

Chapter (4)

Probability Distribution of Two Random Variables

(I) Joint Probability Density Function (J.P.d.f.)

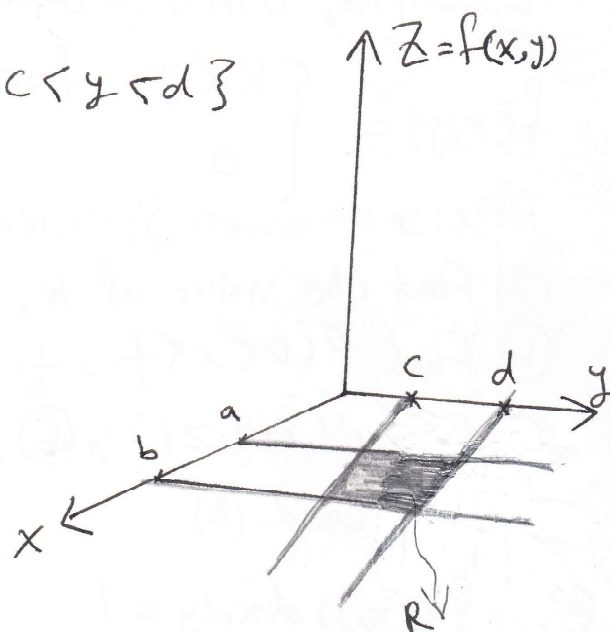
Def. Let X and Y be two c.r.v.'s. A function f defined over XY -plane is a joint p.d.f. of X and Y if for a subset $R \in XY$ -plane, then:

$$P[(x,y) \in R] = \iint_R f(x,y) dx dy$$

if $R = \{ (x,y) ; a \leq x \leq b \text{ \& } c \leq y \leq d \}$

Then

$$\begin{aligned} P[(x,y) \in R] &= \int_c^d \int_a^b f(x,y) dx dy \\ &= \int_a^b \int_c^d f(x,y) dy dx \end{aligned}$$



Note:

- ① $f(x,y)$ is a solid over XY -plane.
- ② R can be $\square, \square, \triangle, \bigcirc, \bigcirc, \dots$
- ③ $P[(x,y) \in R]$ is the volume inside solid of $f(x,y)$.

Properties of Joint p.d.f.

① A joint p.d.f. $f(x,y)$ satisfies two conditions:

① $f(x,y) \geq 0 \quad \forall (x,y) \in R$

② $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$

② Also :

in one dim.
 $f(x)$ is a curve
(a,b): $\overbrace{a \quad b}$
 $P(a \leq x \leq b) = \int_a^b f(x) dx$
Area under the curve

(one dim. probability)
 $\uparrow$
 x

∴ $f(x, y)$ is a function of x and y , from $f(x, y)$, we can find a function of x alone $f_1(x)$ and a function of y alone $f_2(y)$.
s.t.

$f_1(x)$ is called ^{حالة حدية} Marginal p.d.f. of x

where $f_1(x) = \int_{\square} f(x, y) dy$; limit of integral follows limits of y

$f_2(y)$ is called Marginal p.d.f. of y

where $f_2(y) = \int_{\square} f(x, y) dx$; limit of integral follows limits of x

Example 10 Given a joint p.d.f. (J.p.d.f.) $f(x, y)$:

$$f(x, y) = \begin{cases} Kx^2y & \text{for } 0 < x < 2, 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Note: In this example $\iint_{\square} f(x, y) dx dy = \iint_{\square} f(x, y) dy dx$ (y محدود ولا x محدود)

(a) Find the value of K ,

(b) Find $P(0 < x < \frac{1}{3}, \frac{1}{2} < y < 1)$

(c) Find $P(x+y \geq 1)$, (d) Find $f_1(x)$ and $f_2(y)$

Sol. By Cond. (2)

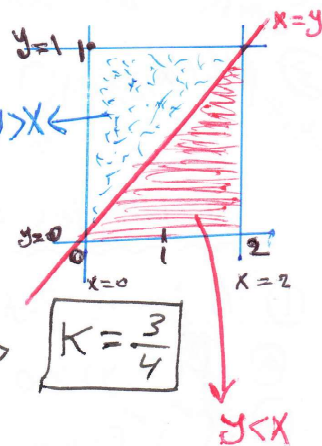
$$(a) \iint_{\square} f(x, y) dx dy = 1$$

$$R = \{(x, y) : 0 < x < 2, 0 < y < 1\}$$

$$\int_0^1 \int_0^2 Kx^2y dx dy = 1 \Rightarrow \int_0^1 Ky \left(\frac{x^3}{3} \Big|_0^2 \right) dy = 1$$

$$\frac{1}{3} \int_0^1 Ky (8-0) dy = 1 \Rightarrow \frac{8K}{3} \int_0^1 y dy = 1$$

$$\frac{8K}{3} \frac{y^2}{2} \Big|_0^1 = 1 \Rightarrow \frac{8K}{6} [1-0] = 1 \Rightarrow \frac{4}{3} K = 1 \Rightarrow K = \frac{3}{4}$$



(y=x) الخط المستقيم هذا لا يعطي المنطقة R
لذلك يجب ان تحدد الأجزاء من
 $y < x, x < y$

$$b) f(x,y) = \begin{cases} \frac{3}{4}x^2y & 0 < x < 2, 0 < y < 1 \\ 0 & \text{o.w} \end{cases}$$

$$P(0 < x < \frac{1}{3}) \frac{1}{2} < y < 2) = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{3}} \frac{3}{4}x^2y \, dx \, dy$$

(are the abscissa
x's are curv's)

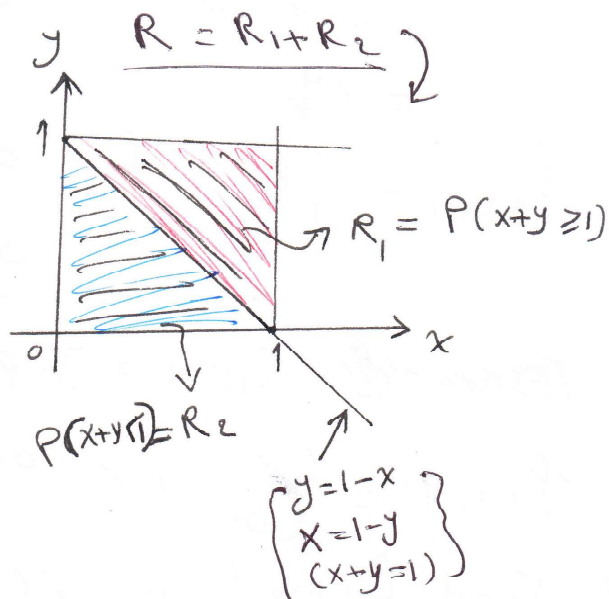
see the figure

$$c) P(x+y \geq 1) ?$$

$$\text{consider } (x+y)=1 \Rightarrow y=1-x$$

x	y=1-x
0	1
1	0

→



$$P(x+y \geq 1) = \text{Volum of } f(x,y) \text{ over } R_1$$

$$P(x+y < 1) = \text{Volum of } f(x,y) \text{ over } R_2$$

$$P(x+y \geq 1) = 1 - P(x+y < 1)$$

$$P(x+y < 1) = \int_0^1 \int_0^{1-x} \frac{3}{4}x^2y \, dy \, dx$$

$$= \int_0^1 \left[\int_0^{1-x} \frac{3}{4}x^2y \, dy \right] dx = \int_0^1 \frac{3}{4}x^2 \frac{y^2}{2} \Big|_0^{1-x} dx$$

$$= \int_0^1 \frac{3}{8}x^2[(1-x)^2 - 0] dx$$

$$= \frac{3}{8} \int_0^1 x^2[1-2x+x^2] dx$$

$$= \frac{3}{8} \left[\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5} \right] \Big|_0^1$$

$$= \frac{3}{8} \left[\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) - (0-0+0) \right] = \frac{3}{8} \left[\frac{10-15+6}{30} \right]$$

$$= \frac{3}{8} \left(\frac{1}{30} \right) = \frac{1}{80}$$

$$P(x+y < 1) = \frac{1}{80} \Rightarrow P(x+y \geq 1) = 1 - \frac{1}{80} = \frac{79}{80}$$

d) To find $f_1(x)$ & $f_2(y)$

$$f_1(x) = \int_0^1 f(x,y) \, dy$$

$$f_1(x) = \int_0^1 \frac{3}{4} x^2 y \, dy = \frac{3}{4} x^2 \frac{y^2}{2} \Big|_0^1 = \frac{3}{8} x^2 [1-0] = \frac{3}{8} x^2$$

$$f_1(x) = \begin{cases} \frac{3}{8} x^2 & \text{for } 0 \leq x \leq 2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_0^2 f(x,y) \, dx$$

$$= \int_0^2 \frac{3}{4} x^2 y \, dx = \frac{3}{4} y \frac{x^3}{3} \Big|_0^2 = \frac{1}{4} y [8-0] = 2y$$

$$f_2(y) = \begin{cases} 2y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases} \quad \left\{ \begin{array}{l} \text{exp.} \\ P(Y < \frac{1}{3}) = \int_0^{\frac{1}{3}} 2y \, dy \\ = y^2 \Big|_0^{\frac{1}{3}} \\ = \frac{1}{9} \end{array} \right\}$$

Example 2 Given a J.P.d.f.

$$f(x,y) = \begin{cases} 2 & \text{for } 0 \leq x \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $f_1(x)$, $f_2(y)$, $P(X > \frac{1}{2})$, $P(Y < \frac{1}{2})$ & $P(X < \frac{1}{3})$.

Sol.

