Chapter Three
Expectation and Variance

Ch. 3 – part 1 [Introduction & Definition & Examples.]

$\boxed{\text{Expectation}}$

Introduction

In chapter zero ....⟶
we define the **mean** & **variance**
الوسط
المعدل
التباين

For the Data

ظاهرة
تحت الدراسة
$\rightarrow X$

data

x_1
$\vdots$
x_N

population

الوسط $\mu_X = \dfrac{\sum^{N} x_i}{N}$

التباين $\sigma_X^2 = \dfrac{\sum (x_i - \mu_X)^2}{N}$

X

data

x_1
$\vdots$
x_n

sample

$\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

$S_X^2 = \dfrac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$

Def. [Expectation value, Mean, Average]

If X is a r.v. with p.f. $f(x)$
Then $E(X)$ is called "Expectation of X"
or "Expected value of X" or "Mean of X"
and denoted by μ_X and define as below

$$E(X) = \begin{cases} \sum_{\forall x_i \in R_X} x_i\, f(x_i) & \text{when } X \text{ is d.r.v. with p.m.f. } f(x) \\ \int x\, f(x)\, dx & \text{when } X \text{ is c.r.v. with p.d.f. } f(x) \end{cases}$$

Remark
1. The value of $E(X)$ is constant
2. $E(X)$ exists if $E(|X|) < \infty$
3. $E(X)$ is not probability value.

ex.1 Given a p.m.f. of X as below

$$f(x) = \begin{cases} \frac{x}{6} & \text{for } x = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(X)$? → $\boxed{\text{① sketch } f(x) \quad \text{② Find } p(X \geq 2)}$ مع نفس أ السابقة

Sol. X is d.r.v. $\Rightarrow R_X = \{1,2,3\}$
by def. of $E(X)$

$$E(X) = \sum_{\forall x_i} x_i \, f(x_i)$$

توضيح

x	1	2	3
$f(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$$= \sum_{x=1}^{3} x \, f(x)$$

$$= 1\left(\tfrac{1}{6}\right) + 2\cdot\left(\tfrac{2}{6}\right) + 3\left(\tfrac{3}{6}\right)$$

$$= \frac{1+4+9}{6} = \frac{14}{6} = \frac{7}{3}$$

ex.2 Given a p.d.f of X as below

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } 0 < x < 4 \\ 0 & \text{o.w} \end{cases} \leftarrow \text{uniform dist.}$$

Find $E(X)$? → ① Sketch $f(x)$ ② Find $P(1 < X < 3)$

Sol. X is c.r.v., $R_X = (0,4)$
by def. of $E(X)$

$$E(X) = \int_0^4 x \cdot f(x)\,dx = \int_0^4 x \cdot \frac{1}{4}\,dx$$

$$= \frac{1}{4} \frac{x^2}{2} \Big]_0^4 = \frac{1}{8}[16 - 0] = \boxed{2}$$

ex.3 (H.W) Toss a Coin two times
Let $X \equiv$ Number of heads

* **Find $E(X)$**

نجيب اولاً اشتقاق $f(x)$ ثم بعد ذلك نجيب الـ $E(X)$

ex.4 Given a p.d.f. as below

$$f(x) = \begin{cases} \frac{a}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$$

(1) Find the value of a to valid p.d.f.
(2) Does $E(X)$ exist.

Sol. لانها p.d.f

$$1 = \int_1^\infty \frac{a}{x^2}\,dx = a\int_1^\infty \frac{1}{x^2}\,dx = a\left[-x^{-1}\right]_1^\infty$$

$$= -a\left[\frac{1}{x}\right]_1^\infty = -a\left[\underbrace{\frac{1}{\infty}}_{صفر} - 1\right]$$

$$\boxed{1 = a}$$

$$\therefore f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w} \end{cases}$$

$$E(X) = \int_1^\infty x\,\frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x}\,dx = \ln x\,\Big]_1^\infty$$

$$= \left[\underbrace{\ln\infty}_{\infty} - \underbrace{\ln 1}_{صفر}\right]$$

$e^\infty = \infty \to \ln\infty = \infty$
$e^0 = 1 \to \ln 1 = 0$

$= \infty$

So $E(X)$ not Exists.

Expectation of a function

Def. Let X be a r.v. with p.f. $f(x)$ and Let $g(x)$ be a function of X then

$$E[g(x)] = \begin{cases} \sum_{\forall x} g(x) f(x) & \text{if } X \text{ is d.r.v. with p.m.f. } f(x) \\ \int_x g(x) f(x)\,dx & \text{if } X \text{ is c.r.v. with p.d.f. } f(x) \end{cases}$$

ex. 1 Given p.m.f. of X as below

$$f(x) = \begin{cases} \frac{x}{3} & \text{for } x = 0,1,2 \\ 0 & \text{o.w.} \end{cases}$$

H.W.

Find $E(x^2)$, $E(2X)$, $E(\sqrt{x})$, $E(4)$

Sol.

$* \; E(x^2) = \sum_{x=0}^{2} x^2 \cdot \frac{x}{3} = \sum_{x=0}^{2} \frac{x^3}{3} = \frac{1}{3}[0^3 + 1^3 + 2^3] = \frac{9}{3} = 3$

$* \; E(2X) = \sum_{x=0}^{2} (2x)\left(\frac{x}{3}\right) = \frac{2}{3} \sum_{x=0}^{2} x^2 = \frac{2}{3}[0^2 + 1^2 + 2^2] = \frac{2}{3}[5] = \frac{10}{3}$

$= 2 \sum_{x=0}^{2} x \left(\frac{x}{3}\right)$

$\boxed{E(2X) = 2\,E(X)} \Longrightarrow E(aX) = a\,E(X)$

معامل X (ثابت)

$* \; E(4) = \sum_{x=0}^{2} (4)\left(\frac{x}{3}\right) = \frac{4}{3}\cancel{[0+1+2]} = 4$ الثابت نفسه

$\boxed{E(4) = 4} \Longrightarrow E(b) = b$

ثابت

ex. Given a p.d.f as follow

$$f(x) = \begin{cases} \frac{1}{2} & 0 < x < 2 \\ 0 & o.w. \end{cases}$$

uniform dist. القمر

Find $E(X^3)$, $E(4X)$, $E(5)$

Sol.

* $E(X^3) = \int_0^2 x^3 \left(\frac{1}{2}\right) dx = \frac{1}{2} \frac{x^4}{4}\Big]_0^2 = \frac{1}{8}[16-0] = 2$

* $E(4X) = \int_0^2 (4x)\left(\frac{1}{2}\right) dx = \cancel{2} \frac{x^2}{\cancel{2}}\Big]_0^2 = 4$

$$\boxed{E(4X) = 4E(X)}$$

* $E(5) = \int_0^2 (5)\left(\frac{1}{2}\right) dx = \frac{5}{2} x\Big]_0^2 = \frac{5}{\cancel{2}}[\cancel{2}]$
$= 5$

or $= 5 \underbrace{\int_0^2 \frac{1}{2} dx}_{=1 \text{ p.d.f}} = 5$

Remark If X is a r.v. (d.r.v. or c.r.v.) with p.f (p.m.f. or p.d.f.) and when $a, b \in \mathbb{R}$, then

1. $E(aX) = aE(X)$.
2. $E(b) = b$.

⁂

Chapter Three – part 2

ch.3 – Part 2 | [Properties of Expectation]

From part 1 → ① $E(aX) = a\,E(X)$
② $E(b) = b$
$a, b \in R$ (Constant)

Th. 1 If X is a r.v. with p.f and when $a, b \in R$, then

$$E(\underbrace{aX+b}_{g(x)}) = aE(X) + b$$

proof
- Case 1: when X is d.r.v. with p.m.f $f(x)$
- case 2: when X is c.r.v with p.d.f $f(x)$

Th. 2 If X be a r.v. with p.f $f(x)$ and when $U(x)$ and $V(x)$ are two functions of X, then

$$E[U(X) \pm V(X)] = E(U(X)) \pm E(V(X))$$

proof

Hint let $\boxed{g(X) = U(X) \pm V(X)}$

$$E[g(x)] = \sum\!\!\!\!\int g(x)\, f(x)\, dx$$

and so on - - -

NotE It's true for more than two functions

Ex. 1 Given a p.m.f. $f(x)$ as below

$$f(x) = \begin{cases} \frac{x}{3} & \text{for } x = 0, 1, 2 \\ 0 & \text{o.w.} \end{cases}$$

Then Find ① $E(X)$, ③ $E(2X^2 + X)$, ④ $E[(X+2)^2]$
② $E(X^2)$, ⑤ $E(1/2)$

Sol. ① $E(X) = \sum_{x=0}^{2} x \cdot \frac{x}{3} = \frac{1}{3}[0^2 + 1^2 + 2^2] = \frac{5}{3}$

② $E(X^2) = \sum_{x=0}^{2} x^2 \cdot \frac{x}{3} = \frac{1}{3}[0^3 + 1^3 + 2^3] = \frac{9}{3} = 3$

③ $E(2X^2 + X) = E(2X^2) + E(X)$

$\overset{\text{by th. 2}}{=} 2E(X^2) + E(X)$

$= 2(3) + \frac{5}{3} = \frac{23}{3}$

④ $E[(X+2)^2] = E(X^2 + 4X + 4)$

$= E(X^2) + 4E(X) + 4 \quad \frac{12}{3}$

توزعت على اكثر
مكانوا اليتي

$= \frac{9}{3} + 4\left(\frac{5}{3}\right) + \frac{12}{3}$

$= \frac{41}{3}$

⑤ $E(1/2) = 1/2$

Ex.2 If X has a uniform dist. on $(0,1)$ then Find $E(X^3+2X^2)$ and $E(X+1)^2$

Sol. $\because X \sim$ unif. $(0,1)$

$\therefore f(x) = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{o.w} \end{cases}$ p.d.f

- $E(X) = \int_0^1 x \cdot 1 \, dx = \frac{x^2}{2}\Big]_0^1 = \boxed{1/2}$
- $E(X^2) = \int_0^1 x^2 \cdot 1 \, dx = \frac{x^3}{3}\Big]_0^1 = \boxed{1/3}$
- $E(X^3) = \int_0^1 x^3 \cdot 1 \, dx = \frac{x^4}{4}\Big]_0^1 = \boxed{1/4}$

- $E(X^3+2X^2) = E(X^3) + 2E(X^2)$

$= \frac{1}{4} + 2 \cdot \frac{1}{3}$

$= \frac{3+8}{12} = \frac{11}{12}$

- $E(X+1)^2 = E[X^2+2X+1]$

$= E(X^2) + 2E(X) + 1$

$= \frac{1}{3} + \cancel{2}\left(\frac{1}{\cancel{2}}\right) + 1$

$= \frac{2+6+6}{6} = \frac{14}{6} = \frac{7}{3}$

or $= \frac{1}{3} + 1 + 1$

$= 7/3$

Part 22

Th. 3 If X is a r.v., then

Ch. 3 - Part 22

(i) If $\exists\, a$ such that $P(X \geq a) = 1$, then $E(X) \geq a$.

(ii) If $\exists\, b$ such that $P(X \leq b) = 1$, then $E(X) \leq b$

(iii) If $\exists\, a, b$ such that $P(a \leq X \leq b) = 1$, then $a \leq E(X) \leq b$

proof ?

- case (1) when X is d.r.v. with p.m.f. $f(x)$
- case (2) when X is c.r.v with p.d.f. $f(x)$

ex. 1 Given a $\boxed{\text{p.d.f.}}$ of X as below

(ع) تطبيق

$$f(x) = \begin{cases} e^{-x} & x > 0 \quad (0, \infty) \longleftarrow R_X \\ 0 & o.w \end{cases}$$

$$P(X \geq 0) = \int_0^\infty f(x)\,dx = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big]_0^\infty$$

$$= -[e^{-\infty} - e^{0}] = 1$$

$$\boxed{P(X \geq a) = 1} \longrightarrow \boxed{a = 0}$$

$$E(X) = \int_0^\infty x \cdot e^{-x}\,dx$$

Int. by parts.

$u = x \rightarrow du = dx$
$dv = e^{-x}dx \rightarrow v = -e^{-x}$
$uv] - \int v\,du$

$$= -x e^{-x}\Big]_0^\infty + \int_0^\infty e^{-x}\,dx$$

$$= -e^{-x}\Big]_0^\infty = -[e^{-\infty} - e^{0}]$$

$$\boxed{E(X) = 1}$$

$$\rightarrow E(X) = 1 > 0 = a \quad \boxed{E(X) > 0}$$

$$\Longrightarrow \boxed{E(X) > a}$$

ex.2 (ii) $f(x) = \begin{cases} e^{x} & x < 0 \\ 0 & \text{o.w} \end{cases}$ $(-\infty, 0] = R_X$ $\boxed{\text{p.d.f.}}$??

$- P(X \leq 0) = \int_{-\infty}^{0} e^{x}\,dx = e^{x}\Big]_{-\infty}^{0} = [e^{0} - e^{-\infty}] = 1$

$\boxed{b = 0} \to \boxed{P(X \leq 0) = 1}$

Int. by part

$E(X) = \int_{-\infty}^{0} x\, e^{x}\,dx$

$\boxed{\begin{array}{ll} u = x & du = dx \\ dv = e^{x}dx & \to v = e^{x} \end{array}}$

$= x e^{x}\Big]_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}\,dx$

$= -e^{x}\Big]_{-\infty}^{0} = -[e^{0} - e^{-\infty}] = \boxed{-1}$

$\boxed{E(X) = -1} \to \boxed{E(X) < 0 = b}$

$\therefore \boxed{E(X) \leq b}$

✓

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ex.3 (iii) تطبيق Given a p.d.f as below

$$f(x) = \begin{cases} \frac{1}{2} & 0<x<2 \\ 0 & o.w \end{cases}$$

p.d.f uniform dist. on (0, 2)

$$P(\overset{a}{0} \le X \le \overset{b}{2}) = \int_0^2 \frac{1}{2}\,dx = \frac{1}{2}x\Big]_0^2 = \frac{1}{2}[2-0] = 1$$

$$\boxed{P(a \le X \le b) = 1}$$

$$E(X) = \int_0^2 x \cdot \frac{1}{2}\,dx = \cdots = 1$$

الحل الأول

$$\boxed{\underset{a}{0} \le \underset{1}{E(X)} \le \underset{b}{2}}$$

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ex.4 (iii) تطبيق Given a p.m.f. of X as below

$$f(x) = \begin{cases} \frac{x}{3} & \text{for } x=0,1,2 \\ 0 & o.w. \end{cases}$$

$$P(\underset{a}{0} \le X \le \underset{b}{2}) = \sum_{x=0}^{2} \frac{x}{3} = \frac{1}{3}[0+1+2] = 1$$

$$\boxed{P(a \le X \le b) = 1}$$

$$E(X) = \sum_{x=0}^{2} x \cdot \underbrace{\frac{x}{3}}_{f(x)} = \frac{1}{3}\sum_{x=0}^{2}(x^2) = \frac{1}{3}[0^2+1^2+2^2]$$

$$= \frac{5}{3} = 1.66$$

$$\boxed{0 \le \underset{1.66}{E(X)} \le 2}$$

$$\boxed{a \le E(X) \le b}$$

Th. 4 Ch.3-part23

If X be a r.v. with p.f. and if $p(X \geq a) = 1$ and $E(X) = a$, then $\boxed{P(X > a) = 0}$ and $\boxed{P(X = a) = 1}$

proof by two cases

ex. Given a p.m.f.

$$f(x) = \begin{cases} 1 & \text{for } X = 2 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \geq \overset{a}{2}) = P(X = 2) = 1$$

$$E(X) = \sum_{\forall x} x \cdot f(x) =$$

$$= [\overset{\cdots -1\times 0}{\cdots 0} + \overset{0\times 0}{0} + \overset{1\times 0}{0} + 2\cdot(1) + \overset{3\times 0}{0} + \overset{4\times 0}{0} + \cdots]$$

$$\boxed{E(X) = \boxed{2}^{a}} \longrightarrow \boxed{E(X) = a}$$

$$P(X > 2) = \sum_{X=3}^{\infty} f(x) = \underset{\text{zero}}{f(3)} + \underset{\text{zero}}{f(4)} + \underset{\text{zero}}{f(5)} + \cdots = 0$$

$$\boxed{P(X > \overset{a}{2}) = 0} \longrightarrow \boxed{P(X > a) = 0}$$

$$\boxed{P(X = \overset{a}{2}) = f(2) = 1} \longrightarrow \boxed{P(X = a) = 1}$$

- Try $\boxed{a = 0}$ in example above

$$P(X \geq 0) = P(X = 0 \cup X = 1 \cup X = 2 \cdots) = P(X = 2) = 1$$

but $E(X) = 2 \longrightarrow \boxed{E(X) \neq 0}$

$$P(X > 0) = 1 \quad , P(X = 0) = 0$$

لايتحقق المثال

H.W $\boxed{\text{Mean of } X \equiv \text{Average of } X = \bar{X} \text{ or } E(X)}$ (8)

$\boxed{1}$ prove that $E(X-\bar{X}) = 0$

<u>Hint</u> $\boxed{\bar{X} = E(X)}$

$\boxed{2}$ If $E(X)$ exist, then prove that

$$\left(E(X)\right)^2 \leq \left(E(|X|)\right)^2 \leq E(X^2)$$

<u>Hint</u> ① $|X| \geq X$

② $\underbrace{\left[\,|X| - E(|X|)\right]^2}_{\text{مربع كمية}} \geq 0$

$\boxed{3}$ prove that $E(X-\bar{X})^2 = E(X^2) - [E(X)]^2$

$\boxed{\text{Var}(X) \quad \text{التباين}}$

موضوع part3 ⟵ المواضيع المتعلقة حول التباين

<u>Hint</u> $\boxed{E(X) = \bar{X}}$

$\boxed{4}$ Suppose that for $k = 1, 2, 3, \ldots$

$P(X=k) = P(X=-k) = \frac{1}{2c}\frac{1}{k^2}$; $c = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Then Find $E(X)$ if exists?

