } x \geq 0$$
$$= 0 \quad \text{O.W}$$




$$E(x) = \int_0^{\infty} Xf(x)dx = \int_0^{t} xf(x)dx + \int_t^{\infty} xf(x)dx \geq \int_t^{\infty} xf(x)dx \geq \int_t^{\infty} tf(x)dx. \quad [x \geq t]$$

$$\therefore E(x) \geq \int_t^{\infty} tf(x)dx \Rightarrow E(x) \geq t\int_t^{\infty} f(x)dx$$

$$E(x) \geq t \cdot p(x \geq t)$$

$$\therefore p(x \geq t) \leq \frac{E(x)}{t}$$

Case "2": If x is a d.r.v. with a p.m.f. f(x) by the same method.

Theorem "12": "Chebyshev inqualities"

Let x be a r.v. where V(x) exists,then:

1. $p(|x-\mu| \geq t) \leq \dfrac{V(x)}{t^2}$, $t > 0$, 2. $p(|x-\mu| < t) \geq 1 - \dfrac{V(x)}{t^2}$, $t > 0$,

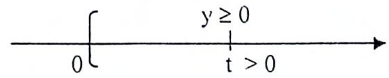
Note: 1. $\dfrac{V(x)}{t^2}$ *is called the upper buond of* $p(|x-\mu| \geq t)$

2. $1 - \dfrac{V(x)}{t^2}$ *is called the lower buond of* $p(|x-\mu| < t)$

Proof: 1. $V(x) = E[(x-\mu)^2] \geq 0$

$$Let\ y = (x-\mu)^2 \geq 0 \Rightarrow E(y) = E[(x-\mu)^2 = V(x) \geq 0$$

$\therefore$ by Markov inquality, we get




$$p(y \geq t^2) \leq \frac{E(y)}{t^2} \Rightarrow p[(x-\mu)^2 \geq t^2] \leq \frac{v(x)}{t^2}$$

$$p[|x-\mu| \geq t] \leq \frac{V(x)}{t^2}$$

2. $p(A^c) = 1 - p(A)$

$$p[|x-\mu| \geq t] = 1 - p[|x-\mu| < t] \leq \frac{V(x)}{t^2}$$

$$p[|x-\mu| < t] \geq 1 - \frac{V(x)}{t^2}$$

Ex. "1": If $x \sim$ unif. $(-\sqrt{3}, \sqrt{3})$ then:

a. Find the upper bound of $p(|x-\mu| \geq \frac{3}{2})$

b. Find the value of $p(|x-\mu| \geq \frac{3}{2})$

Sol.:

a. $$f(x) = \begin{cases} \dfrac{1}{\sqrt{3}+(\sqrt{3})} = \dfrac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < x < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$$

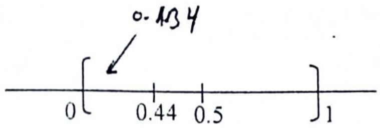
$$u.b = \frac{V(x)}{t^2}, \qquad t = \frac{3}{2}$$

$$E(x) = \int_{-\sqrt{3}}^{\sqrt{3}} X \frac{1}{2\sqrt{3}} dx = \frac{1}{4\sqrt{3}} X^2 \Big|_{-\sqrt{3}}^{\sqrt{3}} = 0$$

$$E(x^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \frac{1}{2\sqrt{3}} dx = \frac{1}{6\sqrt{3}} x^3 \Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{6\sqrt{3}}[(\sqrt{3})^3 - (-\sqrt{3})^3] = \frac{1}{6\sqrt{3}}[3\sqrt{3} + 3\sqrt{3}] = 1$$

$\therefore V(x) = E(x^2) - [E(x)]^2 = 1 - 0 = 1$

$\therefore u.b = \frac{V(x)}{t^2} = \frac{1}{(\frac{3}{2})^2} = \frac{4}{9} = 0.44$




b. $p(|x - \mu| \geq \frac{3}{2}) = p(|x - 0| \geq \frac{3}{2}) = p(|x| \geq \frac{3}{2})$

$= 1 - p(|x| < \frac{3}{2}) = 1 - p(-\frac{3}{2} < x < \frac{3}{2}) = 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} dx$