$R = \{(x,y) : 0 \leq x \leq y \leq 1\}$, Consider $y = x$, then:

$$R = \begin{cases} 0 \leq x \leq y & ; x < y \leq 1 \\ 0 \leq x \leq 1 & ; 0 \leq y \leq 1 \end{cases}$$

$$f_1(x) = \int_x^1 2 \, dy \quad \text{for } 0 \leq x \leq 1$$

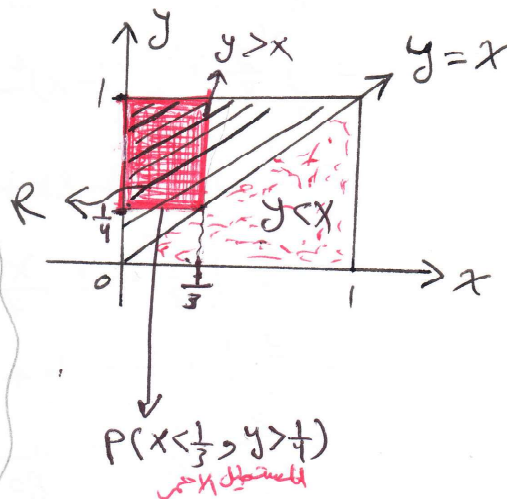
$$= 2y \Big|_x^1 = 2[1-x]$$

$$f_1(x) = \begin{cases} 2(1-x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_0^y 2 \, dx \quad \text{for } 0 \leq y \leq 1$$

$$= 2x \Big|_0^y = 2y$$

$$f_2(y) = \begin{cases} 2y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$



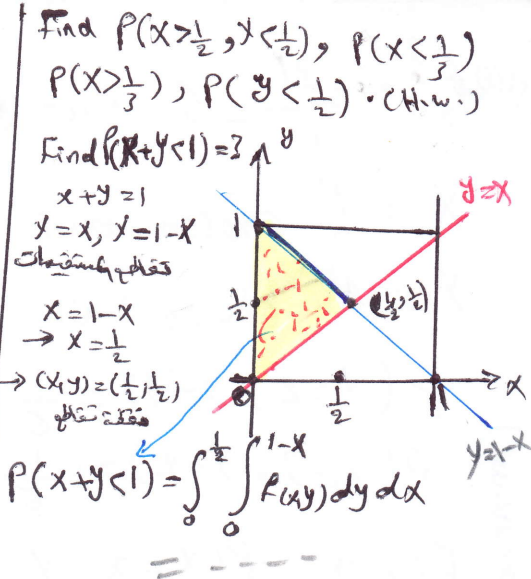
(H.w. Prove that f_1 & f_2 are pr. functions.)

$$P(X > \frac{1}{3}) = \int_{\frac{1}{3}}^1 f_1(x) dx$$

$$= \int_{\frac{1}{3}}^1 2(1-x) dx = \dots$$

$$P(Y < \frac{1}{2}) = \int_0^{\frac{1}{2}} 2y dy = \dots = \frac{1}{4}$$

$$P(X < \frac{1}{3}, Y > \frac{1}{4}) = \int_{\frac{1}{4}}^{\frac{1}{3}} \int_0^{\frac{1}{3}} 2 dx dy = \dots$$



II Joint Probability Mass Function (J.P.M.F.)

Def. Let X and Y be two d.v.v. A function $f(x,y)$ is a joint p.m.f. of x and y iff

$$f(x,y) = P(X=x, Y=y) \quad \forall (x,y) \in \mathbb{R}$$

$\rightarrow$ $\left(\begin{array}{l} X \text{ is d.v.v.} \\ f(x) \text{ is a p.m.f.} \\ f(x) = P(X=x) \end{array} \right)$

and satisfies two conditions:

- (a) $f(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}$
 (b) $\sum_y \sum_x f(x,y) = 1$

Also,

$$f_1(x) = \sum_y f(x,y), \text{ limit of summation follows limit of } y.$$

$$f_2(y) = \sum_x f(x,y), \text{ limit of summation follows limit of } x.$$

where $f_1(x)$ and $f_2(y)$ are called Marginal p.m.f. of x and y respectively.

Example A box has (3) balls (1), (2), (3). Choose two balls one by one without replac.

$\leftarrow$ i.e. (A box has (3) balls which are numbered 1, 2, 3.)

Sol.

$$\boxed{\begin{matrix} \textcircled{1} & \textcircled{2} \\ & \textcircled{3} \end{matrix}} \xrightarrow[\text{without repl.}]{(1-1)} \boxed{00}$$

$n=3 \qquad \qquad \qquad k=2$

$$S = \left\{ \begin{matrix} (1,2) & (2,1) & (3,1) \\ (1,3) & (2,3) & (3,2) \end{matrix} \right\}; S \text{ has } \underline{6} \text{ elts.}$$

$$n = \frac{3!}{1!} = 6 \text{ elts. (كم حالة لسحب 2 من 3 بدون ارجاع)}$$

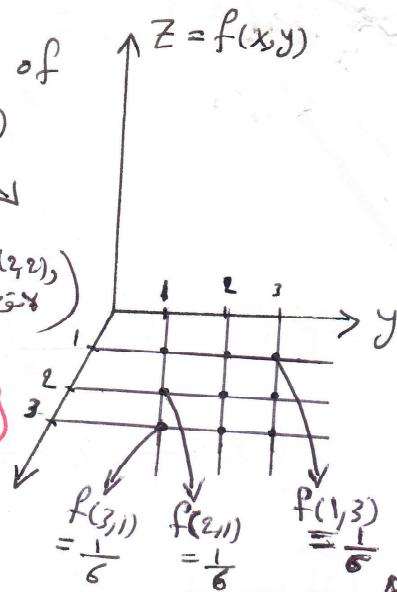
$X = 1^{\text{st}}$ chosen ball ; $x = 1, 2, 3$

$Y = 2^{\text{nd}}$ chosen ball ; $y = 1, 2, 3$

Graph of $f(x, y)$

(i.e. $f(1,1), f(2,2), f(3,3)$)
لا توجد
ربون ارجاع

H.W.
 $P(x, y)$
المسألة
with replacement
بإرجاع



$$P(X=2, Y=1) = \frac{1}{6} = f(2,1)$$

$$P(X=3, Y=2) = \frac{1}{6} = f(3,2)$$

$$f(x, y) = P(X=x, Y=y) = \frac{1}{6}$$

$$f(x, y) = \begin{cases} \frac{1}{6} & \text{for } x=1,2,3; y=1,2,3 \text{ and } x \neq y \\ 0 & \text{o.w.} \end{cases}$$

Find: $f_1(x), f_2(y), (H.W.) P(X=2), P(X>2), P(Y=1), P(X \geq 2), P(Y \leq 2)$

Example 2: Given a J.P.M.F. $f(x, y)$

$$f(x, y) = \begin{cases} \frac{1}{21}(x+y) & \text{for } x=1,2,3 \\ & y=1,2 \\ 0 & \text{o.w.} \end{cases}$$

Find $P(X \geq 2, Y \leq 2), f_1(x), f_2(y)$

$$\begin{aligned} \text{Sol. } P(X \geq 2, Y \leq 2) &= \sum_{y=1}^2 \sum_{x=2}^3 \frac{1}{21}(x+y) \\ &= \frac{1}{21} \sum_{y=1}^2 [(2+y) + (3+y)] \\ &= \frac{1}{21} \sum_{y=1}^2 [2y+5] = \frac{1}{21} [(2(1)+5) + 2(2)+5] \\ &= \frac{1}{21} [7+9] = \frac{16}{21} \end{aligned}$$

$$P(X=3, Y=1) = f(3,1) = \frac{1}{21}(3+1) = \frac{4}{21}$$

$$f_1(x) = \sum_y f(x, y)$$

$$\begin{aligned} f_1(x) &= \sum_{y=1}^2 \frac{1}{21}(x+y) = \frac{1}{21} [(x+1) + (x+2)] \\ &= \frac{1}{21} [3+2x] \end{aligned}$$

$$f_1(x) = \begin{cases} \frac{1}{21} [3+2x] & \text{for } x=1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} P(X \geq 2) &= \sum_{x=2}^3 f_1(x) \\ &= f_1(2) + f_1(3) \\ &= \frac{3}{6} + \frac{3}{6} \\ &= \frac{6}{6} = 1 \end{aligned}$$

$$\text{with rep.} \rightarrow n(5) = 2 \times 2 = 4$$

$$f(x, y) = \begin{cases} \frac{1}{9} & \text{for } x=1,2, y=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Find } P(X \geq 2, Y \leq 2) = \sum_{x=2}^2 \sum_{y=1}^2 f(x, y)$$

$$f_2(y) = \sum_x f(x,y)$$

$$= \sum_{x=1}^3 \frac{1}{2!} (x+y)$$

$$= \frac{1}{2!} [(1+y) + (2+y) + (3+y)]$$

$$= \frac{1}{2!} (6+3y) = \frac{3}{2!} (y+2) = \frac{1}{2} (y+2)$$

$$f_2(y) = \begin{cases} \frac{1}{2} (y+2) & \text{for } y=1,2 \\ 0 & \text{o.w.} \end{cases}$$

Example Given a J.P.d.f.

$$f(x,y) = \begin{cases} cx^2y & \text{for } x^2 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the value of c (b) Find $P(X \leq y)$

(نجد أولاً صيغة x و y)

Sol.