<u>Hint</u> $E(X) = \sum_{k=-\infty}^{-1} k\, P(X=k) + \sum_{k=1}^{\infty} k\, P(X=k)$

Chapter Three - part 3
Variance of random variables

Ch.3-part31 [Definition & Examples & Remarks]

Def [Variance]

If X be a r.v., then the variance of X denoted by $Var(X)$ or $V(X)$ or σ_X^2 is defined as below

$$Var(X) = \sigma_X^2 = E\left[(X - E(X))^2\right], \quad \boxed{E(X) = \mu_X}$$

or

$$Var(X) = \sigma_X^2 = E\left[(X - \mu_X)^2\right]$$

Remark (1) Since $(X-\mu_X)^2 \geq 0 \quad \forall x$

then by $\boxed{th.4}$, $E[(X-\mu_X)^2] \geq 0$

$$\therefore \boxed{Var(X) = \sigma_X^2 \geq 0}$$

(2) $\underset{\text{constant}}{Var(b)} = E\left[(b - E(b))^2\right]$

$$= E[b-b]^2 = 0$$

(3) $Var(aX) = E\left[(aX - E(aX))^2\right]$

$$= E\left[(aX - aE(X))^2\right]$$

$$= E\left[a^2 (X - E(X))^2\right]$$

$$= a^2 E\left[(X - E(X))^2\right] = \boxed{a^2 Var(X)}$$

ex.1 Given a p.m.f. of a d.r.v. X as below

$$f(x) = \begin{cases} \frac{x}{3} & \text{for } x = 1, 2 \\ 0 & \text{o.w.} \end{cases}$$

Find Var(X) ?

Sol. $E(X) = \sum_{x=1}^{2} x \cdot \frac{x}{3} = \frac{1}{3}[1^2 + 2^2] = \frac{5}{3}$

$$\therefore Var(X) = E[(X - E(X))^2]$$

$$= \sum_{x=1}^{2} (x - \tfrac{5}{3})^2 \cdot \frac{x}{3}$$

$$= (1 - \tfrac{5}{3})^2 \cdot \frac{1}{3} + (2 - \tfrac{5}{3})^2 \cdot \frac{2}{3}$$

$$= \frac{4}{9} \cdot \frac{1}{3} + \frac{1}{9} \cdot \frac{2}{3}$$

$$= \frac{4+2}{27} = \frac{6}{27} = \frac{2}{9} > 0$$

ex.2 Given a p.d.f. of a c.r.v. X as below

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

Var(X) = ?

$$E(X) = \int_0^1 x \cdot 3x^2 \, dx$$

$$= \int_0^1 3x^3 \, dx$$

$$= \frac{3}{4} x^4 \Big]_0^1$$

$$= \boxed{3/4}$$

$$Var(X) = E[(X - \tfrac{3}{4})^2]$$

$$= \int_0^1 (x^2 - \tfrac{3}{2}x + \tfrac{9}{16}) \cdot 3x^2 \, dx$$

$$= \int_0^1 [3x^4 - \tfrac{9}{2}x^3 + \tfrac{27}{16}x^2] \, dx$$

$$= \frac{3}{5}x^5 - \frac{9}{8}x^4 + \frac{27}{16 \times 3}x^3 \Big]_0^1$$

$$= \frac{3}{5} - \frac{9}{8} + \frac{9}{16}$$

$$= \frac{48 - 90 + 45}{80} = \frac{3}{80} > 0$$

Th.5 If X be a r.v., then

$$Var(X) = E[X^2] - [E(X)]^2$$

proof باستخدام التعريف

نستطيع برهان
$Var(b) = 0$
$Var(aX) = a^2 Var(X)$

Th.6 If X be a r.v., and

$Y = aX + b \; ; \; a, b \in R$, then

$$Var(Y) = a^2 \, Var(X)$$

proof

باستخدام التعريف
او النظرية 5 اعلاه

ex.1 نفس الامثلة السابقة

$f(x) = \begin{cases} \frac{x}{3} & \text{for } x = 1,2 \\ 0 & o.w \end{cases}$

We find $E(X) = 5/3$

Now $E(X^2) = \sum_{x=1}^{2} x^2 \cdot \frac{x}{3} = \frac{1}{3} \sum_{x=1}^{2} x^3$

$= \frac{1}{3}[1^3 + 2^3]$

$= \frac{1}{3}(9) = 3$

$\therefore Var(X) = E(X^2) - [E(X)]^2$

$= 3 - (\frac{5}{3})^2$

$= 3 - \frac{25}{9}$

$= 2/9 > 0$

ex.2 $f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & o.w. \end{cases}$

we find $E(X) = 3/4$

Now, $E(X^2) = \int_0^1 x^2 \cdot (3x^2) \, dx$

$E(X^2) = \frac{3}{5} x^5 \Big]_0^1 = 3/5$

$\therefore Var(X) = E(X^2) - [E(X)]^2$

$= \frac{3}{5} - (\frac{3}{4})^2$

$= \frac{3}{5} - \frac{9}{16}$

$= \frac{48 - 45}{80} = \frac{3}{80} > 0$

Th. 7 $Var(X) = 0$ $\boxed{\text{iff}}$ $\exists\ k$ (constant)
such that $P(X = k) = 1$

proof

$\Leftarrow$ $P(X = k) = 1$ T.P $Var(X) = 0$

$\because X = k \rightarrow E(X) = E(k) = k$

$\therefore Var(X) = E[(X - E(X))^2]$

$= E[(k - k)^2]$

$= 0$

$\Rightarrow$ $Var(X) = 0$ T.P. $P(X = k) = 1$ $\boxed{E(X) = \mu = k}$

$\because Var(X) = 0$

$\therefore E[(X - \mu)^2] = 0$

let $y = (X - \mu)^2$

$\therefore E(y) = 0$

$\therefore$ by th.4 $[E(X) = a \longrightarrow P(X = a) = 1]$

$\therefore E(y) = 0 \longrightarrow P(y = 0) = 1$

$\therefore P[(X - \mu)^2 = 0] = 1$

$P[X - \mu = 0] = 1$

$E(X) = E(k) = k$

$P[X = \mu] = 1$

$P[X = k] = 1$

H.W. (1) If X be a c.r.v. with p.d.f $f(x)$, where
$f(X) > 0$ for $0 < X < b < \infty$
Then prove that

$$E(X) = \int_0^b [1 - F(x)]\, dx$$

Sol. Hint: Assume $f(x)dx = dF(x)$ then use integration by parts

(2) Given a p.d.f.

$$f(X) = \frac{1}{\pi(1+X^2)} \quad \text{for } -\infty < X < \infty$$

Cauchy distribution

Test the existence of $E(X)$

Chapter three - part 4

Existence of Expectation and Variance

We say,

A random variable X has expectation iff $|X|$ has expectation.

i.e., $\boxed{E(X) \text{ exists } (<\infty)}$ iff $\boxed{E(|X|) < \infty}$

ex.1 Given a p.d.f. as below

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{for } 1 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

? تحقق $f(x)$ is p.d.f

$$E(|X|) = E(X) = \int_1^\infty x \cdot \frac{1}{x^2}\, dx$$

? $\boxed{x > 1}$ +ve

$$= \int_1^\infty \frac{1}{x}\, dx$$

$$= \ln x \Big]_1^\infty$$

$$= \underbrace{\ln(\infty)}_{\infty} - \underbrace{\ln(1)}_{\text{zero}}$$

$$= \infty$$

$E(|X|) \not< \infty$

$E(X)$ is not exists.

ex. 2 Given a p.d.f.

تسمى Cauchy dist. $\rightarrow f(x) = \dfrac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$

Test the existence of $E(X)$??

Sol

$$E(|X|) = \int_{-\infty}^{\infty} \underbrace{|x|}_{+ve} \cdot \frac{1}{\underbrace{\pi(1+x^2)}_{+ve}}\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{2x}{1+x^2}\, dx$$

توضيح
$e^{\infty} = \infty$
$\ln\infty = \infty$

$$= \frac{1}{\pi} \ln(1+x^2) \Big]_0^{\infty}$$

$$= \frac{1}{\pi}\Big[\underbrace{\ln(\infty)}_{\infty} - \underbrace{\ln(1)}_{\text{zero}}\Big] = \infty$$

توضيح
$e^{0} = 1$
$\ln 1 = 0$

$\therefore$ $E(X)$ **does not** exist.

مهم

Remark As we know

$$V(X) = E(X^2) - [E(X)]^2$$

[تعريف التباين]

Then

$V(X)$ exists iff $E(X)$ and $E(X^2)$ exists.

H.W. $f(x) = \begin{cases} \frac{1}{x^3} & \frac{1}{\sqrt{2}} < x < \infty \\ 0 & o.w \end{cases} \longrightarrow$ p.d.f ?

Find $E(X)$ & $Var(X)$ if exists

Examples

1 prove or disprove

$$E[g(x)] = g[E(x)]$$

given p.d.f.

$$f(x) = \begin{cases} 2x & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Let $\boxed{g(x) = \frac{1}{x}}$

$\Rightarrow$ To show that if $E[g(x)] = E(\frac{1}{x}) \overset{?}{=} \frac{1}{E(x)} \overset{?}{=} g[E(x)]$

① $E(\frac{1}{x}) = \int_0^1 \frac{1}{x} \cdot (2x)\,dx = 2x \Big]_0^1 = 2$

② $E(x) = \int_0^1 x \cdot (2x)\,dx = \frac{2}{3} x^3 \Big]_0^1 = \frac{2}{3} \Rightarrow \frac{1}{E(x)} = \frac{3}{2}$

So

$\Rightarrow E[g(x)] = E(\frac{1}{x}) = 2 \neq \frac{3}{2} = \frac{1}{E(x)} = g[E(x)]$

Note Try to test $g(x) = x^2$

and $g(x) = |x|$

$\boxed{2}$ If X is a c.r.v. has a p.d.f $f(x)$ where $f(x) > 0$ for $0 < x < b < \infty$ and zero otherwise

Show that
$$E(X) = \int_0^b [1 - F(x)]\, dx$$

Sol.

$$E(X) = \int_0^b x \cdot \boxed{f(x)\, dx}$$

$$\boxed{\frac{dF(x)}{dx} = f(x) \qquad dF(x) = f(x)\, dx}$$

$$= \int_0^b x \cdot dF(x)$$

$$= uv\Big]_0^b - \int_0^b v\, du$$

$$\boxed{\text{let } u = x \longrightarrow du = dx \qquad dv = dF(x) \longrightarrow v = F(x)}$$

$$= x F(x)\Big]_0^b - \int_0^b F(x)\, dx$$

$$= b F(b) - \int_0^b F(x)\, dx$$

$$\boxed{F(x) = P(X \leq x)}$$

$$= b \cdot P(X \leq b) - \int_0^b F(x)\, dx$$

$$= b \cdot \underbrace{\int_0^b f(x)\, dx}_{=1} - \int_0^b F(x)\, dx$$

$$= \int_0^b dx - \int_0^b F(x)\, dx$$

$$= \int_0^b [1 - F(x)]\, dx$$

Moments of random variables

Def. Let X be a r.v. (either d.r.v. or c.r.v.) and let K be any positive integer, then

(1) $E(X^K)$ is called the K^{th} non-centeral moment of X (or non-central moment of order K)

(2) $E[(X-E(X))^k] = E[(X-\mu)^k]$ is called K^{th} central moment of X

Remarks (1) $E(X^1) = \mu_X$ = mean of X $\equiv 1^{st}$ non-cenetral moment.

(2) $E(X^2) = 2^{nd}$ non-central moment

(3) $E(X^k) = K^{th}$ non-central moment

(4) $E[(X-\mu)^1] \equiv 1^{st}$ central moment

$\longrightarrow = E(X) - E(X) = 0$

(5) $E[(X-\mu)^2] = 2^{nd}$ central moment

= Variance

Existence of Moments

$E(X^k)$ exists iff $E(|X|^k) < \infty$

Th. 8 If $E(X^k)$ exists, then $E(X^j)$ exists for all $j<k$; $k, j \in I^+$.

ex. 1 Given a p.m.f of d.r.v. X as below

$$f(x) = \begin{cases} 1/2 & \text{if } x=1 \\ 1/3 & \text{if } x=2 \\ 1/6 & \text{if } x=3 \\ 0 & \text{o.w.} \end{cases} \Rightarrow \begin{matrix} ① \; E(X^3) = ? \\ ② \; E(X^2) = ? \end{matrix}$$

Sol. ① To Find the third (3^{rd}) non-central moment

$$E(X^3) = \sum_{x=1}^{3} x^3 f(x)$$

$$= 1^3 \cdot \frac{1}{2} + 2^3 \cdot \frac{1}{3} + 3^3 \cdot \frac{1}{6}$$

$$= \frac{1}{2} + \frac{8}{3} + \frac{27}{6}$$

$$= \frac{3+16+27}{6} = \frac{46}{6} = \frac{23}{3}$$

② To Find the (2^{nd}) non-central moment

$$E(X^2) = \sum_{x=1}^{3} x^2 \cdot f(x)$$

$$= 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{3} + 3^2 \cdot \frac{1}{6}$$

$$= \frac{1}{2} + \frac{4}{3} + \frac{9}{6}$$

$$= \frac{3+8+9}{6} = \frac{20}{6} = \frac{10}{3}$$

ex. Given a p.m.f. of d.r.v. X

$$f(x) = \begin{cases} 3/4 & \text{if } x=1 \\ 1/4 & \text{if } x=2 \\ 0 & \text{o.w.} \end{cases}$$

Find 3^{rd} central moments

i.e. $E[(X-E(X))^3] = ?$

Sol. We need to find $E(X)=\mu$

$$\mu = E(X) = \sum_{x=1}^{2} x \cdot f(x) = 1 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4} = \frac{5}{4}$$

$\therefore$

$$E[(X-E(X))^3] = \sum_{x=1}^{2} (x-\tfrac{5}{4})^3 f(x)$$

$$= (1-\tfrac{5}{4})^3 \cdot \frac{3}{4} + (2-\tfrac{5}{4})^3 \cdot \frac{1}{4}$$

$$= (-\tfrac{1}{4})^3 \cdot \frac{3}{4} + (\tfrac{3}{4})^3 \cdot \frac{1}{4}$$

$$= -\frac{3}{4^4} + \frac{27}{4^4} = \frac{24}{256} = \frac{3}{32}$$

Note $E[(X-E(X))^3] = E[X^3 - 3E(X)X^2 + 3(E(X))^2X^1 - (E(X))^3]$

$= E(X^3) - 3E(X) \cdot E(X^2) + 3(E(X))^2 E(X) - (E(X))^3$

We can Find $E(X)$, $E(X^2)$, $E(X^3)$

We can Find the non-central moment to Find the central moment

<u>ex.</u> If a r.v., such that

$E(X)=1$, $E(X^2)=2$ and $E(X^3)=5$

then Find the 3rd central moment?