$= 1 - \frac{1}{2\sqrt{3}} x \Big|_{-\frac{3}{2}}^{\frac{3}{2}} = 1 - \frac{1}{2\sqrt{3}}(\frac{3}{2} + \frac{3}{2}) = 1 - \frac{3}{2\sqrt{3}} = 1 - \frac{\sqrt{3}}{2}$

$= 1 - 0.866 = 0.134$

Ex. "2": Given a p.d.f $f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$

a. Find the lower bound of $p(\frac{5}{4} < x < \frac{11}{4})$

b. Find the value of $p(\frac{5}{4} < x < \frac{11}{4})$

Sol.: a. $L.b = 1 - \frac{V(x)}{t^2}$

$E(x) = \int_0^3 x \cdot \frac{2x}{9} dx = \frac{2}{27} x^3 \Big|_0^3 = \frac{2}{27}(27 - 0) = 2$

$E(x^2) = \int_0^3 x^2 \cdot (\frac{2x}{9}) dx = \frac{2}{36} x^4 \Big|_0^3 = \frac{1}{18}(81) = \frac{9}{2} = 4.5$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$= 4.5 - 4 = 0.5 = \frac{1}{2}$

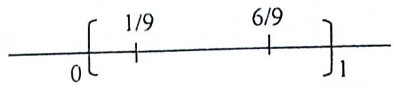
$p(\frac{5}{4} < x < \frac{11}{4}) = p(\frac{5}{4} - 2 < x - 2 < \frac{11}{4} - 2) = p(-\frac{3}{4} < x - 2 < \frac{3}{4})$

$= p(|x - 2| < \frac{3}{4})$

$\therefore t = \frac{3}{4}$

$\therefore L.b = 1 - \frac{V(x)}{t^2} = 1 - \frac{\frac{1}{2}}{\frac{9}{16}} = 1 - \frac{1}{2} \times \frac{16}{9} = 1 - \frac{8}{9} = \frac{1}{9}$

b. $p(\frac{5}{4} < x < \frac{11}{4}) = \int_{\frac{5}{4}}^{\frac{11}{4}} \frac{2x}{9} dx$




$= \frac{1}{9} x^2 \Big|_{\frac{5}{4}}^{\frac{11}{4}} = \frac{1}{9}[\frac{121}{16} - \frac{25}{16}] = \frac{1}{9}[\frac{96}{16}] = \frac{6}{9}$

## Median of Distribution of r.v.:

Def.: The median (m) is a value of x such that satisfing the two following inqualities:

$p(x < m) < \frac{1}{2}$ & $p(x \le m) \ge \frac{1}{2}$

By properties of c.d.f F(x)

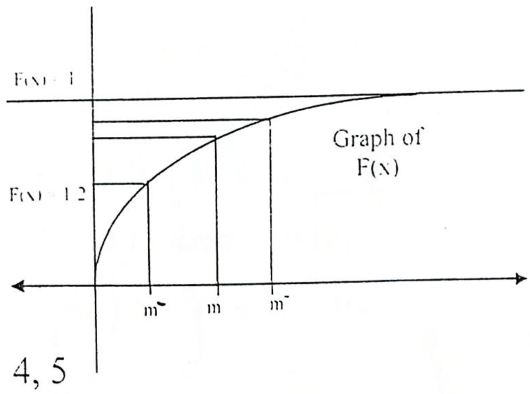
$p(x < m) = F(\overline{m})$, $\quad p(x \le m) = F(m) = F(m^-)$

$i.e.: p(x < m) = F(\overline{m}) < \frac{1}{2}$. $\quad p(x \le m) = F(m^-) \ge \frac{1}{2}$

Note ① If $\overline{m} = m = \overline{m}^{+}$ then

$p(x \le m) = F(m) = \frac{1}{2}$

② The value of median is uniqu




Ex.: Given a p.m.f $\quad f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1,\ 2,\ 3,\ 4,\ 5 \\ 0 & \text{o.w.} \end{cases}$

Find the median of x.

Sol.: $\quad p(x < m) < \frac{1}{2}$ & $p(x \le m) \ge \frac{1}{2}$

Suppose that m = 1

$p(x < 1) = 0 < \frac{1}{2}$, $\quad p(x \le 1) = f(1) = \frac{1}{15} \not\ge \frac{1}{2}$

$\therefore m \ne 1$

## Markov Inequality

* for $p(X \geq 0) = 1$, then

$$p(X \geq t) \leq \frac{E(X)}{t} \quad \text{for } t > 0$$

## Chebyshev Inequalities

Let X be a r.v., where V(x) exists, then

① $p(|X-\mu| \geq t) \leq \frac{V(X)}{t^2}$ for $t > 0$

u.b.