$$R = \{(x,y) : 0 \leq x^2 \leq y \leq 1\} = \begin{cases} 0 \leq x^2 \leq y, x^2 \leq y \leq 1 \\ 0 \leq x^2 \leq 1, 0 \leq y \leq 1 \end{cases}$$

$$\text{suppose } y = x^2 \Rightarrow x = \pm \sqrt{y}$$

$$|x| = \sqrt{x^2}$$

$$0 \leq x^2 \leq y$$

$$|x| \leq \sqrt{y}$$

$$-\sqrt{y} \leq x \leq \sqrt{y}$$

$$x^2 \leq 1$$

$$|x| \leq 1$$

$$-1 \leq x \leq 1$$

$$0 \leq x^2 \leq y \leq 1$$

$$0 \leq y \leq 1$$

$$y = x^2$$

$$x = \pm \sqrt{y}$$

$$R = \begin{cases} -\sqrt{y} \leq x \leq \sqrt{y} ; x^2 \leq y \leq 1 \\ -1 \leq x \leq 1 ; 0 \leq y \leq 1 \end{cases}$$

x	$y = x^2$
0	0
$\pm \frac{1}{2}$	$\frac{1}{4}$
± 1	1

Consider $y = x^2$

$y = x^2, y = x$: إيجاد نقاط التقاطع

$$x^2 = x$$

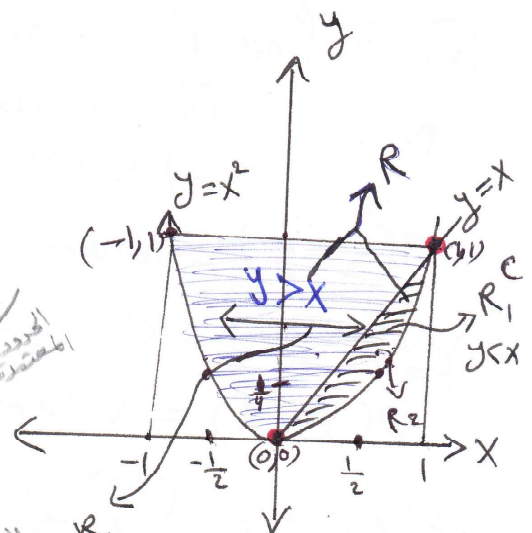
$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0, x = 1$$

$$\rightarrow (0,0), (1,1)$$

(7)



$R_1 (x \leq y)$

$$R = x^2 \leq y \leq 1$$

$$0 \leq x^2 \leq y \leq 1$$

$$(R = R_1 + R_2 = R_1 + R_1^c)$$

(a) By Cond. (2)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx = 1$$

$$\int_{-1}^1 \int_{x^2}^1 c x^2 y dy dx = 1$$

$$1 = \frac{c}{2} \int_{-1}^1 x^2 y^2 \Big|_{x^2}^1 dx = \frac{c}{2} \int_{-1}^1 x^2 (1 - x^4) dx$$

$$1 = \frac{c}{2} \int_{-1}^1 [x^2 - x^6] dx = \frac{c}{2} \left[\frac{x^3}{3} - \frac{x^7}{7} \right]_{-1}^1$$

$$1 = \frac{c}{2} \left[\left(\frac{1}{3} - \frac{1}{7} \right) - \left(-\frac{1}{3} - \frac{-1}{7} \right) \right]$$

$$1 = \frac{c}{2} \left[\left(\frac{1}{3} - \frac{1}{7} \right) + \left(\frac{1}{3} - \frac{1}{7} \right) \right]$$

$$1 = \frac{c}{2} \left(\frac{2}{3} - \frac{2}{7} \right) = c \left(\frac{1}{3} - \frac{1}{7} \right) = c \left(\frac{7-3}{21} \right) = \frac{4}{21} c$$

$$\Rightarrow \boxed{c = \frac{21}{4}}$$

$$\therefore f(x,y) = \begin{cases} \frac{21}{4} x^2 y & \text{for } x^2 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

(b) To find $P(X \leq Y)$:

$$R = \left\{ \begin{array}{l} -\sqrt{y} \leq x \leq \sqrt{y}, \quad x^2 \leq y \leq 1 \\ -1 \leq x \leq 1, \quad 0 \leq y \leq 1 \end{array} \right\}$$

$P(X \leq Y)$ = volum of $f(x,y)$ over R ,

but $P(Y \geq X) = 1 - P(Y < X)$

$P(Y < X)$ = volum of $f(x,y)$ over R^c

$$P(Y < X) = \int_0^1 \int_{x^2}^x \frac{21}{4} x^2 y dy dx$$

$$= \frac{21}{8} \int_0^1 x^2 y^2 \Big|_{x^2}^x dx$$

$$= \frac{21}{8} \int_0^1 x^2 [x^2 - x^4] dx = \frac{21}{8} \int_0^1 [x^4 - x^6] dx$$

$$= \frac{21}{8} \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1$$

$$= \frac{21}{8} \left[\left(\frac{1}{5} - \frac{1}{7} \right) - (0-0) \right]$$

$$= \frac{21}{8} \left[\frac{7-5}{35} \right] = \frac{21}{8} \cdot \frac{2}{35} = \frac{3}{20}$$

$$\begin{aligned} \therefore P(X \leq Y) &= 1 - P(X > Y) \simeq \int_0^1 (x^2 - x) dx \quad (\text{المساحة بين المنحنين}) \\ &= 1 - P(Y < X) \\ &= 1 - \frac{3}{20} = \frac{17}{20} \end{aligned}$$

(c) To find $f_1(x)$ and $f_2(y)$

$$R = \left\{ \begin{array}{l} -\sqrt{y} \leq x \leq \sqrt{y}, \quad x^2 \leq y \leq 1 \\ -1 \leq x \leq 1, \quad 0 \leq y \leq 1 \end{array} \right\}$$

$$f_1(x) = \int_{x^2}^1 \frac{21}{4} x^2 y dy \quad \text{for } -1 \leq x \leq 1$$

$$= \frac{21}{4} x^2 \frac{y^2}{2} \Big|_{x^2}^1 = \frac{21}{8} x^2 (1 - x^4)$$

$$f_1(x) = \begin{cases} \frac{21}{8} x^2 (1 - x^4) & \text{for } -1 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4} x^2 y dx \quad \text{for } 0 \leq y \leq 1$$

$$= \frac{21}{4} y \frac{x^3}{3} \Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{7}{4} y \left[y^{\frac{3}{2}} + y^{\frac{3}{2}} \right] = \frac{7}{2} y^{\frac{5}{2}}$$

$$f_2(y) = \begin{cases} \frac{7}{2} y^{\frac{5}{2}} & 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

H.w. Find $P(X+Y < 1)$.

Stochastic Independence

استقلال الاحتمال

Def: X and Y are stochastically independence iff:

$$f_1(x) f_2(y) = f(x, y) \quad \forall (x, y)$$

Denoted by (s. indep.)

Example 6: Given a J.P.m.f

$x \backslash y$	1	2	3	4	$f_1(x)$
1	$f(1,1) = .1$	$f(1,2) = 0$	$f(1,3) = .1$	$f(1,4) = 0$	$f_1(1) = .1 + .1 = .2$
2	$f(2,1) = .3$	$f(2,2) = 0$	$f(2,3) = .1$	$f(2,4) = .2$	$f_1(2) = .3 + .1 + .2 = .6$
3	$f(3,1) = 0$	$f(3,2) = .2$	$f(3,3) = 0$	$f(3,4) = 0$	$f_1(3) = .2$
$f_2(y)$	$f_2(1) = .4$	$f_2(2) = .2$	$f_2(3) = .2$	$f_2(4) = .2$	$\sum_{x=1}^3 f_1(x) = \sum_{y=1}^4 f_2(y) = 1$

$f(x, y) = P(X=x, Y=y)$
 a. T.P. $f_1(x) \cdot f_2(y) = f(x, y) \quad \forall (x, y)$

Note
 (Prove that) $\sum_{\forall x} f_1(x) = 1 \rightarrow f_1(x)$ is p.m.f.
 $\sum_{\forall y} f_2(y) = 1 \rightarrow f_2(y)$ is p.m.f.

$$f_1(1) \cdot f_2(1) = (.2)(.4) = .08$$

$$f(1,1) = .1$$

$$\therefore f_1(1) f_2(1) \neq f(1,1) \text{ \& } f_1(2) f_2(2) \neq f(2,2)$$

$\therefore X$ and Y are not s. indep.

b. Find $P(X \geq 2, Y \leq 3)$

$$\begin{aligned}
 P(X \geq 2, Y \leq 3) &= \sum_{y=1}^3 \sum_{x=2}^3 f(x, y) \\
 &= f(2, 1) + f(2, 2) + f(2, 3) + f(3, 1) + f(3, 2) + f(3, 3) \\
 &= .03 + 0 + .1 + 0 + .2 + 0 \\
 &= .6
 \end{aligned}$$

$$\begin{aligned}
 \text{c. Find } P(X \geq 2) &= \sum_{x=2}^3 f_1(x) \\
 &= f_1(2) + f_1(3) \\
 &= 0.6 + 0.2 = 0.8
 \end{aligned}$$

$$\begin{aligned}
 \text{d. } P(Y \leq 3) &= \sum_{y=1}^3 f_2(y) \\
 &= f_2(1) + f_2(2) + f_2(3) \\
 &= .4 + .2 + .2 = .8
 \end{aligned}$$

Uniform Distribution of two R.V.'s

Def. If a c.r.v. X and Y have a uniform distribution on a region R , then the joint p.d.f. of X and Y is:

$$f(x, y) = \frac{1}{\text{Area}(R)} \quad \text{for } (x, y) \in R$$

Example X and Y have a uniform dist. on a triangle Δ with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$. Find $f(x, y)$.

Sol.

R is a triangle Δ

$$y \geq x \quad x \geq 0$$

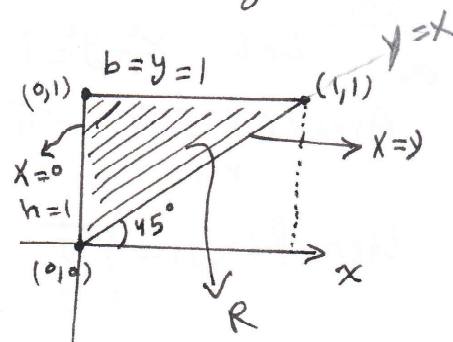
$$y \leq 1$$

$$\Rightarrow 0 \leq x \leq y \leq 1$$

$$R = \{(x, y); 0 \leq x \leq y \leq 1\}$$

$$\begin{aligned}
 \text{Area}(\Delta) &= \frac{1}{2} \cdot h \cdot b \\
 &= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}
 \end{aligned}$$

(11)



also
dependent

$$f(x,y) = \frac{1}{\text{Area}(R)} = \frac{1}{\frac{1}{2}} = \begin{cases} 2 & \text{for } 0 \leq x \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Example: A point (x,y) is chosen from a region R which is bounded by a curve $y=x^2$ and a line $y=x$, find $f(x,y)$.