<u>Sol.</u> 3rd central moment $\equiv E[(X-\mu)^3]$

$$\therefore E[(X-\mu)^3] = E[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3]$$

$$= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3$$

$$= 5 - 3\cdot 1\cdot 2 + 3\cdot 1^2\cdot 1 - 1^3$$

$$= 5 - 6 + 3 - 1 = 1$$

<u>ex.</u> Given a p.d.f of X as below

$$f(x) = \begin{cases} 1/2 & \text{for } 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

Find the 1st non-central moment (1) and 2nd central moment (2)

<u>Sol.</u>

(1) $E(X) = \int_0^2 x \cdot \frac{1}{2}\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right]_0^2 = \frac{1}{4}[4-0] = 1$

(2) $E[(X-E(X))^2] = \text{variance of } X = E[(X-1)^2]$

$$= \int_0^2 (x-1)^2 \cdot \frac{1}{2}\,dx = \frac{1}{2}\int_0^2 [x^2-2x+1]\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{2x^2}{2} + x\right]_0^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - 4 + 2\right] = \frac{1}{2}\left[\frac{8}{3} - \frac{6}{3}\right]$$

$$= \frac{1}{2}\cdot\frac{2}{3} = 1/3$$

$\boxed{\text{or}}$ To Find var(X)

$\text{Var}(X) = E(X^2) - [E(X)]^2$ $\boxed{E(X^2) = 4/3,\ E(X) = 1}$

Ch. 3 - part 6-1
Moment Generating Function

Def A moment generating function (M.g.f.) of a r.v. X is a function that determins all moments of X denoted by $M_X(t)$; $t \in (-h, h)$ and defined as below

$$M_X(t) = \begin{cases} E(e^{tX}) = \sum_x e^{tx} f(x) \ ; \text{if } X \text{ is d.r.v. with p.m.f. } f(x) \\ E(e^{tX}) = \int_x e^{tx} f(x)\,dx \ ; \text{if } X \text{ is c.r.v. with p.d.f. } f(x) \end{cases}$$

ex.1 Given a p.m.f. of d.r.v. X

$$f(x) = \begin{cases} \frac{x}{6} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

Find $M_X(t)$

Sol.

$$M_X(t) = E[e^{tX}] = \sum_{x=1}^{3} e^{tx} \cdot f(x)$$

$$= \sum_{x=1}^{3} e^{tx} \cdot \left(\frac{x}{6}\right)$$

$$= \frac{1}{6} \left\{ 1 \cdot e^{t} + 2 \cdot e^{2t} + 3 e^{3t} \right\}$$

for $t \in (-\infty, \infty)$

ملاحظة: في السؤال اعلاه اذا اعطي $M_X(t)$ فهل من الممكن ايجاد الدالة $f(x)$ ؟ نعم بالمقارنة مع التعريف

<u>ex.2</u> Given a p.d.f. of a c.r.v. X

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \leftarrow (0,\infty) \\ 0 & \text{o.w.} \end{cases}$$

Find $M_X(t)$ and <u>Sketch</u> it's graph.

<u>Sol.</u>

$$M_X(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx}(e^{-x})\,dx = \int_0^\infty e^{-x+tx}\,dt$$

$$= \int_0^\infty e^{-\underline{(1-t)}x}\,dx \qquad (1-t) \text{ هو معامل } x$$

$(1-t)$ is coefficient of X

This integral <u>exists</u> only when $(1-t) > 0$??

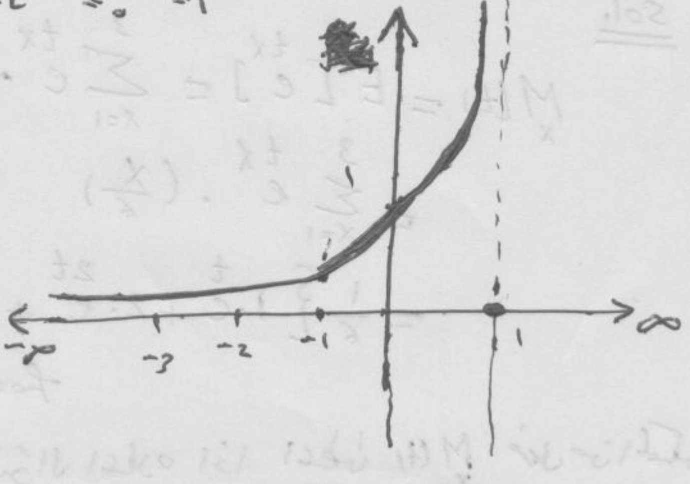
<u>i.e.;</u> $\boxed{t < 1}$ مجال $M_X(t)$ الدالة

$$\therefore M_X(t) = \frac{-1}{1-t} e^{-(1-t)x} \Big]_0^\infty$$

$$= \frac{-1}{1-t}\left[\underbrace{e^{-\infty}}_{=0} - \underbrace{e^{0}}_{=1}\right] = \frac{1}{1-t} \quad \underline{\underline{\text{for } t < 1}}$$

رسم الدالة

t	$M_X(t)$
1	∞
0	1
-1	$\frac{1}{2} = 0.5$
-2	$\frac{1}{3} = 0.33$
-3	$1/4 = 0.25$
⋮	
$-\infty$	0



(11)

ex.1 توضيح (3)

Ch.3 - part-1

if

$$M_X(t) = \frac{1}{6}\left[e^{t} + 2e^{2t} + 3e^{3t} \right] \quad ; \; t \in (-\infty, \infty)$$

then $f(x) = ??$

$\because$ $M_X(t) = E[e^{tX}]$, X is d.r.v.

$$= \sum_{x=1}^{3} e^{tx} \cdot f(x)$$

$M_X(t)$ نقارن مع التعريف اعلاه لـ

$$= e^{t} \cdot f(1) + e^{2t} \cdot f(2) + e^{3t} \cdot f(3)$$

$$\Rightarrow f(1) = \frac{1}{6}$$

$$f(2) = \frac{2}{6}$$

$$f(3) = 3/6$$

$$\therefore \quad f(x) = \begin{cases} \frac{x}{6} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

$$\int_0^\infty e^{-(1-t)x}\,dx$$

ex.2 تابع
Ch.3 - Part-1

This integral exists only when $(1-t)>0$

i.e.; $t<1$ أي أن

$$\int_0^\infty e^{ax}\,dx = \frac{1}{a}e^{ax}\Big]_0^\infty = \frac{1}{a}\Big[\underbrace{e^{\infty}}_{=\infty} - \underbrace{e^{0}}_{=1}\Big] = \infty \text{ غير معرف}$$

$$\int_0^\infty e^{-ax}\,dx = -\frac{1}{a}e^{-ax}\Big]_0^\infty = \frac{-1}{a}\Big[\underbrace{e^{-\infty}}_{=\text{صفر}} - \underbrace{e^{0}}_{=1}\Big] = \frac{1}{a} \text{ معرف}$$

Moment Generating function (m.g.f.)

Th.9 If a r.v. X has m.g.f. $M_X(t)$, then

$M_X(0)=1$, $M'_X(0)=E(X)$, $M''_X(0)=E(X^2)$, ...,

$M_X^{(k)}(0)=E(X^k)$.

proof Hint (by Macclaurin series of e^X)

$$e^X = 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^k}{k!}$$

ex Given $M_X(t) = \frac{1}{1-2t}$ for $t < \frac{1}{2}$

Find Mean and Variance

Sol. $M_X(t) = \frac{1}{1-2t} = (1-2t)^{-1}$

$M'_X(t) = -(1-2t)^{-2}(-2) = 2(1-2t)^{-2}$

$E(X) = M'_X(0) = 2(1-2\times 0)^{-2} = 2 \Longleftrightarrow \boxed{\text{Mean}}$

$M''_X(t) = -4(1-2t)^{-3}(-2) = 8(1-2t)^{-3}$

$E(X^2) = M''_X(0) = 8(1-2\times 0)^{-3} = 8$

$\text{Var}(X) = E(X^2) - [E(X)]^2$ تعريف التباين

$= 8 - 2^2$

$= 4$

②

Th. 10 Let X be a r.v. with m.g.f. $M_X(t)$ and if $\boxed{y = aX + b}$, then the m.g.f. of y is

$$M_y(t) = e^{bt} \cdot M_X(at)$$

proof **Hint** $M_y(t) = E[e^{ty}]$

$$= E[e^{t(aX+b)}]$$

ex. Given $M_X(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$

Let $y = 1 - 2X$; Find $M_y(t)$?

Sol $Y = 1 - 2X \Rightarrow \boxed{a = -2}, \boxed{b = 1}$

by th. 10 $M_y(t) = e^{bt} \cdot M_X(at)$

$$= e^{t} M_X(-2t)$$

for $t < 1/3$

$\because M_X(t) = \frac{1}{1-3t}$ for $-2t < \frac{1}{3}$

$\therefore M_X(-2t) = \frac{1}{1-3(-2t)}$ for $t > -\frac{1}{6}$

$M_X(-2t) = \frac{1}{1+6t}$

$\Rightarrow M_y(t) = e^{t} \cdot \frac{1}{1+6t}$ for $t > -\frac{1}{6}$

نوضح ③
Ch.3 - part 6-2

ex. $f(x) = \begin{cases} \frac{x}{3} & \text{for } x = 1, 2 \\ 0 & \text{o.w.} \end{cases}$

Find ① $E(X)$ by two methods

② Find $E(Y)$, $Y = 2X + 2$ by two methods

Sol. ①

ⓐ الطريقة الأولى

$$E(X) = \sum_{x=1}^{2} x \cdot f(x)$$

$$= \sum_{x=1}^{2} x \cdot \frac{x}{3} = \frac{1}{3} \sum_{x=1}^{2} x^2$$

$$= \frac{1}{3}[1^2 + 2^2] = 5/3$$

ⓑ الطريقة الثانية

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{2} e^{tx} \cdot \frac{x}{3}$$

$$= \frac{1}{3}[e^t + 2e^{2t}]$$

$$M'_X(t) = \frac{1}{3}[e^t + 2(2)e^{2t}]$$

$$= \frac{1}{3}[e^t + 4e^{2t}]$$

$$E(X) = M'_X(0) = \frac{1}{3}[\underbrace{e^0}_{=1} + 4 \cdot \underbrace{e^0}_{=1}]$$

$$= \frac{5}{3}$$

Sol. 2 $Y = 2X + 2$

(a) الطريقة الأولى

$$E(Y) = 2\,E(X) + 2$$
$$= 2\left[\frac{5}{3}\right] + 2$$
$$= \frac{10}{3} + 2 = \frac{16}{3}.$$

(b) الطريقة الثانية

$$M_Y(t) = e^{2t} \cdot M_X(2t) \quad ; \quad M_X(t) = \frac{1}{3}\left[e^{t} + 2e^{2t}\right]$$
$$M_X(2t) = \frac{1}{3}\left[e^{2t} + 2e^{4t}\right]$$
$$= e^{2t} \cdot \left[\frac{1}{3}\left(e^{2t} + 2e^{4t}\right)\right]$$
$$= \frac{1}{3}\left[e^{4t} + 2e^{6t}\right]$$
$$M'_Y(t) = \frac{1}{3}\left[4e^{4t} + 12e^{6t}\right]$$
$$E(Y) = M'_Y(0) = \frac{1}{3}\left[4e^{0} + 12e^{0}\right]$$
$$= \frac{1}{3}\left[4 + 12\right]$$
$$= \frac{16}{3}$$

By th. (10) $\Rightarrow M_y(t) = e^{bt} M_x(at) = e^{t} M_x(-2t)$

$\because M_x(t) = \dfrac{1}{1-3t}$ for $t < \dfrac{1}{3}$

$M_x(-2t) = \dfrac{1}{1-3(-2t)} = \dfrac{1}{1+6t}$ for $t > \dfrac{-1}{6}$

$M_y(t) = e^{t} \cdot \dfrac{1}{1+6t}$ for $t > \dfrac{-1}{6}$

The bounded of probability: حدود الاحتمالية

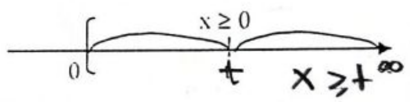
Theorem "11": ((Markov inquality))

If x is ar.v. and if $P(x \geq 0) = 1$ then $\quad P(x \geq t) \leq \dfrac{E(x)}{t}$, for $t > 0$

Proof: case "1": if x is a c.r.v with a p.d.f f(x)

$$\because p(x \geq 0) = 1 \Rightarrow \int_0^{\infty} f(x)dx = 1$$

$$f(x) \begin{cases} > 0 & \text{for } x \geq 0 \\ = 0 & \text{O.W} \end{cases}$$



$$E(x) = \int_0^{\infty} Xf(x)dx = \int_0^{t} xf(x)dx + \int_t^{\infty} xf(x)dx \geq \int_t^{\infty} xf(x)dx \geq \int_t^{\infty} tf(x)dx. \quad [x \geq t]$$

$$\therefore E(x) \geq \int_t^{\infty} tf(x)dx \Rightarrow E(x) \geq t\int_t^{\infty} f(x)dx$$

$$E(x) \geq t \cdot p(x \geq t)$$

$$\therefore p(x \geq t) \leq \frac{E(x)}{t}$$

Case "2": If x is a d.r.v. with a p.m.f. f(x) by the same method.