② $p(|X-\mu| < t) \geq 1 - \frac{V(X)}{t^2}$ for $t > 0$

**Ex** $X$ is a r.v. with $M_X(t)$ for $-h<t<h$

show that

- $P(X \geq a) \leq e^{-at} M_X(t)$ for $0<t<h$
- $P(X \leq a) \leq e^{-at} M_X(t)$ for $-h<t<0$

**Sol.** $M_X(t) = E(e^{tX})$

$Y = e^{tX} > 0$ (دالة اسية موجبة دائماً)

$[P(Y \geq 0) = 1]$.

also, $e^{at} > 0$ ( )

by Markov Inequality

$[P(X \geq 0) = 1$ & for $\boxed{t>0}]$

$$P[X \geq t] \leq \frac{E(X)}{t}$$

بدله نضع $Y$

$e^{at}$ بدلاً عنه

ونطبق النظرية

$$\therefore P[Y \geq e^{at}] \leq \frac{E(Y)}{e^{at}}$$

$$P[e^{tX} \geq e^{at}] \leq \frac{E[e^{Xt}]}{e^{at}}$$

$$P[tX \geq at] \leq \frac{M_X(t)}{e^{at}}$$

① for $t \in (0,h) \rightarrow P[X \geq a] \leq e^{-at} \cdot M_X(t)$

② for $t \in (-h,0) \rightarrow P[X \leq a] \leq e^{-at} M_X(t)$

**Note**

$P(X \geq a) \leq e^{-at} M_X(t)$ for $0<t<h$

$P(X \leq a) \leq e^{-at} \cdot M_X(t)$ for $-h<t<0$

These Inequalities are called (Bernstien Inequalities)

(Bernstien Inequal.)

Ex If $X \sim unif(-1, 2)$, then

Show that $M_X(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$

Sol

$$f(x) = \begin{cases} \frac{1}{2-(-1)} = \frac{1}{3} & \text{for } -1 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

$$M_X(t) = E[e^{tx}] = \int_{-1}^{2} e^{tx} \cdot \frac{1}{3} dx = \frac{1}{3} \int_{-1}^{2} e^{tx} dx$$

$$= \frac{1}{3t} e^{tx} \Big]_{-1}^{2} = \frac{1}{3t} [e^{2t} - e^{-t}]$$

for $t = 0 \longrightarrow M_X(t) = E[e^{0x}] = E[1] = 1$

$$\therefore M_X(t) = \begin{cases} \frac{1}{3t}[e^{2t} - e^{-t}] & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

# Exercises About Probability Bounded

**Q1:** Given a p.f.

$$f(-1) = \frac{1}{8}, \quad f(0) = \frac{6}{8}, \quad f(1) = \frac{1}{8}$$

and $f(x) = 0$ (otherwise)

Show that the value of upper bounded and exact value of $P(|X| \geq 2\sigma_X)$ are identical (متطابقة).

**Sol.**

$$E(X) = \sum_{x=-1}^{1} x\, f(x)$$

$$= (-1)\, f(-1) + (0) f(0) + (1) f(1)$$

$$= -\frac{1}{8} + 0 + \frac{1}{8}$$

$$= \text{Zero}$$

$$\therefore \boxed{E(X) = \mu_X = 0}$$

$$E(X^2) = \sum_{x=-1}^{1} x^2 f(x)$$

$$= (-1)^2 f(-1) + (0)^2 f(0) + (1)^2 f(1)$$

$$= \frac{1}{8} + 0 + \frac{1}{8}$$

$$= \frac{2}{8} = \frac{1}{4}$$

$$\sigma_X^2 = V(X) = E(X^2) - (E(X))^2 \geq 0$$

$$= \frac{1}{4} - (0)^2$$

$$= \frac{1}{4} > 0$$

$$\longrightarrow \sigma_X = +\sqrt{\sigma_X^2} = +\sqrt{V(X)} = +\sqrt{\frac{1}{4}} = \frac{1}{2}$$

$\therefore P(|X| \geq 2\sigma) = P(|X-0| \geq \cancel{2}(\frac{1}{\cancel{2}}))$