Sol. Sketch $y=x$ and $y=x^2$

A point $(x,y) \in R$ must be above the curve $y=x^2$ and under the line $y=x$, then $x^2 \leq y \leq x$.

Also that point must be between $0 \leq x \leq 1$, $0 \leq y \leq 1$

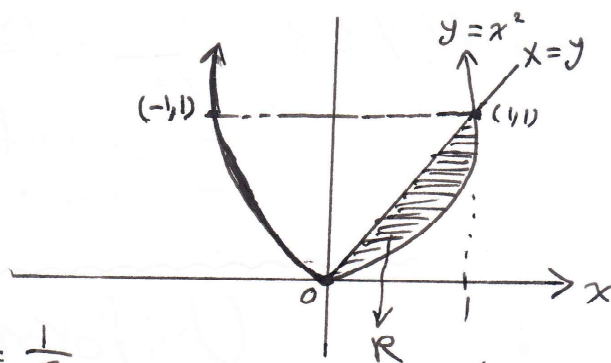
$$R = \{(x,y); 0 \leq x^2 \leq y \leq x \leq 1\}$$

$$\text{Area}(R) = \int_0^1 \int_{x^2}^x dy dx = \int_0^1 |x - x^2| dx$$

$$= \int_0^1 (x - x^2) dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$f(x,y) = \frac{1}{\frac{1}{6}} = \begin{cases} 6 & \text{for } 0 \leq x^2 \leq y \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

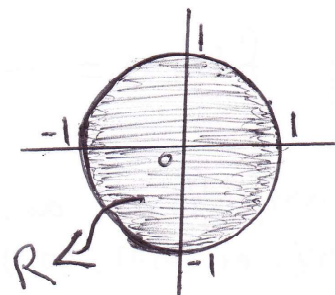


Example: A point (x,y) is chosen from a region R where

$$R = \{(x,y); x^2 + y^2 \leq 1\}$$

a. find $f(x,y)$

b. find $f_1(x)$ & $f_2(y)$



Sol. a. Let $x^2 + y^2 = 1 \Rightarrow r=1$

$$\text{Area}(R) = \int_{r=1} \pi r^2 = \pi \quad (\text{دائرة نصف قطرها } r)$$

$$\text{then } f(x,y) = \begin{cases} \frac{1}{\pi} & \text{for } x^2 + y^2 \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

∴ We must find $f_1(x)$ & $f_2(y)$:

$$f_1(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \quad \text{for } -1 \leq x \leq 1$$

$$= \frac{1}{\pi} y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \frac{1}{\pi} [\sqrt{1-x^2} + \sqrt{1-x^2}] = \begin{cases} \frac{2}{\pi} \sqrt{1-x^2} & \text{for } -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx \quad \text{for } -1 \leq y \leq 1$$

$$= \frac{1}{\pi} x \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = \begin{cases} \frac{2}{\pi} \sqrt{1-y^2} & \text{for } -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \\ 0 & \text{o.w.} \end{cases}$$

$$f_1(x) \cdot f_2(y) = \frac{2}{\pi} \cdot \frac{2}{\pi} \sqrt{(1-y^2)(1-x^2)} = \frac{4}{\pi^2} \sqrt{(1-y^2)(1-x^2)}$$

$$f(x,y) \neq f_1(x) f_2(y) \quad (\text{since } f(x,y) = \frac{1}{\pi})$$

Conditional Function and Conditional Probability

Def. Let X and Y be two r.v.'s $f(y|x)$ denoted the conditional p.d.f or p.m.f. of y given $X=x$.

$f(x|y)$ denoted the conditional p.d.f. or p.m.f. of x given $y=y$.

where

$$f(y|x) = \frac{f(x,y)}{f_1(x)} \quad ; \quad f_1(x) \neq 0$$

$$f(x|y) = \frac{f(x,y)}{f_2(y)} \quad ; \quad f_2(y) \neq 0$$

$$\Leftrightarrow \left(\begin{array}{l} \text{مقارنة بالاحتمال} \\ \text{الشروطي} \\ P(A|B) = \frac{P(A \cap B)}{P(B)} \end{array} \right)$$

Note:

$$P(a < x < b | y=c) = \int_a^b f(x|y=c) dx$$

$$= \sum_{x=a}^b f(x|y=c)$$

$$P(c < y < d | x = a) = \int_c^d f(y|x=a) dy$$

$$= \sum_{y=c}^d f(y|x=a)$$

Example 8 Given a J.P.d.f.

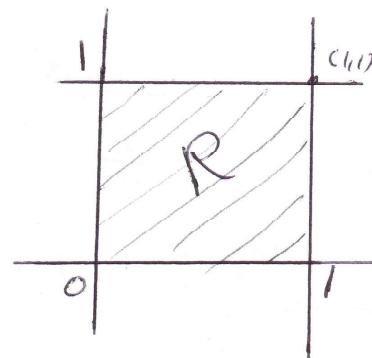
$$f(x,y) = \begin{cases} 8xy & \text{for } 0 < x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

(1) find $f(x|y)$ & $f(y|x)$

(2) find $P(x > \frac{1}{4} | y = \frac{1}{2})$, $P(y < \frac{1}{4} | x = \frac{1}{2})$.

Sol. $R = \{(x,y) ; 0 < x < y < 1\}$

$$R = \begin{cases} 0 < x < y & x < y < 1 \\ c < x < 1 & 0 < y < 1 \end{cases}$$



(1) we must find $f_1(x)$ and $f_2(y)$

$$f_1(x) = \int_x^1 8xy dy = 8x \left[\frac{y^2}{2} \right]_x^1 = 4x(1-x^2)$$

$$= 4x(1-x^2) \text{ for } 0 < x < 1$$

$x \neq 1$ (حتى لا يكون المقام صفرًا) $(1-x^2)$

$$f_2(y) = \int_0^y 8xy dx = \frac{8}{2} y x^2 = 4y^3 \text{ for } 0 < y < 1$$

$$f(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{8xy}{4y^3} = \begin{cases} \frac{2x}{y^2} & \text{for } \begin{cases} 0 < x < y \\ 0 < y < 1 \end{cases} \\ 0 & \text{o.w.} \end{cases}$$

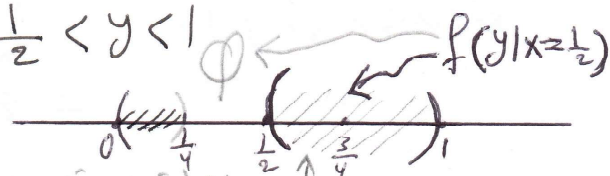
$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$$

∴ $f(y|x) = \begin{cases} \frac{2y}{1-x^2} & \text{for } x < y < 1 \text{ و } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

دالة مقسومة على y
(حتى لا تكون المقومة صفرًا)
بمجرد حذف x تكون المقومة

$$(2) P(y < \frac{1}{4} | x = \frac{1}{2}) = \int_0^{\frac{1}{4}} f(y|x=\frac{1}{2}) dy = \int_0^{\frac{1}{4}} 0 dy = 0$$

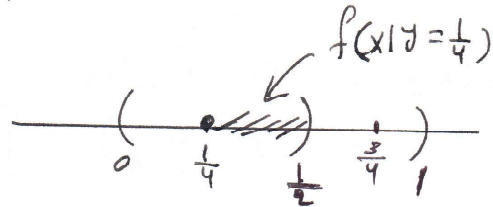
$$x < y < 1, x = \frac{1}{2} \Rightarrow \frac{1}{2} < y < 1$$



$$P(X > \frac{1}{4} | Y = \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x|y=\frac{1}{2}) dx$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{2x}{y} dx$$

← (f(x|y=1/2) = joint pdf / marginal pdf)



$$0 < x < y, y = \frac{1}{2} \Rightarrow 0 < x < \frac{1}{2}$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{2x}{(\frac{1}{2})^2} dx = 4 \cdot 2 \cdot \frac{x^2}{2} \Big|_{\frac{1}{4}}^{\frac{1}{2}}$$

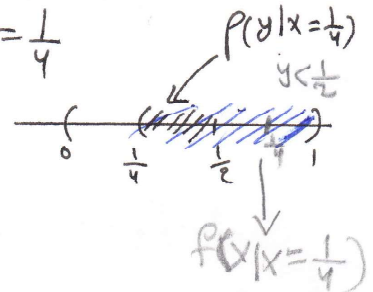
$$= 4 \left[\frac{1}{4} - \frac{1}{16} \right] = 4 \cdot \left(\frac{3}{16} \right) = \frac{3}{4}$$

If $P(X > \frac{1}{4} | Y = \frac{3}{4})$
 $0 < x < y, y = \frac{3}{4}$
 $\Rightarrow$
 $0 < x < \frac{3}{4}$
 $\Rightarrow \frac{1}{4} < x < \frac{3}{4}$
 (عوض)

$$P(Y < \frac{1}{2} | X = \frac{1}{4}) = ?$$

$$x < y < 1, x = \frac{1}{4}$$

$$\frac{1}{4} < y < 1$$



$$P(Y < \frac{1}{2} | X = \frac{1}{4}) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(y|x=\frac{1}{4}) dy$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{2y}{1-x^2} dy = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{2y}{1-(\frac{1}{4})^2} dy$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{2y}{15/16} dy = \frac{16}{15} \cdot 2 \cdot \frac{y^2}{2} \Big|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{16}{15} \left(\frac{1}{4} - \frac{1}{16} \right) = \frac{16}{15} \cdot \frac{3}{16} = \frac{1}{5}$$

Example: If $X \sim \text{unif}(0,1)$ and
 $f(y|x) = \begin{cases} \frac{1}{1-x} & \text{for } x < y < 1, 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

(a) find $f(x,y)$?