Theorem "12": "Chebyshev inqualities"

Let x be a r.v. where V(x) exists,then:

1. $p(|x-\mu| \geq t) \leq \dfrac{V(x)}{t^2}$, $t > 0$, 2. $p(|x-\mu| < t) \geq 1 - \dfrac{V(x)}{t^2}$, $t > 0$,

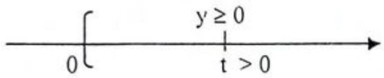
Note: 1. $\dfrac{V(x)}{t^2}$ *is called the upper buond of* $p(|x-\mu| \geq t)$

2. $1 - \dfrac{V(x)}{t^2}$ *is called the lower buond of* $p(|x-\mu| < t)$

Proof: 1. $V(x) = E[(x-\mu)^2] \geq 0$

$$Let\ y=(x-\mu)^2 \geq 0 \Rightarrow E(y)=E[(x-\mu)^2 = V(x) \geq 0$$

$\therefore$ by Markov inquality, we get



$$p(y \geq t^2) \leq \frac{E(y)}{t^2} \Rightarrow p[(x-\mu)^2 \geq t^2] \leq \frac{v(x)}{t^2}$$

$$p[|x-\mu| \geq t] \leq \frac{V(x)}{t^2}$$

2. $p(A^c) = 1 - p(A)$

$$p[|x-\mu| \geq t] = 1 - p[|x-\mu| < t] \leq \frac{V(x)}{t^2}$$

$$p[|x-\mu| < t] \geq 1 - \frac{V(x)}{t^2}$$

Ex. "1": If x ~ unif. $(-\sqrt{3}, \sqrt{3})$ then:

a. Find the upper bound of $p(|x-\mu| \geq \frac{3}{2})$

b. Find the value of $p(|x-\mu| \geq \frac{3}{2})$

Sol.:

a. $$f(x) = \begin{cases} \frac{1}{\sqrt{3}+(\sqrt{3})} = \frac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < x < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$$

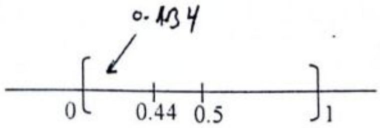
$$u.b = \frac{V(x)}{t^2}, \qquad t = \frac{3}{2}$$

$$E(x) = \int_{\sqrt{3}}^{\sqrt{3}} X \frac{1}{2\sqrt{3}} dx = \frac{1}{4\sqrt{3}} X^2 |_{-\sqrt{3}}^{\sqrt{3}} = 0$$

$$E(x^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \frac{1}{2\sqrt{3}} dx = \frac{1}{6\sqrt{3}} x^3 |_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{6\sqrt{3}}[(\sqrt{3})^3 - (-\sqrt{3})^3] = \frac{1}{6\sqrt{3}}[3\sqrt{3}+3\sqrt{3}] = 1$$

$\therefore V(x) = E(x^2) - [E(x)]^2 = 1 - 0 = 1$

$\therefore u.b = \dfrac{V(x)}{t^2} = \dfrac{1}{(\frac{3}{2})^2} = \dfrac{4}{9} = 0.44$



b. $p(|x-\mu| \geq \frac{3}{2}) = p(|x-0| \geq \frac{3}{2}) = p(|x| \geq \frac{3}{2})$

$= 1 - p(|x| < \frac{3}{2}) = 1 - p(-\frac{3}{2} < x < \frac{3}{2}) = 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} dx$

$= 1 - \frac{1}{2\sqrt{3}} x \Big|_{-\frac{3}{2}}^{\frac{3}{2}} = 1 - \frac{1}{2\sqrt{3}}(\frac{3}{2} + \frac{3}{2}) = 1 - \frac{3}{2\sqrt{3}} = 1 - \frac{\sqrt{3}}{2}$

$= 1 - 0.866 = 0.134$

Ex. "2": Given a p.d.f $f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$

a. Find the lower bound of $p(\frac{5}{4} < x < \frac{11}{4})$

b. Find the value of $p(\frac{5}{4} < x < \frac{11}{4})$

Sol.: a. $L.b = 1 - \dfrac{V(x)}{t^2}$

$E(x) = \int_0^3 x \cdot \frac{2x}{9} dx = \frac{2}{27} x^3 \Big|_0^3 = \frac{2}{27}(27 - 0) = 2$

$E(x^2) = \int_0^3 x^2 \cdot (\frac{2x}{9}) dx = \frac{2}{36} x^4 \Big|_0^3 = \frac{1}{18}(81) = \frac{9}{2} = 4.5$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$= 4.5 - 4 = 0.5 = \frac{1}{2}$

$p(\frac{5}{4} < x < \frac{11}{4}) = p(\frac{5}{4} - 2 < x - 2 < \frac{11}{4} - 2) = p(-\frac{3}{4} < x - 2 < \frac{3}{4})$

$= p(|x-2| < \frac{3}{4})$

$\therefore t = \frac{3}{4}$

$\therefore L.b = 1 - \dfrac{V(x)}{t^2} = 1 - \dfrac{\frac{1}{2}}{\frac{9}{16}} = 1 - \frac{1}{2} \times \frac{16}{9} = 1 - \frac{8}{9} = \frac{1}{9}$

b. $p(\frac{5}{4} < x < \frac{11}{4}) = \int_{\frac{5}{4}}^{\frac{11}{4}} \frac{2x}{9} dx$

$$= \frac{1}{9} x^2 \Big|_{\frac{5}{4}}^{\frac{11}{4}} = \frac{1}{9}[\frac{121}{16} - \frac{25}{16}] = \frac{1}{9}[\frac{96}{16}] = \frac{6}{9}$$

Median of Distribution of r.v.:

Def.: The median (m) is a value of x such that satisfing the two following inqualities:

$$p(x < m) < \frac{1}{2} \quad \& \quad p(x \le m) \ge \frac{1}{2}$$

By properties of c.d.f F(x)

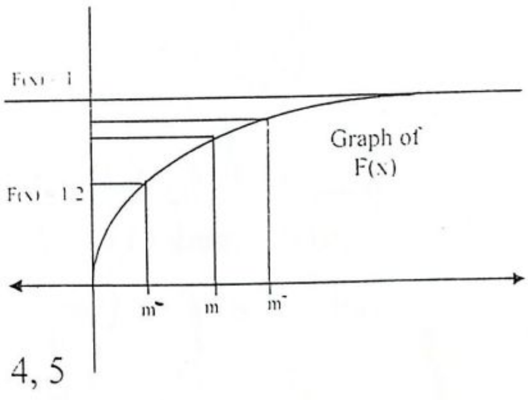
$p(x < m) = F(\overline{m}), \quad p(x \le m) = F(m) = F(m^-)$

$i.e.: p(x < m) = F(\overline{m}) < \frac{1}{2}. \quad p(x \le m) = F(m^-) \ge \frac{1}{2}$

Note ① *If* $\overline{m} = m = m^+$ *then*

$$p(x \le m) = F(m) = \frac{1}{2}$$

② *The value of median is uniqu*



Ex.: Given a p.m.f $f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$

Find the median of x.

Sol.: $p(x < m) < \frac{1}{2} \quad \& \quad p(x \le m) \ge \frac{1}{2}$

Suppose that m =1

$p(x < 1) = 0 < \frac{1}{2}, \quad p(x \le 1) = f(1) = \frac{1}{15} \not\ge \frac{1}{2}$

$\therefore m \ne 1$

Markov Inequality

* for $p(X \geq 0) = 1$, then

$$p(X \geq t) \leq \frac{E(X)}{t} \quad \text{for } t > 0$$

Chebyshev Inequalities

Let X be a r.v., where $V(X)$ exists, then

① $p(|X - \mu| \geq t) \leq \frac{V(X)}{t^2}$ u.b. for $t > 0$

② $p(|X - \mu| < t) \geq 1 - \frac{V(X)}{t^2}$ for $t > 0$

Ex X is a r.v.! with $M_X(t)$ for $-h<t<h$

show that

- $P(X \geq a) \leq e^{-at} M_X(t)$ for $0<t<h$
- $P(X \leq a) \leq e^{-at} M_X(t)$ for $-h<t<0$

Sol. $M_X(t) = E(e^{tX})$

$Y = e^{tX} > 0$ (دالة اسية موجبة دائماً)

$[P(Y \geq 0) = 1]$ •

also, $e^{at} > 0$ ( )

by Markov Inquality

$[P(X \geq 0) = 1$ & for $\boxed{t > 0}]$

$$P[X \geq t] \leq \frac{E(X)}{t}$$

بدلاً عنه Y e^{at} بدلاً عنه

ونطبق النظرية

$$\therefore P[Y \geq e^{at}] \leq \frac{E(Y)}{e^{at}}$$

$$P[e^{tX} \geq e^{at}] \leq \frac{E[e^{Xt}]}{e^{at}}$$

$$P[tX \geq at] \leq \frac{M_X(t)}{e^{at}}$$

① for $t \in (0,h) \rightarrow P[X \geq a] \leq e^{-at} \cdot M_X(t)$

② for $t \in (-h,0) \rightarrow P[X \leq a] \leq e^{-at} M_X(t)$

Note

$P(X \geq a) \leq e^{-at} M_X(t)$ for $0<t<h$

$P(X \leq a) \leq e^{-at} \cdot M_X(t)$ for $-h<t<0$

There Inqualities are called (Bernstien Inqualities)

(Bernstien Inqual.)

Ex If $X \sim unif(-1, 2)$, then

Show that $M_X(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$

Sol

$$f(x) = \begin{cases} \frac{1}{2-(-1)} = \frac{1}{3} & \text{for } -1 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

$$M_X(t) = E[e^{tx}] = \int_{-1}^{2} e^{tx} \cdot \frac{1}{3} dx = \frac{1}{3} \int_{-1}^{2} e^{tx} dx$$

$$= \frac{1}{3t} e^{tx} \Big]_{-1}^{2} = \frac{1}{3t} [e^{2t} - e^{-t}]$$

for $t = 0 \longrightarrow M_X(t) = E[e^{0x}] = E[1] = 1$

so $M_X(t) = \begin{cases} \frac{1}{3t} [e^{2t} - e^{-t}] & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$

Exercises About Probability Bounded

Q_1: Given a p.f.

$f(-1) = \frac{1}{8}$, $f(0) = \frac{6}{8}$, $f(1) = \frac{1}{8}$

and $f(x) = 0$ (otherwise)

Show that the value of upper bounded and exact value of $P(|X| \geq 2\sigma_x)$ are identical (متطابقة).

Sol.

$$E(X) = \sum_{x=-1}^{1} x\, f(x)$$

$$= (-1) f(-1) + (0) f(0) + (1) f(1)$$

$$= -\frac{1}{8} + 0 + \frac{1}{8}$$

$$= \text{Zero}$$

$$\therefore \boxed{E(X) = \mu_x = 0}$$

$$E(X^2) = \sum_{x=-1}^{1} x^2 f(x)$$

$$= (-1)^2 f(-1) + (0)^2 f(0) + (1)^2 f(1)$$

$$= \frac{1}{8} + 0 + \frac{1}{8}$$

$$= \frac{2}{8} = \frac{1}{4}$$

$$\sigma_x^2 = V(X) = E(X^2) - (E(X))^2 \geq 0$$

$$= \frac{1}{4} - (0)^2$$

$$= \frac{1}{4} > 0$$

$$\longrightarrow \sigma_x = +\sqrt{\sigma_x^2} = +\sqrt{V(X)} = +\sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\therefore P(|X| \geq 2\sigma) = P(|X-0| \geq \cancel{2}(\tfrac{1}{\cancel{2}}))$$

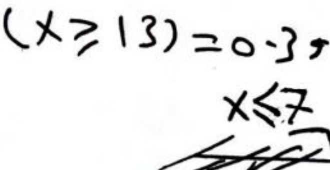
By $\begin{pmatrix}\text{Chebyshov} \\ \text{ineq.}\end{pmatrix}$ $\left\{ P(|X-\mu_x| \geq t) \leq \frac{V(X)}{t^2} \right\}$

$$\rightarrow \boxed{t=1}\;,\; \boxed{\mu_x = 0}$$

Then $\quad U.b. = \frac{V(x)}{t^2} = (\frac{1}{4}) / (1)^2 = \frac{1}{4}$

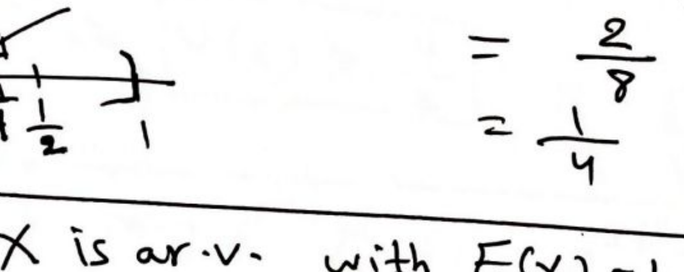
The exact value of bove pr.:

$$\begin{aligned} P(|X| \geq 2\sigma_x) &= P(|X| \geq \cancel{2}(\tfrac{1}{\cancel{2}})) \\ &= P(|X| \geq 1) \\ &= 1 - P(|X| < 1) \\ &= 1 - P(-1 < X < 1) \\ &= 1 - P(X=0) \\ &= 1 - f(0) \\ &= 1 - \frac{6}{8} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$



Q2: If X is a r.v. with $E(X) = 10$, $P(X \leq 7) = 0.2$, $P(X \geq 13) = 0.3$, then show that $V(X) \geq \frac{9}{2}$.

<u>sol.</u>



$$P(X \leq 7) + P(7 < X < 13) + P(X \geq 13) = 1$$

$$0.2 + P(7 < X < 13) + 0.3 = 1$$

$$P(7 < X < 13) = 1 - 0.2 - 0.3 = 1 - 0.5 = 0.5 \in [0,1].$$

$$P(7-10 < X-10 < 13-10) = P(-3 < X-10 < 3)$$

$$= P(|X-10| < 3)$$

$$0.5 = P(|X-10| < 3) \geq 1 - \frac{V(X)}{t^2}, \; t > 0$$

$$P(|X - \mu_x| < t) \geq 1 - \frac{V(X)}{t^2}, \; t > 0 \rightarrow \boxed{t = 3}$$

$$\frac{1}{2} \geq 1 - \frac{V(X)}{9}$$

$$-\frac{1}{2} \geq - \frac{V(X)}{9}$$

$$\frac{1}{2} \leq \frac{V(X)}{9} \Longrightarrow \frac{9}{2} \leq V(X)$$

$$\Rightarrow \therefore \boxed{V(X) \geq \frac{9}{2}}$$

#

Q_3: Given a m.g.f. $M_X(t) = e^{3t^2 + 2t}$ for $-\infty < t < \infty$

Find the L.b. of $P(\frac{-1}{2} < X < \frac{9}{2})$.

Sol.

$$M_X(t) = E(e^{tX}) = e^{3t^2 + 2t}$$

$$L.b. = 1 - \frac{V(X)}{t^2}, \; t > 0$$

$$V(X) = E(X^2) - (E(X))^2 \geq 0$$

$$= M_X''(0) - (M_X'(0))^2 \geq 0$$

$M_x(t) = e^{3t^2+2t}$

$M_x'(t) = (6t+2)\, e^{3t^2+2t}$

$M_x'(t=0) = (6(0)+2)\, e^{3(0)^2+2(0)} = 1$

$\boxed{E(x) = 2}$

$M_x''(t) = (6t+2)^2 e^{3t^2+2t} + (6)\, e^{3t^2+2t}$

$M_x''(t=0) = (\cancel{6(0)}+2)^2 \underbrace{e^0}_{=1} + (6)\underbrace{e^0}_{=1}$

$\boxed{E(X^2) = 10}$

$\therefore V(x) = E(x^2) - (E(x))^2$
$= 10 - (2)^2$
$= 6$

$\therefore \boxed{V(x) = \sigma_x^2 = 6}$

$P(-\frac{1}{2} < x < \frac{9}{2}) = P(-\frac{1}{2} - 2 < x-2 < \frac{9}{2} - 2)$
$= P(\frac{-5}{2} < x-2 < \frac{5}{2})$
$= P(|x-2| < \frac{5}{2})$
$= P(|X-\mu| < t) \Rightarrow \boxed{t = \frac{5}{2}}$

$L.b. = 1 - \frac{6}{(\frac{25}{4})} = 1 - 6(\frac{4}{25})$
$= 1 - \frac{24}{25} = \boxed{\frac{1}{25}}$

#

Q4: If $X \sim unif(0,2)$, find the u.b. of
$P(|X-\mu_x| \geq 2)$.