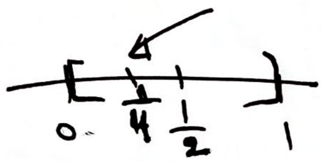
By (Chebyshov inq.) $\left\{ P(|X-\mu_x| \geq t) \leq \frac{V(X)}{t^2} \right\}$

$\rightarrow \boxed{t=1}, \boxed{\mu_x = 0}$

Then $U.b. = \frac{V(X)}{t^2} = (\frac{1}{4}) / (1)^2 = \frac{1}{4}$

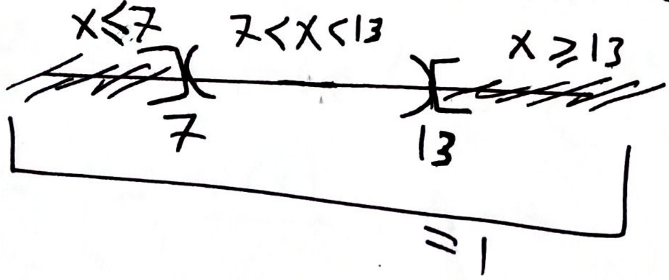
The exact value of bove Pr.:

$$P(|X| \geq 2\sigma_x) = P(|X| \geq \cancel{2}(\frac{1}{\cancel{2}}))$$
$$= P(|X| \geq 1)$$
$$= 1 - P(|X| < 1)$$
$$= 1 - P(-1 < X < 1)$$
$$= 1 - P(X=0)$$
$$= 1 - f(0)$$
$$= 1 - \frac{6}{8}$$
$$= \frac{2}{8}$$
$$= \frac{1}{4}$$




Q2: If X is a r.v. with $E(X) = 10$, $P(X \leq 7) = 0.2$, $P(X \geq 13) = 0.3$, then show that $V(X) \geq \frac{9}{2}$.

sol.



$$P(X \le 7) + P(7 < X < 13) + P(X \ge 13) = 1$$

$$0.2 + P(7 < X < 13) + 0.3 = 1$$

$$P(7 < X < 13) = 1 - 0.2 - 0.3 = 1 - 0.5 = 0.5 \in [0,1].$$

$$P(7-10 < X-10 < 13-10) = P(-3 < X-10 < 3)$$
$$= P(|X-10| < 3)$$

$$0.5 = P(|X-10| < 3) \ge 1 - \frac{V(X)}{t^2}, \quad t > 0$$

$$P(|X-\mu_x| < t) \ge 1 - \frac{V(X)}{t^2}, \; t > 0 \rightarrow \boxed{t=3}$$

$$\frac{1}{2} \ge 1 - \frac{V(X)}{9}$$

$$-\frac{1}{2} \ge -\frac{V(X)}{9}$$

$$\frac{1}{2} \le \frac{V(X)}{9} \Rightarrow \frac{9}{2} \le V(X)$$

$$\Rightarrow \therefore \boxed{V(X) \ge \frac{9}{2}}$$

#

---

$Q_3$: Given a m.g.f. $M_X(t) = e^{3t^2 + 2t}$ for $-\infty < t < \infty$
Find the L.b. of $P(\frac{-1}{2} < X < \frac{9}{2})$.

Sol.

$$M_X(t) = E(e^{tX}) = e^{3t^2 + 2t}$$

$$\text{L.b.} = 1 - \frac{V(X)}{t^2}, \quad t > 0$$

$$V(X) = E(X^2) - (E(X))^2 \ge 0$$
$$= M_X''(0) - (M_X'(0))^2 \ge 0$$

$$M_x(t) = e^{3t^2+2t}$$

$$M_x'(t) = (6t+2)\, e^{3t^2+2t}$$

$$M_x'(t=0) = (6(0)+2)\, e^{3(0)^2+2(0)} = 1$$

$$\boxed{E(x) = 2}$$

$$M_x''(t) = (6t+2)^2 e^{3t^2+2t} + (6)\, e^{3t^2+2t}$$

$$M_x''(t=0) = (\cancel{6(0)}+2)^2 \underbrace{e^0}_{=1} + (6)\, \underbrace{e^0}_{=1}$$