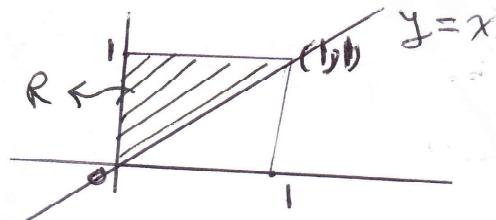
$$\because X \sim \text{unif}(0,1)$$

$$f_1(x) = \frac{1}{1-0} = 1 \Rightarrow f_1(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$f(x,y) = f_1(x) \cdot f(y|x)$$

$$= (1) \left(\frac{1}{1-x} \right) = \begin{cases} \frac{1}{1-x} & \text{for } x < y < 1, 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$



b. Find $P(x > \frac{1}{2} | y = \frac{3}{4}) = \int_0^y f(x|y = \frac{3}{4}) dx$

$$f(x|y) = \frac{f(x,y)}{f_2(y)}$$

$$f_2(y) = \int_0^y f(x,y) dx = \int_0^y \frac{1}{1-x} dx = -\ln(1-x) \Big|_0^y$$

$$= -[\ln(1-y) - \ln(1-0)]$$

$$= -\ln(1-y) \text{ for } 0 < y < 1$$

$$\therefore f_2(y) = -\ln(1-y) \text{ for } 0 < y < 1$$

$$\therefore f(x|y) = \frac{\frac{1}{1-x}}{-\ln(1-y)}$$

$$f(x|y) = \begin{cases} \frac{1}{-\ln(1-y)} \cdot \frac{1}{1-x} & \text{for } 0 < x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$P(x > \frac{1}{2} | y = \frac{3}{4}) = \int_{\frac{1}{2}}^y f(x|y = \frac{3}{4}) dx$$

$$0 < x < y, y = \frac{3}{4}$$

$$\Rightarrow 0 < x < \frac{3}{4}$$

بشكل عام

$x > \frac{1}{2}$ شرط

$$\Rightarrow \frac{1}{2} < x < \frac{3}{4}$$



$$= \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{-1}{\ln(1-y)} \cdot \frac{1}{1-x} dx = \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{-1}{\ln(1-\frac{3}{4})} \cdot \left(\frac{1}{1-x} \right) dx$$

$$= \frac{-1}{\ln 1 - \ln 4} \cdot [-\ln(1-x)]_{\frac{1}{2}}^{\frac{3}{4}}$$

$$= \frac{+1}{-\ln 4} [\ln(1-\frac{3}{4}) - \ln(1-\frac{1}{2})]$$

$$= \frac{-1}{\ln 4} [\ln(\frac{1}{4}) - \ln(\frac{1}{2})]$$

$$= \frac{-1}{\ln 2^2} [\underbrace{\ln 1}_{\text{Zero}} - \ln 4) - (\underbrace{\ln 1}_{\text{Zero}} - \ln 2)]$$

$$= \frac{-1}{2 \ln 2} (-2 \ln 2 + \ln 2) = \frac{-1}{2 \ln 2} (-\ln 2) = \frac{1}{2}$$

H.w Find $P(Y < \frac{3}{4} | X = \frac{1}{2})$

التوقع والبيان Expectation of two Random Variables

Def. $E(xy)$ is the expectation or the mean of both x and y such that:

$$E(xy) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) dx dy \quad \text{if } x \text{ and } y \text{ are c.r.v.}$$

$$= \sum_x \sum_y xy f(x,y) \quad \text{if } x \text{ and } y \text{ are d.r.v.}$$

Note: ① $E(x) \equiv \text{mean of } x = \mu_x$

s.t. $E(x) = \int_{-\infty}^{\infty} x f_1(x) dx \quad \text{if } x \text{ is c.r.v.}$
 $= \sum_x x f_1(x) \quad \text{if } x \text{ is d.r.v.}$

② $E(y) \equiv \text{mean of } y = \mu_y$

s.t. $E(y) = \int_{-\infty}^{\infty} y f_2(y) dy \quad \text{if } y \text{ is c.r.v.}$
 $= \sum_y y f_2(y) \quad \text{if } y \text{ is d.r.v.}$

③ (a) $V(x) = E(x^2) - [E(x)]^2$

(b) $V(y) = E(y^2) - [E(y)]^2$

④ Covariance of both x & y

We use the symbol $\text{Cov}(x,y)$ (it's mean covariance), s.t.
 $\text{Cov}(x,y) = E\{[x - E(x)][y - E(y)]\}.$

Note: $\text{Cov}(x,y) \geq 0$ or $\text{Cov}(x,y) < 0$

Theorem (1) If X and Y are Random Variables, then:

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

Proof $\text{Cov}(X, Y) = E\{[X - E(X)][Y - E(Y)]\}$

$$\begin{aligned} &= E\{XY - YE(X) - XE(Y) + E(X) \cdot E(Y)\} \\ &= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

Theorem (2) If X and Y are independent, then

$$E(XY) = E(X)E(Y) \quad (\text{or } \text{Cov}(X, Y) = 0)$$

Proof Case ① If X and Y are c.r.v.

$$\because X \text{ \& } Y \text{ are indep.} \Rightarrow f_1(x)f_2(y) = f(x, y)$$

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy) f(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_1(x) f_2(y) dx dy \\ &= \int_{-\infty}^{\infty} x f_1(x) dx \cdot \int_{-\infty}^{\infty} y f_2(y) dy \\ &= E(X) \cdot E(Y) \end{aligned}$$

Case ② By the same method.

Note By Th. 2 above, if X & Y are indep., then

$$\text{Cov}(X, Y) = 0$$

Example ② Given a J. p. d. f.

$$f(x, y) = \begin{cases} 2 & \text{for } 0 < y < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Q: Are X & Y independent?

جواب السؤال هذا لا

① If $\text{Cov}(X, Y) = 0$

② If $E(XY) = E(X)E(Y)$

③ If X & Y are indep. r.v.

i.e. $f(x, y) = f_1(x)f_2(y)$

(Ans)

$$R = \left\{ \begin{array}{ll} 0 \leq y \leq x & , \quad y \leq x \leq 1 \\ 0 \leq y \leq 1 & , \quad 0 \leq x \leq 1 \end{array} \right\}$$

$$\begin{aligned} E(xy) &= \int_0^1 \int_y^1 xy \cdot f(x,y) dx dy = \int_0^1 \int_0^x xy f(x,y) dy dx \\ &= \int_0^1 \int_y^1 xy(2) dx dy \\ &= \int_0^1 2y \left. \frac{x^2}{2} \right|_y^1 dy = \int_0^1 y(1-y^2) dy \\ &= \int_0^1 [y - y^3] dy = \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 \\ &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \end{aligned}$$

$$E(x) = \int_0^1 x f_1(x) dx, \quad E(y) = \int_0^1 y f_2(y) dy$$

$$f_1(x) = \int_0^x 2 dy = \int_0^x 2 dy = 2y \Big|_0^x = 2x \Rightarrow f_1(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_y^1 f(x,y) dx = \int_y^1 2 dx = 2x \Big|_y^1 = 2[1-y] \Rightarrow f_2(y) = \begin{cases} 2(1-y) & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(x) = \int_0^1 x f_1(x) dx = \int_0^1 x(2x) dx = \int_0^1 2x^2 dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$\begin{aligned} E(y) &= \int_0^1 y f_2(y) dy = \int_0^1 2y(1-y) dy = \int_0^1 (2y - 2y^2) dy = y^2 - \frac{2}{3} y^3 \Big|_0^1 \\ &= 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

$$\therefore \text{Cov}(X, y) = E(xy) - E(x)E(y)$$

$$= \frac{1}{4} - \left(\frac{2}{3}\right) \cdot \left(\frac{1}{3}\right) = \frac{9-8}{36} = \frac{1}{36}$$

$\text{Cov}(X, y) \neq 0 \Rightarrow X$ and y are dependent.

& $\text{Cov}(X, y) \geq 0$ all ways.

Correlation Coefficient

Def. when X and Y are dependent and we want to know how much X depends on Y we use the correlation to measure it, denoted by " $\rho_{x,y}$ " read (Rho) where:

$$\rho_{x,y} = \frac{\text{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$

i.e. $\text{Cov}(X,Y) \neq 0$ if X & Y are dependent
where $-1 \leq \rho \leq 1$

H.w. Find $\rho_{x,y}$ from above example.

Conditional Mean and Conditional Variance

$E(Y|X)$ denote to the conditional mean of Y given $X=x$.

$E(X|Y)$ denote to the conditional mean of X given $Y=y$.

where $E(Y|X) = \int_{\square} y f(y|x) dy$, limit of integral follows, limit of Y .

$E(X|Y) = \int_{\square} x f(x|y) dx$, limit of integral follows, limit of X .

Note Since $f(y|x)$ defined for interval of Y interm of X , then $E(Y|X)$ is a function of X . Also $E(X|Y)$ is a function of Y .

$V(Y|X)$ denote to the conditional variance of Y given $X=x$.