Sol. $\because X \sim unif(0,2)$

$\therefore f(x) = \begin{cases} \frac{1}{2} & \text{for } 0<x<2 \\ 0 & \text{o.w.} \end{cases}$

$$\mu_x = E(x) = \int_0^2 x\, f(x)\, dx$$
$$= \int_0^2 x \left(\tfrac{1}{2}\right) dx$$
$$= \left(\tfrac{1}{2}\right)\frac{x^2}{2}\Big|_0^2$$
$$= \frac{1}{4}\left[2^2 - 0^2\right]$$
$$= \frac{\cancel{4}}{\cancel{4}} = 1$$

$$\rightarrow \boxed{\therefore \mu_x = E(X) = 1}$$

$$E(x^2) = \int_0^2 x^2 f(x)\, dx$$
$$= \frac{1}{2}\int_0^2 x^2\, dx$$
$$= \frac{1}{2}\,\frac{x^3}{3}\Big|_0^2$$
$$= \frac{1}{6}(2^3 - 0^3) = \frac{8}{6} = \boxed{\frac{4}{3}}$$

$$\therefore V(X) = E(x^2) - (E(x))^2$$
$$= \frac{4}{3} - (1)^2$$
$$= \frac{1}{3} > 0$$

$\therefore$ u.b. $= \dfrac{V(x)}{t^2}$, $t > 0$

To find (t) :

$$P(|X-\mu_x| \geq t) = P(|X-1| \geq 2) \quad \therefore \boxed{t=2}$$

$$U.b. = \frac{V(x)}{t^2} = \frac{\frac{1}{3}}{(2)^2} = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

Q7: If $X \sim unif(-\sqrt{3}, \sqrt{3})$ then find the upper bounded of $P(|X-\mu_x| \geq \frac{3}{2})$.

Sol. $\because X \sim unif(-\sqrt{3}, \sqrt{3})$

$$\therefore f(x) = \begin{cases} \dfrac{1}{\sqrt{3}+\sqrt{3}} = \dfrac{1}{2\sqrt{3}} & for\ -\sqrt{3} < X < \sqrt{3} \\ 0 & o.w. \end{cases}$$

$$E(X) = \int_{-\sqrt{3}}^{\sqrt{3}} X f(x)\, dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} x \left(\frac{1}{2\sqrt{3}}\right) dx$$

$$= \left(\frac{1}{2\sqrt{3}}\right) \frac{X^2}{2} \Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{4\sqrt{3}} \left[\underbrace{(\sqrt{3})^2}_{=3} - \underbrace{(-\sqrt{3})^2}_{=3}\right]$$

$$= \frac{1}{4\sqrt{3}} (0)$$

$$= \text{Zero} = \mu_x$$

$$P(|X-0| \geq \underbrace{\tfrac{3}{2}}_{=t}) = P(|X-\mu_x| \geq t) \leq \frac{V(x)}{t^2}$$

$\boxed{\text{To find } V(x)}$

$$E(x^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 f(x)\,dx$$

$$= \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} x^2\,dx$$

$$= \frac{1}{2\sqrt{3}} \left. \frac{x^3}{3} \right|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{1}{6\sqrt{3}} \left[(\sqrt{3})^3 - (-\sqrt{3})^3 \right]$$

$$= \frac{1}{6\sqrt{3}} \left[3\sqrt{3} + 3\sqrt{3} \right]$$

$$= \frac{1}{\cancel{(6\sqrt{3})}} \cancel{(6\sqrt{3})}$$

$$= 1$$

$$\therefore V(x) = E(x^2) - (E(x))^2 \geq 0$$

$$= 1 - 0$$

$$= 1$$

$$\therefore \quad \text{U.b.} = \frac{V(x)}{t^2} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}$$

Q8: If a r.v. X with m.g.f. $M_x(t)$ for $-h<t<h$
Show that:

① $P(X \geq a) \leq e^{-at} M_x(t)$ for $0<t<h$

② $P(X \leq a) \leq e^{-at} M_x(t)$ for $-h<t<0$

Proof: $M_x(t) = E(e^{tx})$,
let $Y = e^{tx}$

$e^{at} > 0$, $a > 0$, $t > 0$

$P(X \geq t) \leq \frac{E(X)}{t}$ (by Markov Inq.)

$\left[P(Y \geq e^{at}) \leq \frac{E(Y)}{e^{at}} \right]$

Now;

$$P(X \geq a) = P(tX \geq ta)$$
$$= P(e^{tX} \geq e^{at})$$

$\therefore P(Y \geq e^{at}) \leq \frac{E(Y)}{e^{at}}$ (by Markov inq.)

$\leq \frac{E(e^{tX})}{e^{at}}$ since $(Y = e^{tX})$

$\therefore \boxed{P(e^{tX} \geq e^{at}) \leq \frac{M_X(t)}{e^{at}}} = e^{-at} M_X(t)$

Now; $P(\cancel{\ln e^{tX}} \geq \cancel{\ln e^{at}}) \leq e^{-at} M_X(t)$

$P(X \geq a) \leq e^{-at} M_X(t)$ —① for $0<t<h$

And $P(\cancel{\ln e^{tX}} \geq \cancel{\ln e^{at}}) \leq e^{-at} M_X(t)$

$P(tX \geq at) \leq e^{-at} M_X(t)$

$P(X \leq a) \leq e^{-at} M_X(t)$ —② for $-h<t<0$

#

(13)

Excercise

(1) Show that $p(|x-\mu| \geq c) \leq \dfrac{E[(x-\mu)^{2k}]}{c^{2k}}$, $c>0$

Proof let $y = (x-\mu)^{2k} \geq 0$ since $\boxed{2k}$ even Integer.

by thm. 13 (Markove inequality)
if $p(x\geq 0)=1 \rightarrow p(x\geq t) \leq \dfrac{E(x)}{t}$, $t>0$

$\therefore p(y \geq c^{2k}) \leq \dfrac{E(y)}{c^{2k}}$ since $\boxed{c^{2k} > 0}$

$p[(x-\mu)^{2k} \geq c^{2k}] \leq \dfrac{E[(x-\mu)^{2k}]}{c^{2k}}$

$p[|x-\mu| \geq c] \leq \dfrac{E[(x-\mu)^{2k}]}{c^{2k}}$ 2k

(2) Given a p.m.f.

$$f(x) = \begin{cases} 1/8 & \text{if } x=-1 \\ 6/8 & \text{if } x=0 \\ 1/8 & \text{if } x=1 \end{cases}$$

(i) Find the u.b. of $p[|x-\mu| \geq 1]$
(ii) Find the value of $p[|x-\mu| \geq 1]$

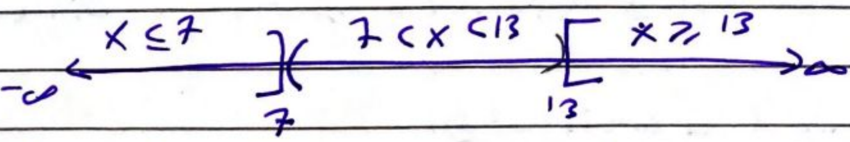
Sol. (i) u.b. $= \dfrac{V(x)}{t^2}$; $t=1$
we must find $V(x)$
$\mu = E(x) = \sum_{x=-1}^{1} x f(x) = (-1)(\frac{1}{8}) + 0(\frac{6}{8}) + 1(\frac{1}{8}) = 0$
$E(x^2) = \sum x^2 f(x) = (-1)^2(\frac{1}{8}) + 0^2(\frac{6}{8}) + 1^2(\frac{1}{8}) = \frac{2}{8} = 1/4$
$V(x) = E(x^2) - [E(x)]^2$
$= \frac{1}{4} - 0 = 1/4$
u.b. $= \dfrac{V(x)}{t^2} = \dfrac{1/4}{1^2} = 1/4$

$[\ \ 0 \quad 1/4 \quad 1/2 \quad 1\ \ ]$

(ii) $p(|x-\mu| \geq 1) = 1 - p[|x-\mu| < 1]$
$= 1 - p[|x-0| < 1]$
$= 1 - p[-1 < x < 1]$
$= 1 - f(0)$
$= 1 - 6/8 = 2/8 = 1/4$

Ex (3) let X be a r.v. where $E(X) = 10$, $p(X \le 7) = 0.2$ and $p(X \ge 13) = 0.3$, show that $V(x) \ge 9/2$

Sol.



$$(X \le 7) \cup (7 < X < 13) \cup (X \ge 13) = S$$

$$p(X \le 7) + p(7 < X < 13) + p(X \ge 13) = P(S) = 1$$

$$0.2 + p(7 < X < 13) + 0.3 = 1$$

$$p(7 < X < 13) = 1 - 0.5 = 0.5 = 1/2 \quad \text{——} \; (1)$$

$$E(x) = \mu_x = 10$$

by th. (14) [Chebyshiv inequality]

$$P[\,|X - \mu| \ge t\,] \le \frac{V(x)}{t^2} \; ; \; t > 0$$

$$P[\,|X - \mu| < t\,] \ge 1 - \frac{V(x)}{t^2} \; , \; t > 0$$

$$p(7 < X < 13) = p(7 - 10 < x - 10 < 13 - 10)$$

(1), we → $\frac{1}{2} = p(-3 < x - 10 < 3)$

$$\frac{1}{2} = p(|x - 10| < 3) \ge 1 - \frac{V(x)}{t^2} \; , \; t = 3$$

$$\frac{1}{2} \ge 1 - \frac{V(x)}{3^2}$$

$$\frac{V(X)}{9} \ge 1 - \frac{1}{2} = 1/2$$

$$\boxed{V(X) \ge \frac{9}{2}}$$

Ex. If x is a r.v. with $p(X \geq b) = 1$, $b \in R$
then $E[(x-b)^2] = 0$ iff $p(x = b) = 1$.

proof let $y = (x-b)^2$

$\Rightarrow$ $E(y) = E[(x-b)^2] = 0$

by th. $p(X \geq b) = 1$ & $\boxed{E(X) = b} \rightarrow p(x = b) = 1$ & $p(x > b) = 0$

$\because$ $E(y) = 0 \longrightarrow p(y = 0) = 1$

$p[(x-b)^2 = 0] = 1$

$p[(x-b) = 0] = 1$

$p[x = b] = 1$

$\Leftarrow$ if $p(x = b) = 1$

$\boxed{x = b} \longrightarrow x - b = 0 \Rightarrow (x-b)^2 = 0$

$\therefore$ $E[(x-b)^2] = 0$

Ex If x is a r.v. such that $p(X < 0) = 0$
then $p(x \geq 2\mu) \leq 1/2$

$$p(X \geq 2\mu) \leq \frac{E(x)}{2\mu} = \frac{\mu}{2\mu} = 1/2$$

EX Given $g(t) = \ln M_x(t)$
Show that $g'(0) = \mu$ and $g''(0) = V(x)$

proof $g(t) = \ln M_x(t)$

$$g'(t) = \frac{1}{M_x(t)} \cdot M'_x(t)$$

$$\boxed{t=0} \rightarrow g'(0) = \frac{M'_x(0)}{M_x(0)} = \frac{E(x)}{1} = \mu$$

$$g''(t) = \frac{M_x(t) \cdot M''_x(t) - M'_x(t) \cdot M'_x(t)}{[M_x(t)]^2}$$

$$\boxed{t=0}, \quad g''(0) = \frac{M_x(0) \cdot M''_x(0) - M'_x(0) \cdot M'_x(0)}{[M_x(0)]^2}$$

$$= \frac{1 \cdot E(x^2) - E(x) \cdot E(x)}{1^2}$$

$$= \frac{E(x^2) - E(x)^2}{1}$$

$$= V(x)$$

Q_1: Given a p.m.f. $f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$

Find the expected Value of X.

Sol.

$$E(x) = \sum_{x=1}^{5} x \cdot f(x) = \sum_{x=1}^{5} \frac{x^2}{15} = \frac{1}{15} \sum_{x=1}^{5} x^2$$

$$= \frac{1}{15}(1+4+9+16+25) = \frac{55}{15} = \frac{11}{3}$$

Q_2: Given a p.d.f. $f(x) = \begin{cases} 3x^2 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

Find $E(x)$.

Sol.

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_0^1 x \cdot (13x^2)\, dx = \frac{13}{4} x^4 \Big|_0^1$$

$$= \frac{13}{4}$$

Q_3: Given a p.d.f. $f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 4 \\ 0 & \text{o.w.} \end{cases}$

Find $E[(x+2)^2]$.

Sol.

$$E[(x+2)^2] = E[(x^2+4x+4)] = E(x^2) + 4E(x) + 4$$

To find $E(x)$ & $E(x^2)$:

$$E(x) = \int_0^4 x \cdot f(x)\, dx = \int_0^4 x \cdot \left(\frac{x}{2}\right) dx = \frac{1}{2}\int_0^4 x^2\, dx = \frac{1}{2}\frac{x^3}{3}\Big|_0^4$$

$$= \frac{1}{6}(4^3 - 0^3) = \frac{64}{6} \simeq 10.67$$

$$E(x^2) = \int_0^4 x^2 \left(\frac{x}{2}\right) dx = \frac{1}{2}\int_0^4 x^3\, dx = \frac{1}{2}\frac{x^4}{4}\Big|_0^4 = \frac{1}{8}(4^4 - 0^4)$$

$$= \frac{256}{8} = 32$$

$$\therefore E[(x+2)^2] = (32) + 4(10.67) + 4 = 78.68$$

Q4: Given a p.m.f. $f(x) = \begin{cases} \frac{x}{10} & \text{for } x = 1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

Find $E(2X^3-1)$, $E[(X-1)^3]$

Sol.

$$E(2X^3-1) = 2E(X^3)-1$$

$$E(X^3) = \sum_{X=1}^{4} X^3 \cdot f(x) = \sum_{X=1}^{4} X^3\left(\frac{X}{10}\right) = \frac{1}{10}\sum_{X=1}^{4} X^4$$

$$= \frac{1}{10}\left[1^4+2^4+3^4+4^4\right] = \frac{354}{10} = 35.4$$

$$\therefore E(2X^3-1) = 2(35.4)-1 = 69.8$$

Now, $E[(X-1)^3] = E(X^3-3X^2+3X-1)$

$$= E(X^3)-3E(X^2)+3E(X)-1$$

$$E(X^2) = \frac{1}{10}\sum_{X=1}^{4} X^2 \cdot X = \frac{1}{10}\sum_{X=1}^{4} X^3$$

$$= \frac{1}{10}\left[1^3+2^3+3^3+4^3\right] = \frac{1}{10}\left[1+8+27+64\right]$$

$$= \frac{100}{10} = 10$$

$$E(X) = \frac{1}{10}\sum_{X=1}^{4} X \cdot X = \frac{1}{10}\sum_{x=1}^{4} X^2$$

$$= \frac{1}{10}\left[1^2+2^2+3^2+4^2\right] = \frac{1}{10}\left[1+4+9+16\right]$$

$$= \frac{30}{10} = 3$$

$$\therefore E[(X-1)^3] = 35.4-3(10)+3(3)-1$$

$$= 35.4-30+8 = 13.4$$

OR

$$\sum_{x=1}^{n} X = \frac{n(n+1)}{2}$$

$$\sum_{x=1}^{4} X = \frac{4(5)}{2} = 10$$

$$\sum_{x=1}^{n} X^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{x=1}^{4} X^2 = \frac{4(5)(9)}{6} = 30$$

$$\sum_{x=1}^{n} X^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{x=1}^{4} X^3 = (10)^2 = 100$$

Q5: Given a p.d.f. $f(x) = \begin{cases} 6x(1-x) & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

If $y = 2-3x$, then find $E(y)$ & $V(y)$.