$$\boxed{E(x^2) = 10}$$

$$\therefore V(x) = E(x^2) - (E(x))^2$$

$$= 10 - (2)^2$$

$$= 6$$

$$\therefore \boxed{V(x) = \sigma_x^2 = 6}$$

$$P\left(-\frac{1}{2} < x < \frac{9}{2}\right) = P\left(-\frac{1}{2} - 2 < x - 2 < \frac{9}{2} - 2\right)$$

$$= P\left(\frac{-5}{2} < x-2 < \frac{5}{2}\right)$$

$$= P\left(|x-2| < \frac{5}{2}\right)$$

$$= P(|x-\mu| < t) \Rightarrow \boxed{t = \frac{5}{2}}$$

$$L.b. = 1 - \frac{6}{\left(\frac{25}{4}\right)} = 1 - 6\left(\frac{4}{25}\right)$$

$$= 1 - \frac{24}{25} = \boxed{\frac{1}{25}}$$

#

Q4: If $X \sim unif(0,2)$, find the u.b. of $P(|X-\mu_x| \geq 2)$.

Sol. $\because X \sim unif(0,2)$

$\therefore f(x) = \begin{cases} \frac{1}{2} & \text{for } 0<x<2 \\ 0 & \text{o.w.} \end{cases}$

$$\mu_x = E(x) = \int_0^2 x\, f(x)\, dx$$

$$= \int_0^2 x \left(\tfrac{1}{2}\right) dx$$

$$= \left(\tfrac{1}{2}\right) \frac{x^2}{2} \Big|_0^2$$

$$= \frac{1}{4}\left[2^2 - 0^2\right]$$

$$= \cancel{\frac{4}{4}} = 1$$

$$\longrightarrow \boxed{\therefore \mu_x = E(X) = 1}$$

$$E(x^2) = \int_0^2 x^2 f(x)\, dx$$

$$= \frac{1}{2}\int_0^2 x^2\, dx$$

$$= \frac{1}{2}\, \frac{x^3}{3} \Big|_0^2$$

$$= \frac{1}{6}\left(2^3 - 0^3\right) = \frac{8}{6} = \boxed{\frac{4}{3}}$$

$$\therefore V(X) = E(x^2) - (E(x))^2$$

$$= \frac{4}{3} - (1)^2$$

$$= \frac{1}{3} > 0$$

$\therefore$ U.b. $= \dfrac{V(x)}{t^2}$ , $t > 0$

<u>To find (t)</u> :

$P(|X - \mu_x| \geq t) = P(|X-1| \geq 2)$ $\therefore \boxed{t = 2}$

$U.b. = \dfrac{V(x)}{t^2} = \dfrac{\frac{1}{3}}{(2)^2} = \dfrac{1}{3} \times \dfrac{1}{4} = \dfrac{1}{12}$

---

Q7: If $X \sim unif(-\sqrt{3}, \sqrt{3})$ then find the upper bounded of $P(|X - \mu_x| \geq \frac{3}{2})$.

<u>Sol.</u>

$\because X \sim unif(-\sqrt{3}, \sqrt{3})$

$\therefore f(x) = \begin{cases} \dfrac{1}{\sqrt{3}+\sqrt{3}} = \dfrac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < X < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$

$$E(X) = \int_{-\sqrt{3}}^{\sqrt{3}} x\, f(x)\, dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} x \left(\frac{1}{2\sqrt{3}}\right) dx$$

$$= \left(\frac{1}{2\sqrt{3}}\right) \frac{x^2}{2} \Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{4\sqrt{3}} \left[\underbrace{(\sqrt{3})^2}_{=3} - \underbrace{(-\sqrt{3})^2}_{=3}\right]$$

$$= \frac{1}{4\sqrt{3}} (0)$$

$$= \text{Zero} = \mu_x$$

$$P(|X - 0| \geq \underbrace{\tfrac{3}{2}}_{=t}) = P(|X - \mu_x| \geq t) \leq \frac{V(x)}{t^2}$$

$\boxed{\text{To find } V(x)}$

$$E(X^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 f(x)\,dx$$

$$= \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} x^2\,dx$$

$$= \frac{1}{2\sqrt{3}} \left. \frac{x^3}{3} \right|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{1}{6\sqrt{3}} \left[ (\sqrt{3})^3 - (-\sqrt{3})^3 \right]$$

$$= \frac{1}{6\sqrt{3}} \left[ 3\sqrt{3} + 3\sqrt{3} \right]$$

$$= \frac{1}{\cancel{(6\sqrt{3})}} \cancel{(6\sqrt{3})}$$

$$= 1$$

$\therefore$ $$V(X) = E(X^2) - (E(X))^2 \geq 0$$

$$= 1 - 0$$

$$= 1$$

$\therefore$ $$\text{U.b.} = \frac{V(X)}{t^2} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}$$