$V(X|Y)$ denote to the conditional variance of X given $Y=y$.

where

$$V(y|x) = E\{[y - E(y|x)]^2 | x\}$$

$$= \int [y - E(y|x)]^2 f(y|x) dy$$

$$V(x|y) = E\{[x - E(x|y)]^2 | y\}$$

$$= \int [x - E(x|y)]^2 f(x|y) dx$$

Example: Given a J.P.d.f $f(x, y)$

$$f(x, y) = \begin{cases} 2 & \text{for } 0 \leq x+y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(x|y), E(y|x), V(x|y), V(y|x)$

Sol.

$$R = \{(x, y); 0 \leq x+y \leq 1\} = \left\{ \begin{array}{l} 0 \leq x \leq 1-y, 0 \leq y \leq 1-x \\ 0 \leq x \leq 1, 0 \leq y \leq 1 \end{array} \right\}$$

$$E(x|y) = \int x f(x|y) dx$$

$$f(x|y) = \frac{f(x, y)}{f_2(y)}$$

$$f_2(y) = \int_0^{1-y} f(x, y) dx = \int_0^{1-y} 2 dx = 2x \Big|_0^{1-y} = 2(1-y)$$

$$f_2(y) = \begin{cases} 2(1-y) & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f(x|y) = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

$$f(x|y) = \begin{cases} \frac{1}{1-y} & \text{for } 0 \leq x \leq 1-y \\ 0 & \text{o.w.} \end{cases}$$

$$E(x|y) = \int_0^{1-y} x \cdot \frac{1}{1-y} dx = \frac{1}{1-y} \cdot \frac{x^2}{2} \Big|_0^{1-y} = \frac{(1-y)^2}{2(1-y)} = \frac{1-y}{2}; 0 \leq y \leq 1$$

$$V(x|y) = \int_0^{1-y} [x - E(x|y)]^2 dx = \int_0^{1-y} \left[x - \frac{(1-y)}{2} \right]^2 dx$$

$$= \int_0^{1-y} \left[x^2 - (1-y)x + \frac{(1-y)^2}{4} \right] dx = \frac{1}{12} (1-y)^2 \text{ for } 0 \leq y \leq 1$$

Find $E(y|x)$ & $V(y|x)$ H.w

Theorem (3) If X and Y are two Random Variables, then:

a. $E[E(Y|X)] = E(Y)$, b. $E[E(X|Y)] = E(X)$

Proof Case (1): If X and Y are c.r.v. with J.P.d.f. $f(x,y)$

$E(Y|X)$ is a function of X .

Suppose that $E(Y|X) = g(x)$

$$\begin{aligned} E[E(Y|X)] &= E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f_1(x) dx = \int_{-\infty}^{\infty} [E(Y|X)] f_1(x) dx \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y f(y|x) dy \right] f_1(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(y|x) f_1(x) dy dx \end{aligned}$$

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f_1(x)} \Rightarrow f(y|x) \cdot f_1(x) = f(x,y) \\ \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y) dx dy &= \int_{-\infty}^{\infty} y f_2(y) dy = E(Y) \end{aligned}$$

Case (2): If X and Y are d.r.v. with J.P.m.f. $f(x,y)$

$E(Y|X)$ is a function of X .

Suppose that $E(Y|X) = g(x)$

$$\begin{aligned} E[E(Y|X)] &= E(g(x)) = \sum_x g(x) \cdot f_1(x) = \sum_x [E(Y|X)] f_1(x) \\ &= \sum_x \left[\sum_y y f(y|x) \right] f_1(x) = \sum_x \sum_y y f(y|x) f_1(x) \end{aligned}$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} \Rightarrow f(y|x) \cdot f_1(x) = f(x,y)$$

$$\sum_y y \sum_x f(x,y) = \sum_y y f_2(y) = E(Y)$$

b. By the same method.

Example (14) Given a J.P.d.f. $f(x,y)$

$$f(x,y) = \begin{cases} 2 & \text{for } 0 \leq x \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E[E(Y|X)]$.

Sol.

$$E[E(Y|X)] = E(Y) \text{ (by Theorem)}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_2(y) dy \stackrel{f_2(y)}{=} \int_0^y \int_0^y f(x,y) dx = \int_0^y 2 dx = 2x \Big|_0^y = 2y$$

$$f_2(y) = \begin{cases} 2y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(y) = \int_0^1 y(2y) dy = \frac{2}{3} y^3 \Big|_0^1 = \frac{2}{3}$$

Theorem (4) If X and Y are random variables, then:

a. $V(X|Y) = E(X^2|Y) - [E(X|Y)]^2$

b. $V(Y|X) = E(Y^2|X) - [E(Y|X)]^2$

Proof:

$$\begin{aligned} \text{a. } V(X|Y) &= E\{[X - E(X|Y)]^2 | Y\} \\ &= E\{[(X|Y) - E(X|Y)]^2\} \\ &= E\{X^2|Y - 2(X|Y)E(X|Y) + [E(X|Y)]^2\} \\ &= E(X^2|Y) - 2E(X|Y)E(X|Y) + [E(X|Y)]^2 \\ &= E(X^2|Y) - [E(X|Y)]^2 \end{aligned}$$

b. By the same method.

Transformation of two Random Variables

II, Transformation of two d.r.v.

Let X_1 and X_2 are two d.r.v.s with J.P.M.f $f(x_1, x_2) > 0$, Such that A is the space of (X_1, X_2) .

Let $\left. \begin{aligned} Y_1 &= u_1(x_1, x_2) \\ Y_2 &= u_2(x_1, x_2) \end{aligned} \right\}$ be (1-1) transformation of A onto B
(the space of (Y_1, Y_2)).

$\left. \begin{aligned} X_1 &= u_1^{-1}(y_1, y_2) \\ X_2 &= u_2^{-1}(y_1, y_2) \end{aligned} \right\}$ be (1-1) transformation from B to A .

Then the J.P.M.f of (Y_1, Y_2) is $g(y_1, y_2) = f[u_1^{-1}(y_1, y_2), u_2^{-1}(y_1, y_2)]$.

Example: Given a J.P.M.f of X_1 and X_2

$$f(x_1, x_2) = \begin{cases} \left(\frac{2}{3}\right)^{x_1+x_2} \left(\frac{1}{3}\right)^{2-x_1-x_2} & \text{for } (x_1, x_2) = (0,0), (0,1), (1,0), (1,1) \\ 0 & \text{o.w.} \end{cases}$$

Find the J.P.M.f. of $Y_1 = X_1 - X_2$, $Y_2 = X_1 + X_2$.

Sol. $u_1 = Y_1 = X_1 - X_2$

$u_2 = Y_2 = X_1 + X_2$

$$\left. \begin{aligned} Y_1 &= X_1 - X_2 \\ + Y_2 &= X_1 + X_2 \\ \hline X_1 &= \frac{Y_1 + Y_2}{2} \end{aligned} \right\} \Rightarrow g(y_1, y_2)$$
$$\left. \begin{aligned} Y_1 &= X_1 - X_2 \\ - Y_2 &= X_1 + X_2 \\ \hline Y_2 - Y_1 &= 2X_2 \\ X_2 &= \frac{Y_2 - Y_1}{2} \end{aligned} \right\}$$

$$g(y_1, y_2) = \begin{cases} \left(\frac{2}{3}\right)^{\left(\frac{y_1+y_2}{2}\right) + \left(\frac{y_2-y_1}{2}\right)} \left(\frac{1}{3}\right)^{2 - \frac{(y_1+y_2)}{2} - \frac{(y_2-y_1)}{2}} & \text{for } (y_1, y_2) = (0,0), (-1,1), (1,1), (0,2) \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \left(\frac{2}{3}\right)^{y_2} \left(\frac{1}{3}\right)^{2-y_2} & \text{for } (y_1, y_2) = (0,0), (-1,1), (1,1), (0,2) \\ 0 & \text{o.w.} \end{cases}$$

$$g(y) = \sum_{y_2} g(y_1, y_2), \quad y = 0, 1, 2$$

$$= \sum_{y_2} \left(\frac{2}{3}\right)^{y_2} \left(\frac{1}{3}\right)^{2-y_2}$$

$$= \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1} + \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{2-2}$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{2}{9} + \frac{4}{9} = \underline{1}$$

H.w. Let X_1 and X_2 have J.P.M.f. $f(x_1, x_2)$:

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{2} & \text{for } x_1 = x_2 = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

1. Find J.P.M.f. of $Y_1 = X_1 X_2$ & $Y_2 = X_2$
2. Find the Marginal P.M.f. of $Y_2 [g(y_2)]$.

«II» Transformation of two C.R.V.s:

Let X_1 and X_2 are two C.R.V.s with J.P.d.f. $f(x_1, x_2) > 0$, for all $(x_1, x_2) \in A$ (the space of (x_1, x_2)).

Let $\begin{cases} Y_1 = u_1(x_1, x_2) \\ Y_2 = u_2(x_1, x_2) \end{cases}$ be (1-1) transformation from A to B (the space of (y_1, y_2)).

$$\text{If } X_1 = u_1^{-1}(y_1, y_2)$$

$$X_2 = u_2^{-1}(y_1, y_2)$$

Then, the J.P.d.f. of y_1 and y_2 is

$$g(y_1, y_2) = f[u_1^{-1}(y_1, y_2), u_2^{-1}(y_1, y_2)] \cdot |J| \text{ where } J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Example: Let X_1 and X_2 be two C.R.V.s with J.P.d.f.

$$f(x_1, x_2) = \begin{cases} \frac{1}{2\pi} e^{-\frac{(x_1^2 + x_2^2)}{2}} & \text{for } -\infty < x_1 < \infty, -\infty < x_2 < \infty \\ 0 & \text{o.w.} \end{cases}$$

1. Find the J.P.d.f. of $Y_1 = X_1 | X_2$ and $Y_2 = X_2$.
2. Find the Marginal P.d.f. of $Y_1 [g(y_1)]$.

Sol. $y_1 = u_1(x_1, x_2) = \frac{x_1}{x_2}$, $y_2 = u_2(x_1, x_2) = x_2$
 $x_1 = u_1^{-1}(y_1, y_2) = y_1 y_2$, $x_2 = u_2^{-1}(y_1, y_2) = y_2$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$$

$$g(y_1, y_2) = f[u_1^{-1}(y_1, y_2), u_2^{-1}(y_1, y_2)] |J|$$

$$= \frac{y_2}{2\pi} e^{-\frac{(y_1^2 y_2^2 + y_2^2)}{2}} \text{ for } -\infty < y_1 < \infty, -\infty < y_2 < \infty$$

$$g(y_1) = \int_{-\infty}^{\infty} g(y_1, y_2) dy_2 = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(y_1^2 + 1)y_2^2}{2}} dy_2$$

$$= \frac{2}{2\pi} \int_0^{\infty} e^{-\frac{(y_1^2 + 1)y_2^2}{2}} dy_2 = \frac{1}{\pi} \left[\frac{-1}{y_1^2 + 1} \right] e^{-\frac{(y_1^2 + 1)y_2^2}{2}} \Big|_0^{\infty}$$

$$= \frac{-1}{\pi(1+y_1^2)} [e^{-\infty} - e^0] = \frac{1}{\pi(1+y_1^2)} \text{ for } -\infty < y_1 < \infty$$

(Cauchy distribution)

H.W. Let x_1 and x_2 have a J.P.d.f $f(x_1, x_2)$

$$f(x_1, x_2) = \begin{cases} e^{-(x_1 + x_2)} & \text{for } 0 \leq x_1 < \infty, 0 \leq x_2 < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find the J.P.d.f of $y_2 = \frac{x_1}{x_1 + x_2}$ and $y_1 = x_1 + x_2$.