Sol:

$$E(Y) = E(2-3X) = 2-3E(X) \quad \text{(by theorem)}$$

$$E(X) = \int_0^1 x \cdot (6x(1-x))\,dx = \cdots$$

$$V(Y) = V(2-3X) = 9V(X) \quad \text{(by Theorem)}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_0^1 x^2 \cdot 6x(1-x)\,dx = \cdots$$

Q6: Let X be a c.r.v. have a p.d.f $f(x)$ where $f(x)$

$$f(x) > 0 \quad \text{for } 0<x<b$$
$$= 0 \quad \text{o.w.}$$

Show that $E(X) = \int_0^b [1-F(x)]\,dx$

Sol: $E(X) = \int_0^b x \cdot f(x)\,dx$ (by def. of $E(X)$)

$$= \int_0^b x \cdot \frac{dF(x)}{\cancel{dx}} \cancel{dx} \quad \text{by } f(x) = \frac{dF(x)}{dx}$$

& by part integration: $\left(\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du\right)$

let $u = x$ & $dv = dF(x)$

$du = dx$ $\quad V = F(x)$

$$\int_0^b x\,dF(x) = xF(x)\Big|_0^b - \int_0^b F(x)\,dx$$

$$= x\int_0^b f(x)\,dx\Big|_0^b - \int_0^b F(x)\,dx \quad \text{by } F(x) = \int_0^b f(x)\,dx$$

$$= x \cdot 1\Big|_0^b - \int_0^b F(x)\,dx \quad \left(\text{by } \int_0^b f(x)\,dx = 1 \text{ since } f \text{ is p.d.f}\right)$$

$$= (b-0) - \int_0^b F(x)\,dx$$

$$\int_0^b x\, dF(x) = b - \int_0^b F(x)\,dx$$

$$= \int_0^b dx - \int_0^b F(x)\,dx \qquad \left(\text{by } b = \int_0^b dx = x\Big|_0^b = b\right)$$

$$E(x) = \int_0^b (1 - F(x))\,dx$$

Q_7: If X is d.r.v. have p.m.f. $f(x) > 0$ for $x = -1, 0, 1$
$= 0$ o.w.

a. If $f(0) = \frac{1}{2}$, Find $E(X^2)$

b. If $f(0) = \frac{1}{2}$ and $E(X) = \frac{1}{6}$, Find $f(-1)$, $f(1)$

<u>sol.</u>

a. If $f(0) = \frac{1}{2}$

$$E(X^2) = \sum_{x=-1}^{1} x^2 . f(x) = (-1)^2 f(-1) + (0)^2 f(0) + (1)^2 f(1)$$
$$= 1 . f(-1) + 0 + 1 . f(1)$$
$$= f(-1) + f(1) \quad \text{—— (1)}$$

$$\sum_{x=-1,0,1} f(x) = f(-1) + f(0) + f(1) = 1 \;(\text{since } f \text{ is p.m.f.})$$
$$= f(-1) + \frac{1}{2} + f(1) = 1$$
$$\Rightarrow f(-1) + f(1) = \frac{1}{2} \quad \text{—— (2)}$$

put (2) in (1), we get:

$$E(X^2) = f(-1) + f(1) = \frac{1}{2} \Longrightarrow E(X^2) = \frac{1}{2}$$

b. If $f(0) = \frac{1}{2}$ & $E(X) = \frac{1}{6}$

$$\sum_{x=-1}^{1} f(x) = f(-1) + f(0) + f(1)$$
$$= f(-1) + \frac{1}{2} + f(1)$$
$$\Rightarrow f(-1) + f(1) = \frac{1}{2} \quad \text{—— (1)}$$

$$E(X) = \sum_{x=-1,0,1} x . f(x) = (-1) f(-1) + (0) f(0) + (1) f(1)$$
$$= -f(-1) + f(1) \quad \& \; E(X) = \frac{1}{6}$$
$$\therefore \; f(1) - f(-1) = \frac{1}{6} \quad \text{—— (2)}$$

$f(1) = \frac{1}{3}$, $f(-1) = \frac{1}{6}$ ⇐ بجمع المعادلتين (1)، (2)

Q8: Given a p.d.f. $f(x) = \begin{cases} 1-|x| & \text{for } |x|<1 \\ 0 & \text{o.w.} \end{cases}$

a. Find $E(x)$ & $V(x)$

b. If $y = 2-3x$, find $E(y)$ & $V(y)$.

<u>Sol.</u> $f(x) = \begin{cases} 1-|x| & \text{for } -1<x<1 \\ 0 & \text{o.w.} \end{cases}$

$$= \begin{cases} 1+x & \text{for } -1<x<0 \\ 1-x & \text{for } 0\le x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(x) = \int_{-1}^{0} x\cdot(1+x)\,dx + \int_{0}^{1} x\cdot(1-x)\,dx$$

$$= \left(\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right)\Big|_{-1}^{0} + \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_{0}^{1}$$

$$= -\tfrac{1}{6} + \tfrac{1}{6} = 0$$

$$V(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_{-1}^{0} x^2(1+x)\,dx + \int_{0}^{1} x^2(1-x)\,dx$$

$$= \left(\tfrac{1}{3}x^3 + \tfrac{1}{4}x^4\right)\Big|_{-1}^{0} + \left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_{0}^{1}$$

$$= \tfrac{1}{12} + \tfrac{1}{12} = \tfrac{1}{6}$$

$$\sigma_x^2 = V(x) = \tfrac{1}{6} - 0 = \tfrac{1}{6} > 0$$

$$E(Y) = E(2-3x) = 2-3E(x) = 2-3(0) = \underline{\underline{2}}$$

$$V(Y) = V(2-3x) = (-3)^2 V(x) = 9\left(\tfrac{1}{6}\right) = \underline{\underline{\tfrac{3}{2}}} > 0$$

Q9: If $X \sim uniform(a,b)$; $a,b \in \mathbb{R}$. Find the value of 1^{st} centeral moment of X and also find the 2^{nd} centeral moment of X.

<u>Sol.</u> The 1^{st} centeral moment of X is equal to Zero =

$$E[(X-\mu)^1] = E(X - E(x)) = E(X) - E(\mu) = E(X) - \mu$$
$$= \mu - \mu = 0$$

<u>OR</u>

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{o.w.} \end{cases}$$

$$E(x) = \int_a^b x\, f(x)\, dx = \int_a^b x \frac{1}{(b-a)}\, dx = \frac{1}{2(b-a)} x^2 \Big|_a^b$$

$$= \frac{1}{2(b-a)}(b^2-a^2) = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{(b+a)}{2}$$

$$E[(X-\mu)^1] = E(x) - E(\mu) = E(x) - E(E(x))$$
$$= \frac{b+a}{2} - E\left(\frac{b+a}{2}\right)$$
$$= \cancel{\frac{b+a}{2}} - \cancel{\left(\frac{b+a}{2}\right)} = 0$$

2^{nd} centeral moment of X is $E[(X-\mu)^2] = Var(x) = \sigma_x^2$

$$\sigma_x^2 = E(x^2) - (E(x))^2$$
$$= \frac{1}{b-a}\int_a^b x^2 dx - (0)^2$$
$$= \frac{(b^3-a^3)}{3(b-a)} - \text{Zero} = \frac{b^3-a^3}{3(b-a)} > 0 \quad \& \; b>a$$

Q10: Let $\mu = E(x)$ and $\sigma^2 = V(x)$. Show that $E[(X-\mu)^4] \geq \sigma_x^4$ where $E((X-\mu)^4)$ is the 4^{th} centeral moment of X.

i.e. 4^{th} centered moment of X is greater than or equal the square of Variance of X.

Sol. Let $y = (X-\mu)^2 \Rightarrow E(y) = V(x) = \sigma_x^2$

$$V(y) = E(y^2) - [E(y)]^2 \geq 0$$
$$E(y^2) \geq [E(y)]^2$$
$$E[((X-\mu_x)^2)^2] \geq [\sigma_x^2]^2$$
$$E[(X-\mu_x)^4] \geq \sigma_x^4$$

Q_{11}: Find the mean and Variance if exist of Cauchy distribution.

Sol: $X \sim$ Cauchy dist.

$$\therefore f(x) = \frac{1}{\pi(1+x^2)} \quad \text{for } -\infty < x < \infty$$

$$\mu_{|x|} = E(|X|) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|}{(1+x^2)}\,dx \quad (E(x) \text{ exist only when } E(|X|) < \infty).$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{2x}{(1+x^2)}\,dx$$

$$= \frac{1}{\pi} \ln(1+x^2)\Big|_0^{\infty}$$

$$= \frac{1}{\pi}\left[\ln(\infty) - \ln(1)\right]$$

$$= \frac{1}{\pi}(\infty - 0) = \infty \nless \infty$$

$\therefore E(x)$ does not exist.

Therefore $V(x) = E(x^2) - (E(x))^2$ does not exist.

Q_{12}: Show that $E(x^2) \geq (E(x))^2$

Sol. ① when $V(x) = 0$

$$V(x) = E(x^2) - (E(x))^2 = 0$$

$$\therefore E(x^2) = (E(x))^2 \quad \text{— ①}$$

② when $V(x) > 0$

$$V(x) = E(x^2) - (E(x))^2 > 0$$

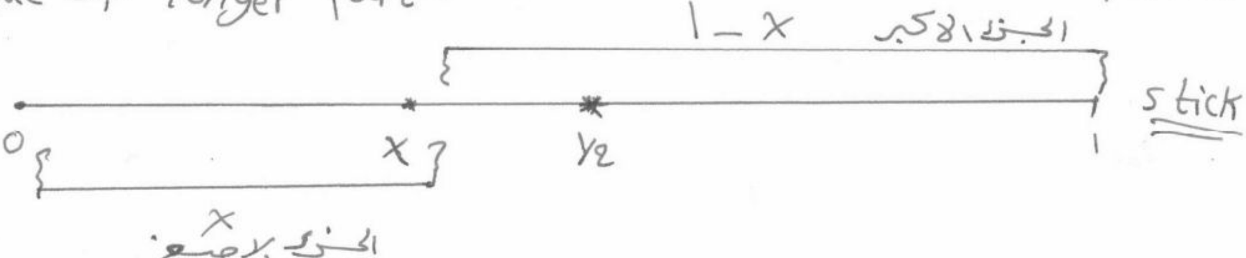
$$\therefore E(x^2) > (E(x))^2 \quad \text{— ②}$$

By ① & ② we get:

$$E(x^2) \geq (E(x))^2$$

Q13: A point X is chosen at random from a stick of length one unite, and then the stick is broken at the chosen point into two unequal parts, find the expected value of longer part.

Sol.



$\because x \in (0, \frac{1}{2})$

$\therefore X \sim unif.\ (0, \frac{1}{2})$

$$f(x) = \begin{cases} \frac{1}{1/2} = 2 & \text{for } 0 < x < \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

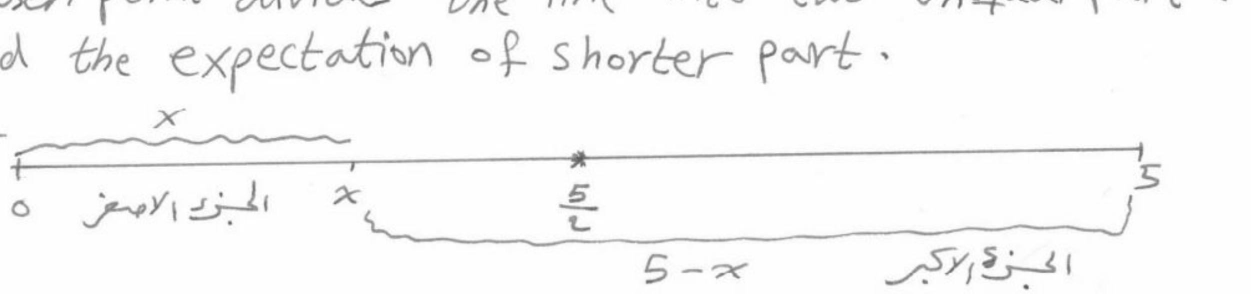
$E(\text{longer part}) = E(1-x) = 1 - E(x)$

$E(x) = \int_0^{\frac{1}{2}} 2x\, dx = \frac{2}{2} x^2 \Big|_0^{1/2} = \frac{1}{4}$

$1 - E(x) = 1 - \frac{1}{4} = \frac{3}{4}$ is the expected value of longer part.

Q14: A point X is chosen on a line of long (5) Cm. This chosen point divider the line into two unequal part. Find the expectation of shorter part.

Sol.



$\because x \in (0, \frac{5}{2})$ then $X \sim unif.(0, \frac{5}{2})$

$$f(x) = \begin{cases} \frac{2}{5} & \text{for } 0 < x < \frac{5}{2} \\ 0 & \text{o.w.} \end{cases}$$

$E(\text{the Shorter Part}) = E(x) = \int_0^{\frac{5}{2}} \frac{2}{5} x\, dx = \frac{5}{4} = 1.25$.

Q15: Show that : (i) $E\left(\frac{X-\mu_x}{\sigma}\right) = 0$

(ii) $E\left[\left(\frac{X-\mu_x}{\sigma}\right)^2\right] = 1$

Proof

(i) $E\left(\frac{X-\mu_x}{\sigma}\right) = \frac{1}{\sigma} E(X-\mu_x)$

$= \frac{1}{\sigma}\left[E(X) - E(\mu_x)\right]$

$= \frac{1}{\sigma}\left[\mu_x - \mu_x\right]$

$= \frac{1}{\sigma}(0) = \text{Zero}$

(ii) $E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2} \underbrace{E\left[(X-\mu)^2\right]}$

$= \frac{1}{\sigma^2}\sigma^2 = 1$

Q16: If the function of X is ;

$f(x) = \begin{cases} \frac{1}{4} & \text{for } x = 2,4,8,16 \\ 0 & \text{o.w.} \end{cases}$

(i) Find the m.g.f. of x ?

(ii) Find the mean and Variance of X ?

Sol.

$M_X(t) = E(e^{tx}) = \sum_{x=2,4,8,16} e^{tx} \cdot f(x)$

$= \left[e^{2t}\left(\frac{1}{4}\right) + e^{4t}\left(\frac{1}{4}\right) + e^{8t}\left(\frac{1}{4}\right) + e^{16t}\left(\frac{1}{4}\right)\right]$

$= \frac{1}{4}\left[e^{2t} + e^{4t} + e^{8t} + e^{16t}\right]$

$M_X'(t) = \frac{1}{4}\left[2e^{2t} + 4e^{4t} + 8e^{8t} + 16e^{16t}\right]$

$M_X'(0) = \frac{1}{4}\left[2+4+8+16\right] = \frac{30}{4} = 7.5$

$\therefore M_X'(0) = E(X) = 7.5$

$$M_x''(t) = \frac{1}{4}\left[4e^{2t} + 16e^{4t} + 64e^{8t} + 256e^{16t}\right]$$

$$M_x''(0) = \frac{1}{4}\left[4 + 16 + 64 + 256\right] = 85$$

$$M_x''(0) = E(X^2) = 85$$

$$\begin{aligned} Var(X) &= E(X^2) - (E(X))^2 \\ &= 85 - (7.5)^2 \\ &= 28.75 \end{aligned}$$

Q17: $f(x) = \begin{cases} \frac{1}{4} & \text{for } x = 0 \\ \frac{1}{2} & \text{for } x = 1 \\ \frac{1}{4} & \text{for } x = 2 \\ 0 & \text{o.w.} \end{cases}$

is a p.m.f.

(i) Find the moment .g. f. of X.

(ii) Find $E(X)$ & $V(X)$.

sol.