---

Q8: If a r.v. X with m.g.f. $M_X(t)$ for $-h<t<h$

Show that:

① $P(X \geq a) \leq e^{-at} M_X(t)$ for $0<t<h$

② $P(X \leq a) \leq e^{-at} M_X(t)$ for $-h<t<0$

Proof: $M_X(t) = E(e^{tX})$,

let $Y = e^{tX}$

$e^{at} > 0, \quad a > 0, \quad t > 0$

$P(X \geq t) \leq \dfrac{E(X)}{t}$ ( by Markov Inq.)

$\left[ P(Y \geq e^{at}) \leq \dfrac{E(Y)}{e^{at}} \right]$

Now;

$$P(X \geq a) = P(tx \geq ta)$$
$$= P(e^{tx} \geq e^{at})$$

$$\therefore P(Y \geq e^{at}) \leq \frac{E(Y)}{e^{at}} \quad \text{(by Markov inq.)}$$

$$\leq \frac{E(e^{tx})}{e^{at}} \quad \text{since } (Y = e^{tx})$$

$$\therefore \boxed{P(e^{tx} \geq e^{at}) \leq \frac{M_X(t)}{e^{at}}} = e^{-at} M_X(t)$$

Now; $P(\cancel{\ln} e^{tx} \geq \cancel{\ln} e^{at}) \leq e^{-at} M_X(t)$

$$P(X \geq a) \leq e^{-at} M_X(t) \quad \text{— ① for } 0 < t < h$$

And $P(\cancel{\ln} e^{tx} \geq \cancel{\ln} e^{at}) \leq e^{-at} M_X(t)$

$$P(tx \geq at) \leq e^{-at} M_X(t)$$

$$P(X \leq a) \leq e^{-at} M_X(t) \quad \text{— ② for } -h < t < 0$$

#

Excercise

(1) Show that $p(|X-\mu| \geq c) \leq \dfrac{E[(X-\mu)^{2k}]}{c^{2k}}$ , $c>0$

Proof let $y = (X-\mu)^{2k} \geq 0$ since $\boxed{2k}$ even Integer.

by thm. B (Markove inequality)

if $p(X \geq 0)=1 \rightarrow p(X \geq t) \leq \dfrac{E(X)}{t}$ , $t>0$

$\therefore p(Y \geq c^{2k}) \leq \dfrac{E(Y)}{c^{2k}}$ since $\boxed{c^{2k} > 0}$

$p[(X-\mu)^{2k} \geq c^{2k}] \leq \dfrac{E[(X-\mu)^{2k}]}{c^{2k}}$

$p[|X-\mu| \geq c] \leq \dfrac{E[(X-\mu)^{2k}]}{c^{2k}}$

2k ✓

(2) Given a p.m.f.

$f(x) = \begin{cases} 1/8 & \text{if } x=-1 \\ 6/8 & \text{if } x=0 \\ 1/8 & \text{if } x=1 \end{cases}$

(i) Find the u.b. of $p[|X-\mu| \geq 1]$

(ii) Find the value of $p[|X-\mu| \geq 1]$

Sol. (i) u.b. $= \dfrac{V(X)}{t^2}$ ; $t=1$

we must find $V(X)$

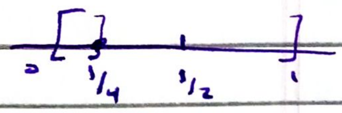
$\mu = E(X) = \sum_{x=-1}^{1} x f(x) = (-1)(\frac{1}{8}) + 0(\frac{6}{8}) + 1(\frac{1}{8}) = 0$

$E(X^2) = \sum x^2 f(x) = (-1)^2(\frac{1}{8}) + 0^2(\frac{6}{8}) + 1^2(\frac{1}{8}) = \frac{2}{8} = 1/4$