H.W. Given J.P.d.f

$$f(x_1, x_2) = \begin{cases} 4x_1 x_2 & \text{for } 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the J.P.d.f of $y_1 = \frac{x_1}{x_2}$ and $y_2 = x_1 x_2$.

Joint Distribution function (J.d.f) of two a.r.v.s X and Y

Def. The J.d.f of X and Y is defined to a function such that for all values of X and Y ($-\infty < x < \infty, -\infty < y < \infty$) then:

$$F(x, y) = P(X \leq x, Y \leq y)$$

Remarks

$$1. P(a < X \leq b, c < Y \leq d) = [F(b, d) - F(a, d)] - [F(b, c) - F(a, c)].$$

نقارن اعلاه للمتغير العشوائى الواحد (X)

$$\left\{ \begin{array}{l} 1. F(x) = P(X \leq x) \end{array} \right.$$

$$\left\{ \begin{array}{l} 2. P(a < X \leq b) = F(b) - F(a) \end{array} \right.$$

$$2. (i) F_1(x) = P(X \leq x) = \lim_{y \rightarrow \infty} P(X \leq x \text{ and } Y \leq y) = \lim_{y \rightarrow \infty} F(x, y)$$

$$(ii) F_2(y) = \lim_{x \rightarrow \infty} F(x, y)$$

3. If X and Y have Cont. J.d.f $F(x, y)$ with J.P.d.f $f(x, y)$, then for any values of X and Y the J.d.f. is

$$F(x, y) = \int_{-\infty}^y \int_{-\infty}^x f(r, s) dr ds \text{ therefore}$$

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}, \quad f(x, y) \text{ diff.}$$

Example 12 Given a J.d.f $F(x, y)$

$$F(x, y) = \begin{cases} 0 & \text{for } x < 0, y < 0 \\ \frac{1}{16} xy(x+y) & \text{for } 0 \leq x \leq 2, 0 \leq y \leq 2 \\ 1 & \text{for } x > 2, y > 2 \end{cases}$$

Find $F_1(x), F_2(y), f(x, y)$.

$$\begin{aligned} \text{sol. } F_1(x) &= \lim_{y \rightarrow \infty} F(x, y) = \lim_{y \rightarrow 2} F(x, y) = F(x, 2) \\ &= \frac{1}{16} x(2)(x+2) = \frac{x(x+2)}{8} \end{aligned}$$

$$F_1(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x(x+2)}{8} & \text{for } 0 \leq x \leq 2 \\ 1 & \text{for } x > 2 \end{cases}$$

$$F_2(y) = \lim_{x \rightarrow \infty} F(x, y) = \lim_{x > 2} F(x, y) = F(2, y) \\ = \frac{1}{16} (2) y (2+y) = \frac{y(2+y)}{8}$$

$$F_2(y) = \begin{cases} 0 & \text{for } y < 0 \\ \frac{y(2+y)}{8} & \text{for } 0 \leq y \leq 2 \\ 1 & \text{for } y > 2 \end{cases}$$

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial F(x, y)}{\partial y} \right]$$

$$F(x, y) = \frac{1}{16} (x^2 y + x y^2)$$

$$f(x, y) = \frac{\partial}{\partial x} \left[\frac{1}{16} (x^2 + 2xy) \right] = \left[\frac{1}{16} (2x + 2y) \right] \\ = \frac{1}{8} (x+y)$$

$$f(x, y) = \begin{cases} \frac{x+y}{8} & \text{for } 0 \leq y \leq 2 \\ 0 & \text{o.w.} \end{cases}$$

Note: We Call:

Univariate: involving one variable quantities. x

وحد المتغير

Bivariate: involving two variables quantities. x, y

ثنائي المتغير

Multivariate: involving more than two variables quantities.

متعدد المتغيرات

$x_1, x_2, \dots, x_n$

Moment Generating Function of two R.V.'s X & Y

Let $f(x, y)$ be the J.P.d.f. of X, Y ; and consider that:

$E[e^{t_1 X + t_2 Y}]$ exists for $|t_1| < h_1, |t_2| < h_2$; h_1, h_2 are positive constants, then :

$E[e^{t_1 X + t_2 Y}]$ is the m.g.f. of X & Y and is denoted

by $M_{X,Y}(t_1, t_2)$ and is defined as follows :

$$M_{X,Y}(t_1, t_2) = E[e^{t_1 X + t_2 Y}] = \begin{cases} \sum_{x,y} e^{t_1 x + t_2 y} f(x, y) & \text{if } X, Y \text{ are d.r.v.s} \\ \iint_{x,y} e^{t_1 x + t_2 y} f(x, y) dy dx & \text{if } X, Y \text{ are c.r.v.s} \end{cases}$$

and

$$M_X(t_1) = M_X(t_1, 0) = E[e^{t_1 X}] = \begin{cases} \sum_{x,y} e^{t_1 x} f_1(x) & \text{if } X \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{t_1 x} f_1(x) dx & \text{if } X \text{ is c.r.v.} \end{cases}$$

$$M_Y(t_2) = M_Y(0, t_2) = E[e^{t_2 Y}] = \begin{cases} \sum_{x,y} e^{t_2 y} f_2(y) & \text{if } Y \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{t_2 y} f_2(y) dy & \text{if } Y \text{ is c.r.v.} \end{cases}$$

and

$$\frac{\partial M(0,0)}{\partial t_1} = M'_X = E(X) ; \quad \frac{\partial M(0,0)}{\partial t_2} = M'_Y = E(Y)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1^2} = E(X^2) ; \quad \frac{\partial^2 M(0,0)}{\partial t_2^2} = E(Y^2)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} = E(XY)$$

(Pg.1)

The variance of X or Y by using $M_{X,Y}(t_1, t_2)$:

$$\begin{aligned}\text{Var}(X) = \sigma_x^2 &= \frac{\partial^2 M(0,0)}{\partial t_1^2} - \left(\frac{\partial M(0,0)}{\partial t_1} \right)^2 \\ &= E(X^2) - [E(X)]^2\end{aligned}$$

$$\begin{aligned}\text{Var}(Y) = \sigma_y^2 &= \frac{\partial^2 M(0,0)}{\partial t_2^2} - \left(\frac{\partial M(0,0)}{\partial t_2} \right)^2 \\ &= E(Y^2) - [E(Y)]^2\end{aligned}$$

Example : Let $f(x,y)$ be the J.P.D.F. of X, Y :

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } 0 < x < \infty, y < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find the M.G.F of X, Y ; also, find $M_x, M_y, E(XY)$, and $\rho_{X,Y}$.

Sol.

$$\begin{aligned}M_{X,Y}(t_1, t_2) &= E[e^{t_1 X + t_2 Y}] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x,y) dy dx \\ &= \int_0^{\infty} \int_0^{\infty} e^{t_1 x + t_2 y} e^{-(x+y)} dy dx \\ &= \left(\int_0^{\infty} e^{-x(1-t_1)} dx \right) \left(\int_0^{\infty} e^{-y(1-t_2)} dy \right) \\ &= \frac{1}{(1-t_1)} \cdot \frac{1}{(1-t_2)} = \frac{1}{(1-t_1)(1-t_2)}\end{aligned}$$