(i) $M_x(t) = E(e^{tx}) = \sum_{x=0}^{2} e^{tx} f(x)$

$$\begin{aligned} &= e^{0\cdot t}\left(\tfrac{1}{4}\right) + e^{t}\left(\tfrac{1}{2}\right) + e^{2t}\left(\tfrac{1}{4}\right) \\ &= \frac{1}{4} + \frac{1}{2}e^{t} + \frac{1}{4}e^{2t} \end{aligned}$$

(ii) $M_x'(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{2t}$

$M_x'(0) = \frac{1}{2} + \frac{1}{2} = 1 = E(X)$

$M_x''(t) = \frac{1}{2}e^{t} + e^{2t}$

$M_x''(0) = \frac{1}{2} + 1 = \frac{3}{2} = E(X^2)$

$$\begin{aligned} \sigma_x^2 &= \mu_x^2 - (\mu_x)^2 \\ &= E(X^2) - (E(X))^2 = \frac{3}{2} - 1 = \frac{1}{2} \end{aligned}$$

Q18: Suppose that the length of time a transistor will work (in a given Circuit) is a random Variable X with p.d.f. f(x):

$$f(x) = \begin{cases} 1000\, e^{-1000x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

The moment g. f. for X is:

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} (1000)\, e^{-1000x}\, dx$$

$$= \int_0^\infty (1000)\, e^{-x(1000-t)}\, dx$$

$$M_X(t) = \frac{1000}{1000-t}\ , \text{ for } t < 1000$$

$$\mu_X = M_X'(0) = \frac{1}{1000} \quad \& \quad \mu_X^2 = M_X''(0) = E(X^2) = \frac{2}{(1000)^2}$$

$$\sigma_X^2 = M_X''(0) - (M_X'(0))^2$$

$$= \frac{2}{(1000)^2} - \frac{1}{(1000)^2} = \frac{1}{(1000)^2}$$

Q19: Let X have the p.d.f.:

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

H.W.

Show that: $\mu_X = \frac{1}{3}$, $\sigma_X^2 = \frac{2}{9}$

Q20: If $X \sim \text{unif}(0,2)$. Find $V(X)$ H.W.

Q21: Given a p.d.f.

$$f(x) = \begin{cases} e^{x} & \text{for } x < 0 \\ 0 & \text{o.w.} \end{cases}$$

→

Find $M_x(t)$ and sketch it's graph. Also, find $E(x)$ by two methods.

Sol.

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{0} e^{tx} e^{x} dx = \int_{-\infty}^{0} e^{x(1+t)} dx$$

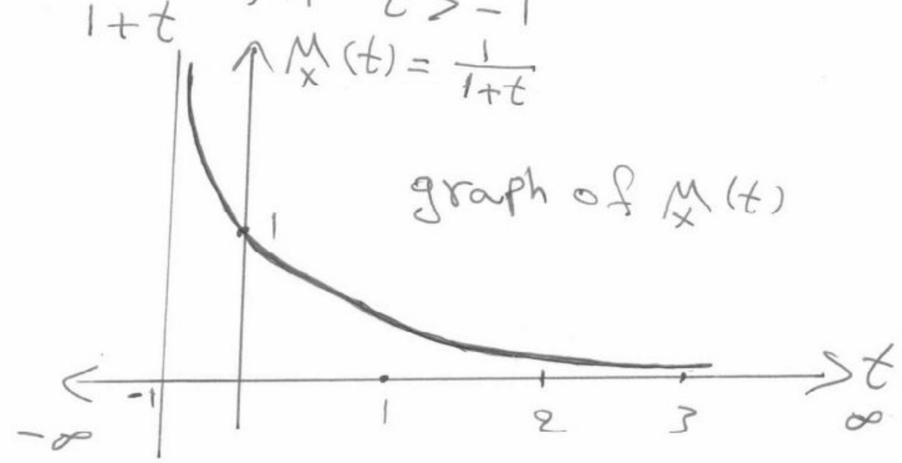
This integration exists only when $(1+t) > 0 \Rightarrow \boxed{t > -1}$

$$M_x(t) = \frac{1}{1+t} e^{x(1+t)} \Big|_{-\infty}^{0} = \frac{1}{1+t}[e^{0} - e^{-\infty}]$$

$$= \frac{1}{1+t}[1-0] = \frac{1}{1+t}$$

$$\therefore M_x(t) = \frac{1}{1+t} \quad \text{for } t > -1$$

t	$M_x(t)$
-1	∞
0	1
1	$\frac{1}{2}$
2	$\frac{1}{3}$
3	$\frac{1}{4}$
$\vdots$	
∞	0



① $E(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{0} x \cdot e^{x} dx$

By : $\int_{-\infty}^{0} u\,dv = uv\Big|_{-\infty}^{0} - \int_{-\infty}^{0} v\,du$

let $u = x \qquad dv = e^{x}$

$du = dx \qquad v = e^{x}$

$\rightarrow$

$$E(x) = xe^x \Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^x \, dx$$

$$= [0 \cdot e^0 - (-\infty) e^{-\infty}] - e^x \Big|_{-\infty}^{0}$$

$$= [0 - 0] - [e^0 - e^{-\infty}]$$

$$= 0 - [1 - 0] = -1$$

$\underline{\underline{\text{OR}}}$ $E(x) = M_x'(0)$

$$M_x(t) = \frac{1}{(1+t)} = (1+t)^{-1} \; ; \quad t > -1$$

$$M_x'(t) = -(1+t)^{-2}(1) = -(1+t)^{-2}$$

$$M_x'(0) = -(1+0)^{-2} = -(1)^{-2} = \underline{\underline{-1}}$$

$$\therefore E(X) = M_x'(0) = -1$$

Q22: Let the p.d.f. of X be given by $f(0) = \frac{3}{10}$, $f(1) = \frac{3}{10}$, $f(2) = \frac{1}{10}$ and $f(3) = \frac{3}{10}$. Compute the mean, Variance and standard deviation of X.

Sol. $R_X = \{0, 1, 2, 3\}$

$\therefore X = 0, 1, 2, 3$ is d.r.v.

$$\text{mean} = E(x) = \sum_{x=0}^{3} x \cdot f(x)$$

$$= 0 \cdot \frac{3}{10} + 1 \cdot \frac{3}{10} + 2 \cdot \frac{1}{10} + 3 \cdot \frac{3}{10} = \frac{17}{10}$$

$$E(x^2) = \sum_{x=0}^{3} x^2 \cdot f(x)$$

$$= 0^2 \cdot \frac{3}{10} + 1^2 \cdot \frac{3}{10} + 2^2 \cdot \frac{1}{10} + 3^2 \cdot \frac{3}{10}$$

$$\sigma_x^2 = Var(x) = E(x^2) - (E(x))^2$$

$\sigma_x = \sqrt{\sigma_x^2}$ ... Complete

Q23: Find the mean and Variance of the distribution that has the distribution function:

$$F(x) = \begin{cases} 0 & \text{for } x < 10 \\ \frac{1}{4} & \text{for } 10 \leq x < 15 \\ \frac{3}{4} & \text{for } 15 \leq x < 20 \\ 1 & \text{for } 20 \leq x \end{cases}$$

Sol.

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x = 10, 20 \\ \frac{2}{4} & \text{for } x = 15 \\ 0 & \text{o.w.} \end{cases}$$

طريقة ايجاد f من F في حالة d.r.v.

$$\begin{cases} F(10) - 0 = \frac{1}{4} - 0 = \frac{1}{4} \\ F(15) - F(10) = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} \\ F(20) - F(15) = \frac{4}{4} - \frac{3}{4} = \frac{1}{4} \end{cases}$$

$E(x) = ?$

$$E(x) = \sum_{x=10}^{20} x \cdot f(x) = 10 \cdot \frac{1}{4} + 15 \cdot \frac{2}{4} + 20 \cdot \frac{1}{4}$$
$$= \frac{10}{4} + \frac{30}{4} + \frac{20}{4} = \frac{60}{4} = 15$$

$$E(x^2) = \sum_{x=10}^{20} x^2 \cdot f(x) = (10)^2 \cdot \left(\frac{1}{4}\right) + (15)^2 \cdot \left(\frac{2}{4}\right) + (20)^2 \cdot \left(\frac{1}{4}\right)$$
$$= \frac{100}{4} + \frac{450}{4} + \frac{400}{4} = \frac{950}{4} = 237.5$$

$$V(x) = E(x^2) - [E(x)]^2$$
$$= 237.5 - (15)^2$$
$$= 237.5 - 225$$
$$= 12.5$$

Q24: Compute the mean and Variance of the sum of the two numbers that occur when a pair of fair dice is rolled one time?

Sol. $R_x = \{2, 3, \ldots, 12\}$, $n(S) = 6^2 = 36$

(15)

$$f(x) = \begin{cases} \frac{1}{36} & \text{for } x = 2 \\ \frac{2}{36} & \text{for } x = 3 \\ \frac{4}{36} & \text{for } x = 4 \\ \vdots & \vdots \\ \frac{1}{36} & \text{for } x = 12 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \sum_{x=2}^{12} x \cdot f(x) = ---$$

$$E(X^2) = \sum_{x=2}^{12} x^2 \cdot f(x) = ---$$

$$\sigma_x^2 = \text{var}(X) = E(X^2) - \left(E(X)\right)^2$$
$$= ---$$

Complete ---

Q25: A single die is rolled one time. Let X be the number of 6's faces that occur. Compute $M_X(t)$ and use it to evaluate the first (3) moments of X.

رقم وجه الزار الظاهر لنا

sol.

$$f(x) = \begin{cases} \frac{1}{6} & \text{for } x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{o.w.} \end{cases}$$

$R_x = \{1, 2, 3, 4, 5, 6\} \longrightarrow X = \underline{1, 2, 3, 4, 5, 6}$ (d.r.v.)

when

1st moment $= E(X) = M_X'(0)$

2nd moment $= E(X^2) = M_X''(0)$

3rd moment $= E(X^3) = M_X'''(0)$

when m.g.f of X is $M_X(t) = \frac{1}{6}\left[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right]$

$$M_X'(t) = \frac{1}{6}\left[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right]$$

$$M_X'(0) = \frac{1}{6}\left[e^0 + 2e^0 + 3e^0 + 4e^0 + 5e^0 + 6e^0\right] \quad (e^0 = 1)$$

$$= \frac{1}{6}\left[1 + 2 + 3 + 4 + 5 + 6\right] = \frac{21}{6} = E(X)$$

and so on for $M_X''(0) = E(X^2)$, $M_X'''(0) = E(X^3)$.

Q26: The number of hours of satisfactory operation that a certain brand of T.V. set will give (without repair) is a random variable X whose probability density function is;

$$f(x) = \begin{cases} 2\left(\frac{1}{3}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Compute $M_x(t)$ and use it to evaluate the first moment of X.

Sol.

$$M_x(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} \cdot \left(2\left(\frac{1}{3}\right)^x\right)$$

$$= 2\sum_{x=1}^{\infty} \left(\frac{e^t}{3}\right)^x$$

$$= 2\left[\frac{e^t}{3} + \left(\frac{e^t}{3}\right)^2 + \left(\frac{e^t}{3}\right)^3 + \dots\right]$$

$$= \frac{2e^t}{3}\underbrace{\left[1 + \left(\frac{e^t}{3}\right) + \left(\frac{e^t}{3}\right)^2 + \dots\right]}_{\text{Geometric series } \forall r = \frac{e^t}{3} < 1}$$

$$= \frac{2e^t}{3}\left[\frac{1}{1 - \frac{e^t}{3}}\right]$$

$$= \frac{2}{\cancel{3}} e^t \left[\frac{\cancel{3}}{3 - e^t}\right] = \frac{2e^t}{3 - e^t}$$

$$M_x'(t) = \frac{(3 - e^t)(2e^t) - 2e^t(-e^t)}{(3 - e^t)^2}$$

$$E(X) = M_x'(0) = \frac{(3)(2) + 2}{3^2} = \frac{8}{9}$$

Q27: Let the $M(t)$ of Y be defined by:

$$M_X(t) = \frac{1}{15}e^{t} + \frac{2}{15}e^{2t} + \frac{3}{15}e^{3t} + \frac{4}{15}e^{4t} + \frac{5}{15}e^{5t}$$

Compute the p.m.f. of X.

Sol. $M_X(t) = \sum_{x=1}^{5} \frac{x}{15}e^{tx} = \sum_{x=1}^{5} e^{tx} \cdot \left(\frac{x}{15}\right)$

$$\therefore f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Q28: If the moment generating function of X is;

$$M_X(t) = \frac{2}{5}e^{t} + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}$$

Find the mean and Variance of X, and define the p.m.f. of X.

Sol.

$$M_X(t) = \frac{2}{5}e^{t} + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t} = \sum_{x=1}^{3} e^{tx} \cdot f(x)$$

$$f(x) = \begin{cases} \frac{2}{5} & \text{for } x = 1, 3 \\ \frac{1}{5} & \text{for } x = 2 \\ 0 & \text{o.w.} \end{cases}$$

$$M'_X(t) = \frac{2}{5}e^{t} + \frac{2}{5}e^{2t} + \frac{6}{5}e^{3t}$$

$$M'_X(0) = \frac{2}{5} + \frac{2}{5} + \frac{6}{5} = \frac{10}{5} = 2$$

$$M''_X(t) = \frac{2}{5}e^{t} + \frac{4}{5}e^{2t} + \frac{10}{5}e^{3t}$$

$$E(X^2) = M''_X(0) = \frac{2}{5} + \frac{4}{5} + \frac{18}{5} = \frac{24}{5}$$

$$\begin{aligned} Var(x) &= M''_X(0) - (M'_X(0))^2 \\ &= \frac{24}{5} - 4 \\ &= \frac{24}{5} - \frac{20}{5} = \frac{4}{5} = 0.8 \end{aligned}$$

Q29: Given a c.d.f.

$$F(x) = \begin{cases} 1-\exp(-\frac{x}{3}) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

(i) Find $P(X>4)$, $P(2 < X \leq 5)$, $P(X=0)$

(ii) Find $M_x(t)$, $E(X)$, $V(X)$.

Sol.

$$F(x) = \begin{cases} 1-e^{-\frac{x}{3}} & \text{for } x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X>4) = 1 - P(X \leq 4)$$
$$= 1 - F(4)$$
$$= 1 - e^{-4/3}$$

$$P(2 < X \leq 5) = F(5) - F(2)$$
$$= [1-e^{-5/3}] - [1-e^{-2/3}]$$

$\because F(x)$ is conts at $x=0 \longrightarrow P(x=0)=0$

OR $P[x=0] = F(0^+) - F(0^-)$
$$= [1-e^0] - 0$$
$$= 0 - 0$$
$$= 0$$

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3} & \text{for } x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$M_x(t) = E(e^{tx}) = \int_0^\infty \frac{1}{3} e^{-x/3} \cdot e^{tx}$$
$$= \frac{1}{3}\int_0^\infty e^{-(\frac{1}{3}-t)x}\, dx$$
$$= \frac{1}{3} \cdot \frac{-1}{\frac{1}{3}-t} \cdot e^{-(\frac{1}{3}-t)x} \Big]_0^\infty$$

$$M_X(t) = \frac{-1}{3(\frac{1}{3}-t)}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{3(\frac{1}{3}-t)} \quad \text{for } t > \frac{1}{3}$$

$$= \frac{\frac{1}{3}}{(\frac{1}{3}-t)}$$

Q30: Let a r.v. has the p.d.f. Show that: $\mu_x = \frac{1}{3}$ & $\sigma_x^2 = \frac{2}{9}$.