$V(X) = E(X^2) - [E(X)]^2$

$= \frac{1}{4} - 0 = 1/4$

u.b. $= \dfrac{V(X)}{t^2} = \dfrac{1/4}{1^2} = 1/4$




(ii) $p(|X-\mu| \geq 1) = 1 - p[|X-\mu| < 1]$

$= 1 - p[|X-0| < 1]$

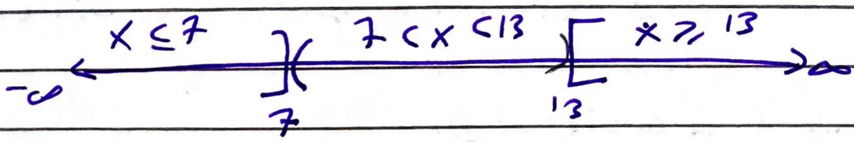
$= 1 - p[-1 < X < 1]$

$= 1 - f(0)$

$= 1 - 6/8 = 2/8 = 1/4$

EX (3) let $X$ be a r.v. where $E(X) = 10$, $p(X \le 7) = 0.2$ and $p(X \ge 13) = 0.3$, show that $V(X) \ge 9/2$

Sol.




$$(X \le 7) \cup (7 < X < 13) \cup (X \ge 13) = S$$

$$P(X \le 7) + P(7 < X < 13) + P(X \ge 13) = P(S) = 1$$

$$0.2 + P(7 < X < 13) + 0.3 = 1$$

$$P(7 < X < 13) = 1 - 0.5 = 0.5 = 1/2 \quad \text{——} \quad (1)$$

$$E(X) = \mu_x = 10$$

by th. (14) [chebyshiv inequalities]

$$P[|X - \mu| \ge t] \le \frac{V(X)}{t^2}, \quad t > 0$$

$$P[|X - \mu| < t] \ge 1 - \frac{V(X)}{t^2}, \quad t > 0$$

$$P(7 < X < 13) = P(7 - 10 < X - 10 < 13 - 10)$$

(1), we → $\frac{1}{2} = P(-3 < X - 10 < 3)$

$$\frac{1}{2} = P(|X - 10| < 3) \ge 1 - \frac{V(X)}{t^2}, \quad t = 3$$

$$\frac{1}{2} \ge 1 - \frac{V(X)}{3^2}$$

$$\frac{V(X)}{9} \ge 1 - \frac{1}{2} = 1/2$$

$$\boxed{V(X) \ge \frac{9}{2}}$$

**Ex.** If $X$ is a r.v. with $P(X \geq b) = 1$, $b \in R$
then $E[(X-b)^2] = 0$ iff $P(X = b) = 1$.

proof let $Y = (X-b)^2$

$\Rightarrow$ $E(Y) = E[(X-b)^2] = 0$

by th. $P(X \geq b) = 1$ & $\boxed{E(X) = b} \longrightarrow P(X = b) = 1$ & $P(X > b) = 0$

$\because$ $E(Y) = 0 \longrightarrow P(Y = 0) = 1$

$P[(X-b)^2 = 0] = 1$

$P[(X-b) = 0] = 1$

$P[X = b] = 1$

$\Leftarrow$ if $P(X = b) = 1$

$\boxed{X = b} \longrightarrow X - b = 0 \Longrightarrow (X-b)^2 = 0$

$\therefore$ $E[(X-b)^2] = 0$

**Ex** If $X$ is a r.v. such that $P(X < 0) = 0$
then $P(X \geq 2\mu) \leq 1/2$

$$P(X \geq 2\mu) \leq \frac{E(X)}{2\mu} = \frac{\mu}{2\mu} = 1/2$$

Ex Given $g(t) = \ln M_x(t)$
Show that $g'(0) = \mu$ and $g''(0) = V(x)$

Proof $g(t) = \ln M_x(t)$

$$g'(t) = \frac{1}{M_x(t)} \cdot M_x'(t)$$

$\boxed{t=0} \rightarrow$ $$g'(0) = \frac{M_x'(0)}{M_x(0)} = \frac{E(x)}{1} = \mu$$

$$g''(t) = \frac{M_x(t) \cdot M_x''(t) - M_x'(t) \cdot M_x'(t)}{[M_x(t)]^2}$$

$\boxed{t=0}$, $$g''(0) = \frac{M_x(0) \cdot M_x''(0) - M_x'(0) \cdot M_x'(0)}{[M_x(0)]^2}$$

$$= \frac{1 \cdot E(x^2) - E(x) \cdot E(x)}{1^2}$$

$$= \frac{E(x^2) - E(x)^2}{1}$$

$$= V(x)$$