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

$$\mu_x = E(X) = \int_{-1}^{1} x \cdot f(x)\,dx$$

$$= \int_{-1}^{1} \frac{1}{2} x(x+1)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1} = \left(\frac{x^3}{6} + \frac{x^2}{4}\right)_{-1}^{1}$$

$$= \left[\left(\frac{1}{6} + \cancel{\frac{1}{4}}\right) - \left(\frac{-1}{6} + \cancel{\frac{1}{4}}\right)\right]$$

$$= \frac{2}{6} = \frac{1}{3}$$

$$E(X^2) = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1} = \left[\frac{x^4}{8} + \frac{x^3}{6}\right]_{-1}^{1}$$

$$= \left[\left(\frac{1}{8} + \frac{1}{6}\right) - \left(\frac{1}{8} + \frac{-1}{6}\right)\right] = \frac{2}{6} = \frac{1}{3}$$

$$Var(x) = \frac{1}{3} - \frac{1}{9} = \frac{3}{9} - \frac{1}{9} = \frac{2}{9}$$

Q31: Let a r.v. X has the p.m.f.

$$f(x) = \begin{cases} \frac{1}{8} & \text{for } x = 0, 3 \\ \frac{3}{8} & \text{for } x = 1, 2 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

$R_x = \{0, 1, 2, 3\}$, X is d.r.v.

$$E(X) = \sum_{x=0}^{3} x \cdot f(x) = 0 \cdot \cancel{f(0)} + 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3)$$
$$= 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8}$$
$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

$$E(X^2) = \sum_{x=0}^{3} x^2 \cdot f(x) = 0^2 \cdot f(0) + 1^2 \cdot f(1) + 2^2 \cdot f(2) + 3^2 \cdot f(3)$$
$$= 1 \cdot \frac{3}{8} + 4 \cdot \frac{3}{8} + 9 \cdot \frac{1}{8}$$
$$= \frac{24}{8}$$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{24}{8} - \left[\frac{3}{2}\right]^2$$
$$= \frac{12}{4} - \frac{9}{4}$$
$$= \frac{3}{4} \geq 0 .$$

Q32: Consider the following C.d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2}(x^3+1) & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find (i) $P(-1 < X < 1)$ (ii) p.d.f. $f(x)$ (iii) $E(X)$, $V(X)$

Sol. F is conts. at $x = 1$ & $x = -1$

(i) $P(-1 < X < 1) = P(-1 < X \leq 1) = F(1) - F(-1)$ (since F is conts. at $x=-1$)
$$= 1 - 0$$
$$= 1$$

OR) $$f(x) = F'(x) = \begin{cases} \frac{3x^2}{2} & \text{for } -1 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$P(-1 < X < 1) = \int_{-1}^{1} \frac{3x^2}{2} \, dx = \frac{3}{2} \frac{x^3}{3} \Big]_{-1}^{1}$$
$$= \frac{1}{2}[1^3 - (-1)^3] = \frac{1}{2}[1+1] = 1$$

$E(X^2) = M_X''(0)$

$M_X''(t) = \frac{1}{4}\left[4e^{2t} + 16e^{4t} + 64e^{8t} + (16)^2 e^{16t}\right]$

$E(X^2) = M_X''(0) = \frac{1}{4}\left[4 + 16 + 64 + 256\right] = \frac{340}{4}$

$Var(X) = \frac{340}{4} - \frac{225}{4} = \frac{115}{4}$

Q34: Team (A) has Probability $(\frac{2}{3})$ of wining whenever it plays. If (A) plays (4) games; Find the probability function that (A) wins.
(1) Exactly 2-games. (2) find $E(X)$ & $V(X)$.
(3) at least 2-games.

Sol.

$R_X = \{0, 1, 2, 3, 4\}$

X يمثل عدد مرات الفوز

$p = \frac{2}{3}$ ← فوز A | عدم فوز A^c → $q = P(A^c) = \frac{1}{3}$

$n = 4$

$X \sim b(4, \frac{2}{3})$, $n = 4$ (اللعبة تكررت اربع مرات)

$$f(x) = P(X = x) = \begin{cases} C_x^n \; p^x (1-p)^{n-x} & \text{for } x = 0, 1, 2, \dots, n \\ 0 & \text{o.w.} \end{cases}$$

$$f(x; 4, \tfrac{2}{3}) = \begin{cases} C_x^4 (\frac{2}{3})^x (\frac{1}{3})^{4-x} & \text{for } x = 0, 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases}$$

$P(X = 2) = f(2) = C_2^4 (\frac{2}{3})^2 (\frac{1}{3})^2$
$= (6) \cdot (\frac{4}{9}) \cdot (\frac{1}{9}) = \frac{24}{81} \in [0, 1]$

$E(X) = np = 4(\frac{2}{3}) = \frac{8}{3}$

$V(X) = np(1-p) = 4(\frac{2}{3})(\frac{1}{3}) = \frac{8}{9}$

$P(\text{win at least 2-games}) = P(X \geq 2) = \sum_{x=2}^{4} f(x) = f(2) + f(3) + f(4)$

complete ---

Q35: Three coins tossed once. Let X be the random variable of heads appears. Find:

(1) pr.m.f of X

(2) pr. that we have three heads.

(3) pr. that we have no. heads.

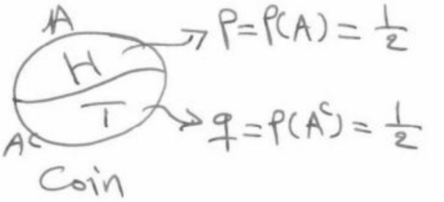
(4) $E(X)$ و $V(X)$.

Sol.

$X \equiv$ no. of head

$R_X = \{0, 1, 2, 3\}$

تجربة العملة تكررت (٣ مرات) $n = 3$



$\therefore X \sim b(3, \frac{1}{2})$

$$(i)\ f(x) = P(X = x) = \begin{cases} C^3_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

$$(ii)\ P(X = 3) = f(3) = C^3_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0 = 1 \cdot \frac{1}{8} \cdot 1 = 1/8$$

$$(iii)\ P(X = 0) = f(0) = C^3_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^3 = 1 \cdot 1 \cdot \left(\frac{1}{8}\right) = 1/8$$

$$(iiii)\ E(X) = n \cdot p = 3 \cdot \frac{1}{2} = 1.5$$

$$V(X) = np(1-p) = 3 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

عباس نجم سلمان
استاذ [redacted]

Ch.3 – Part 11 : X is a r.v. & $a, b \in R$

Theorem

T.P. ① $E(b) = b$ (ثابت) & ② $E(aX) = aE(X)$ (ثابت)

Proof

Case 1 If X is d.r.v. with p.m.f. $f(x)$

by def. of $E(X)$ $\left[E(X) = \sum_{\forall x} x f(x)\right]$

① $E(b) = \sum_{\forall x} b \cdot f(x)$

$= b \underbrace{\sum_{\forall x} f(x)}_{=1 \text{ pmf لأنه}}$

$= b$

② $E(aX) = \sum_{\forall x} (ax) f(x)$

$= a \cdot \underbrace{\sum_{\forall x} x \cdot f(x)}_{E(x)}$

$= a \cdot E(X)$

Case 2 If X is c.r.v. with p.d.f.

by def. of $E(X)$ $\left[E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx\right]$

① $E(b) = \int_{-\infty}^{\infty} b \cdot f(x)\,dx$

$= b \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1 \text{ لأنه p.d.f}}$

$= b$

② $E(aX) = \int_{-\infty}^{\infty} (ax) f(x)\,dx$

$= a \underbrace{\int_{-\infty}^{\infty} x \cdot f(x)\,dx}_{E(x)}$

$= a \cdot E(X)$

ex. $X \sim$ uniform $\{0,1\}$

Find $E(X)$, $E(2X)$ and $E(2)$

Sol. $f(x) = \begin{cases} \frac{1}{2} & \text{for } x = 0,1 \\ 0 & \text{o.w} \end{cases}$

$E(X) = \sum_{x=0}^{1} x f(x) = \sum_{x=0}^{1} x \left(\frac{1}{2}\right)$

or $= [0 \cdot \underbrace{f(0)}_{=1/2} + 1 \cdot \underbrace{f(1)}_{=1/2}]$

$= 1/2$

$\Rightarrow E(2X) = 2E(X) = 2 \cdot \frac{1}{2} = 1$

$\Rightarrow E(2) = 2$

لا نحتاج استخدام التوزيع النظرية تكفي بالوعد

ex $X \sim$ unif $(0,1)$

Find $E(X)$, $E(2X)$ and $E(2)$

Sol. $f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$= \int_0^1 x \cdot 1\,dx = \left.\frac{x^2}{2}\right]_0^1$

$= \left[\frac{1}{2} - 0\right] = 1/2$

$\rightarrow E(2X) = 2E(X) = 2 \cdot \frac{1}{2} = 1$

$\rightarrow E(2) = 2$

Theorem (1) If X is a r.v. with p.f. $f(x)$ such that $E(X)$ exist & $a, b \in R$, then $E(aX+b) = a\,E(X) + b$

proof **case 1** If X is c.r.v. with p.d.f $f(x)$

$$E(aX+b) = \int_{-\infty}^{\infty} [ax+b]\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} ax\, f(x)\, dx + \int_{-\infty}^{\infty} b\, f(x)\, dx$$

$$= a \underbrace{\int_{-\infty}^{\infty} x \cdot f(x)\, dx}_{\text{تعريف } E(X)} + b \underbrace{\int_{-\infty}^{\infty} f(x)\, dx}_{=1} \quad \text{p.d.f}$$

$$= a \cdot E(X) + b$$

proof **case 2** If X is d.r.v. with p.m.f $f(x)$

by the same way using sumation instade of integration

$$E(aX+b) = \sum_{\forall x} (ax+b)\, f(x)$$

$$= \sum_{\forall x} (ax)\, f(x) + \sum_{\forall x} b\, f(x)$$

$$= a \underbrace{\sum_{\forall x} x\, f(x)}_{E(X)} + b \underbrace{\sum_{\forall x} f(x)}_{=1}$$

$$= a\,E(X) + b$$

ex- If $X \sim unif(0,1)$ من خلال السابق

Find $E(2X+3)$

Sol- $f(x) = \begin{cases} 1 & , 0 < x < 1 \\ 0 & , o.w \end{cases}$

من خلال الخاصية ١ $\Longrightarrow$ We compute $E(X) = \frac{1}{2}$

$$E(2X+3) = 2 \cdot E(X) + 3$$
$$= 2 \cdot \frac{1}{2} + 3$$
$$= 4$$

ex.1 Toss a coin twice (H.W)

Let X be a number of head

(1) Find p.m.f. of X ($f(x)$) and sketch it's graph

(2) Find c.d.f. of X [$F(x)$] 〃 〃 〃

(3) Find $P(X>1)$, $P(X<2)$, $P(X=2)$ by two methods

(4) Find [Expected value of X] ; $E(X)$ if exists ??

[mean of X]

(5) Find $E(2X+3)$ & $E(2)$

س١

هل توجد حالات تكون فيها $E(X)$ غير موجودة؟ ادرسها

H-W Page(3) (3) توزيع هندسي

$$f(x) = \begin{cases} p q^{x-1} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{o.w} \end{cases}$$

$$E(X) = \sum_{x=1}^{\infty} x\, p q^{x-1} = p \sum_{x=1}^{\infty} x q^{x-1}$$

$$= p \cdot \frac{d}{dq} \sum_{x=1}^{\infty} q^{x}$$ ← Geometric series convergent when $|q|<1$ $\frac{1}{1-q}$

$$= p \cdot \frac{d}{dq} \frac{q}{1-q}$$

$$= p \cdot \frac{(1-q) - q(-1)}{(1-q)^2}$$ $1-q = p$

$$= p \cdot \frac{1 - \cancel{q} + \cancel{q}}{(1-q)^2}$$

$$= p \cdot \frac{1}{p^2} = \frac{1}{p}$$

The condition for E(X) exist is

$q < 1$ for series convergent.

ex Given a p.m.f. of X as below

$$f(x) = \begin{cases} \frac{x}{6} & \text{for } x = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

Let $y = 2x$

Find $E(y)$ using different methods

① $y = 2X \longrightarrow E(y) = 2E(x)$

$$E(X) = \sum_{x=1}^{3} (x)\left(\frac{x}{6}\right) = \frac{1}{6}\left[1^2 + 2^2 + 3^2\right] = \frac{14}{6} = \frac{7}{3}$$

$$\therefore E(y) = 2\left(\frac{7}{3}\right) = \frac{14}{3}$$

② by transformation

$$y = 2x \longrightarrow x = \frac{y}{2} = [u^{-1}(y)]$$

$$g(y) = f[u^{-1}(y)] = f(x) = \begin{cases} \frac{y/2}{6} = \frac{y}{12} & \text{for } y = 2,4,6 \\ 0 & \text{o.w.} \end{cases}$$

A: x : 1 2 3

B: $y = 2x$: 2 4 6

$$E(y) = \sum_{y=2}^{6} y \cdot \frac{y}{12} = \frac{1}{12}\sum_{y=2}^{6} y^2 = \frac{1}{12}[4 + 16 + 36]$$

$$= \frac{56}{12} = \frac{14}{3}$$

$$\begin{array}{r} 14 \\ 4 \\ \hline 56 \end{array}$$

③
$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{6} & \text{for } 1 \le x < 2 \\ \frac{3}{6} & \text{for } 2 \le x < 3 \\ 1 & \text{for } x \ge 3 \end{cases}$$

$$G(y) = P(Y \le y) = P(2X \le y) = P\left(X \le \frac{y}{2}\right) = F(y/2)$$

$G(2) = F(1) = 1/6$

$G(4) = F(2) = 3/6$

$G(6) = F(3) = 1$

$$G(y) = \begin{cases} 0 & y < 2 \\ 1/6 & 2 \le y < 4 \\ 3/6 & 4 \le y < 6 \\ 1 & y \ge 6 \end{cases}$$

① If $E(X)$ exists, then
$(E(X))^2 \leq (E(|X|))^2 \leq E(X^2)$

proof of $[|X| - E(|X|)]^2 \geq 0$ $\Rightarrow$ by th. 4 of $E(X)$

$0 \leq E[|X| - E(|X|)]^2 = E[\underbrace{(|X|^2)}_{X^2} - 2|X|E(|X|) + (E(|X|))^2]$

$= E(X^2) - 2E(|X|)E(|X|) + [E(|X|)]^2$

$= E(X^2) - [E(|X|)]^2$

$\therefore [E(|X|)]^2 \leq E(X^2)$ ——— ①

$\because |X| \geq X \longrightarrow |X| - X \geq 0$

$\therefore$ by th. 4 $E[|X| - X] \geq 0$

$\therefore E(|X|) - E(X) \geq 0$

$E(|X|) \geq E(X)$

$\Rightarrow [E(|X|)]^2 \geq [E(X)]^2$ ——— ②

$\Rightarrow [E(X)]^2 \leq [E(|X|)]^2$

(6)

$p(X=x) = f(x) = \frac{1}{2c}\frac{1}{x^2}$

التاريخ / / ٢٠١   الموضوع :

$p(X=-k)$

H-W $P(X=k) = \frac{1}{2c}\frac{1}{k^2}$ for $c = \sum_{n=1}^{\infty}\frac{1}{n^2}$

$f(k)$ $\qquad k = \mp 1, \mp 2, \ldots$

$$E(X) = \sum_{k=-\infty}^{-1} k\,P(X=-k) + \sum_{X=1}^{\infty} k\,P(X=k)$$

$$= \sum_{k=-\infty}^{-1} k\frac{1}{2c}\cdot\frac{1}{k^2} + \sum_{k=1}^{\infty} k\frac{1}{2c}\frac{1}{k^2}$$

$$= \frac{1}{2c}\sum_{k=-\infty}^{-1}\frac{1}{k} + \frac{1}{2c}\sum_{k=1}^{\infty}\frac{1}{k}$$

$$= \frac{1}{2c}\left[-\frac{1}{1}-\frac{1}{2}-\frac{1}{3}-\cdots\right] + \frac{1}{2c}\left[+\frac{1}{1}+\frac{1}{2}+\cdots\right]$$

$$= -\frac{1}{2c}\left[\frac{1}{1}+\frac{1}{2}+\cdots\right] + \frac{1}{2c}\left[\frac{1}{1}+\cdots\right]$$

$=$ Zero

or $\quad = \frac{-1}{2c}\sum_{k=+1}^{\infty}\frac{1}{k} + \frac{1}{2c}\sum_{k=1}^{\infty}\frac{1}{k}$

$= $ Zero $\qquad$ exist