(Pg. 2)

$$\text{M.g.f of } X = M_{x,y}(t_1, 0) = M_x(t_1) = \frac{1}{1-t_1}$$

$$\begin{aligned} \mu_x = E(X) &= \left. \frac{\partial M(t_1, 0)}{\partial t_1} \right|_{t_1=0} = \left. \frac{\partial M_x(t_1)}{\partial t_1} \right|_{t_1=0} \\ &= \left. \frac{1}{(1-t_1)^2} \right|_{t_1=0} = \boxed{1} \end{aligned}$$

$$\text{M.g.f of } Y = M_{x,y}(0, t_2) = M_y(t_2) = \frac{1}{1-t_2}$$

$$\begin{aligned} \mu_y = E(Y) &= \left. \frac{\partial M(0, t_2)}{\partial t_2} \right|_{t_2=0} = \left. \frac{\partial M_y(t_2)}{\partial t_2} \right|_{t_2=0} \\ &= \left. \frac{1}{(1-t_2)^2} \right|_{t_2=0} = \boxed{1} \end{aligned}$$

$$\therefore \text{M.g.f of } X, Y = M_{x,y}(t_1, t_2) = \frac{1}{(1-t_1)(1-t_2)}$$

$$\begin{aligned} E(XY) &= \left. \frac{\partial^2 M(t_1, t_2)}{\partial t_1 \partial t_2} \right|_{t_1=t_2=0} \\ &= \left. \frac{\partial}{\partial t_1} \left(\frac{\partial M(t_1, t_2)}{\partial t_2} \right) \right|_{t_1=t_2=0} \\ &= \left. \frac{\partial}{\partial t_1} \left(\frac{1}{(1-t_1)(1-t_2)^2} \right) \right|_{t_1=t_2=0} \\ &= \left. \frac{1}{(1-t_1)^2(1-t_2)^2} \right|_{t_1=t_2=0} \\ &= \boxed{1} \end{aligned}$$

$$\rho_{x,y} = \frac{\text{Cov}(x,y)}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)E(y)}{\sqrt{\sigma_x^2} \sqrt{\sigma_y^2}}$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2$$

$$\sigma_y^2 = E(y^2) - [E(y)]^2$$

$$E(x^2) = \left. \frac{\partial^2 M(t_1, 0)}{\partial t_1^2} \right|_{t_1=0} = -2(1-t_1)^{-3}(-1) \Big|_{t_1=0} = \boxed{2}$$

$$E(y^2) = \left. \frac{\partial^2 M(0, t_2)}{\partial t_2^2} \right|_{t_2=0} = -2(1-t_2)^{-3}(-1) \Big|_{t_2=0} = \boxed{2}$$

$$\sigma_x^2 = 2 - (1)^2 = \boxed{1} \Rightarrow \boxed{\sigma_x = 1}$$

$$\sigma_y^2 = 2 - (1)^2 = \boxed{1} \Rightarrow \boxed{\sigma_y = 1}$$

$$\rho_{x,y} = \frac{1 - (1)(1)}{(1)(1)} = 0$$

$\therefore X$ & Y are independent (since $\text{Cov}(x,y) = 0$).

H.W. Given a J.P.F of X & Y :

$$f(x,y) = \begin{cases} e^{-y-x} & \text{for } 0 \leq x \leq y < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find: ① the M.G.F. of X, Y $[M_{x,y}(t_1, t_2)]$.

② $E(xy)$

③ $f_1(x)$ & $f_2(y)$

④ $M_x = E(x)$ & $M_y = E(y)$ (by two methods).

⑤ $\text{Cov}(X, Y)$.

Note: $R = \left\{ \begin{array}{ll} 0 \leq x \leq y & , x \leq y < \infty \\ 0 \leq x < \infty & , 0 \leq y < \infty \end{array} \right\}$

Transformations

Q1: If X has a uniform dist. on $\{1, 2, 3\}$ then find the p.m.f. of $Y = 4X + 2$.

Sol. $X \sim \text{unif.}(3) ; X = 1, 2, 3$

$$A = \{X ; X = 1, 2, 3\}$$

$$B = \{Y ; Y = 6, 10, 14\}$$

$$Y = 4X + 2 = u(X) \in B$$

$$X = \frac{Y-2}{4} = u^{-1}(Y) \in A$$

$$f(X) = \begin{cases} \frac{1}{3} & \text{for } X = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

$$g(Y) = f[u^{-1}(Y)] = f\left(\frac{Y-2}{4}\right)$$

$$g(Y) = \begin{cases} \frac{1}{3} & \text{for } Y = 6, 10, 14 \\ 0 & \text{o.w.} \end{cases}$$

#

Q2: Given a p.m.f. of X :

$$f(X) = \begin{cases} \left(\frac{1}{8}\right)^X & \text{for } X = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

$$Y = X^3 = u(X) \in B$$

$$X = \sqrt[3]{Y} = u^{-1}(Y) \in A$$

$$A = \{X ; X = 1, 2, 3\}$$

$$B = \{Y ; Y = 1, 8, 27\}$$

$$g(Y) = f[u^{-1}(Y)] = f(\sqrt[3]{Y})$$

$$\therefore g(Y) = \begin{cases} \left(\frac{1}{8}\right)^{\sqrt[3]{Y}} & \text{for } Y = 1, 8, 27 \\ 0 & \text{o.w.} \end{cases}$$

#

(Transformations)

$$Q_3: f(x) = \begin{cases} \frac{x^2}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

Find p.d.f. of $y = x^3$.

Sol. $y = x^3 = u(x) \in B$
 $x = \sqrt[3]{y} = u^{-1}(y) \in A$

$$A = \{x; 0 < x < 3\}$$

$$B = \{y; 0 < y < 27\} \quad (y(0) = 0 \text{ \& } y(3) = 27)$$

$$J = [u^{-1}(y)]' = x' = \frac{dx}{dy}$$

$$J = (\sqrt[3]{y})' = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{y^2}}$$

$$|J| = \left| \frac{1}{3\sqrt[3]{y^2}} \right| = \frac{1}{3\sqrt[3]{y^2}}$$

$$\begin{aligned} f[u^{-1}(y)] &= f[\sqrt[3]{y}] \text{ \& } f(x) = \frac{x^2}{9} \\ &= \frac{(\sqrt[3]{y})^2}{9} = \frac{\sqrt[3]{y^2}}{9} \end{aligned}$$

$$\therefore g(y) = f[u^{-1}(y)] |J|$$

$$= \frac{(\sqrt[3]{y^2})}{9} \cdot \frac{1}{3(\sqrt[3]{y^2})} = \frac{1}{27}$$

$$\therefore g(y) = \begin{cases} \frac{1}{27} & \text{for } 0 < y < 27 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore y \sim \text{unif}(0, 27)$$

#

(Transformations)

Q4: Given a.p.d.f. of X :

$$f(x) = \begin{cases} 2xe^{-x^2} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find p.d.f. of $Y = X^2$.

Sol. $Y = X^2 = u(X) \in A$
 $X = \sqrt{Y} = u^{-1}(Y) \in B$

$$A = \{X; 0 < X < \infty\}$$

$$B = \{Y; 0 < Y < \infty\} \longrightarrow \begin{cases} Y(0) = 2(0)e^0 = 0 \\ Y(\infty) = 2(\infty)e^{-\infty} = \infty \end{cases}$$

$$\begin{aligned} f[u^{-1}(y)] &= f[\sqrt{y}] ; f(x) = 2xe^{-x^2} \\ &= 2\sqrt{y} e^{-(\sqrt{y})^2} \\ &= 2\sqrt{y} e^{-y} \end{aligned}$$

$$\begin{aligned} J &= [u^{-1}(y)]' = x' = \frac{dx}{dy} \\ &= (\sqrt{y})' = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}} \end{aligned}$$

$$|J| = \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2\sqrt{y}}$$

$$\begin{aligned} \therefore g(y) &= f[u^{-1}(y)] |J| \\ &= (2\sqrt{y} e^{-y}) \left(\frac{1}{2\sqrt{y}} \right) = e^{-y} \end{aligned}$$

$$\therefore g(y) = \begin{cases} e^{-y} & \text{for } 0 < y < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore Y \sim \text{EXP}(\lambda=1) ; \lambda=1$$

(Transformations)

Exercises (H.W.)

Q₁: $X \sim \text{unif.}(0,1)$ & $Y = -2 \ln X$; then show that:
 $Y \sim \chi^2(2)$ or $Y \sim G(1,2)$.

Q₂: If $X \sim \text{unif.}(0,1)$ & $Y = -4 \ln X$; then find p.d.f. of Y .

Q₃: If $X \sim P(3)$; $Y = 2X$; find p.d.f. of Y .

Q₄: If $X \sim b(3, \frac{1}{2})$; $Y = X^2$; find p.f. of Y .

Q₅: If $X \sim f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

& $Y = 8X^3$, then find the p.d.f. of Y .

Q: Given a p.d.f. of X :

$$f_1(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

① Let $Y = X(4-X)$ be a transformation.

Find a p.d.f. of Y ($f_2(y)$) & C.d.f. $G(y)$.

② Let $Y = X(2-X)$ be a transformation.

Find a p.d.f. of Y ($f_2(y)$) & C.d.f. $G(y)$.

Sol. ① $Y = X(4-X) = 4X - X^2 = u(x) \in B$

$$A = \{x; 0 < x < 2\}$$

(Transformations)

Q₁: Let X_1 & X_2 have J.P.M.F. $f(X_1, X_2)$:

$$f(X_1, X_2) = \begin{cases} \frac{X_1 X_2}{2} & \text{for } X_1 = X_2 = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

① Find J.P.M.F. of $Y_1 = X_1 X_2$ & $Y_2 = X_2$.

② Find the marginal p.m.f. of Y_2 ($g(Y_2)$).

Sol.

$$A = \{ (X_1, X_2) ; X_1 = 1, 2, 3 \text{ \& } X_2 = 1, 2, 3 \}$$

$$B = \{ (Y_1, Y_2) ; Y_1 = 1, 2, 3, 4, 6, 9, \text{ \& } Y_2 = 1, 2, 3 \}$$

$$Y_1 = u_1(X_1, X_2) = X_1 X_2$$

$$Y_2 = u_2(X_1, X_2) = X_2$$

$$X_1 = u_1^{-1}(Y_1, Y_2) = Y_1 / Y_2$$

$$X_2 = u_2^{-1}(Y_1, Y_2) = Y_2$$

$$\begin{aligned} \therefore g(Y_1, Y_2) &= f[u_1^{-1}(Y_1, Y_2), u_2^{-1}(Y_1, Y_2)] \\ &= \begin{cases} \left(\frac{Y_1}{Y_2}\right) \left(\frac{Y_2}{2}\right) = Y_1/2 & \text{for } Y_1 = 1, 2, 3, 4, 6, 9 \\ & Y_2 = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases} \end{aligned}$$

$$\textcircled{2} \quad g(Y_2) = \sum_{\forall Y_1} g(Y_1, Y_2) = \sum_{Y_1 = 1, 2, 3, 4, 6, 9} \left(\frac{Y_1}{2}\right)$$

$$= \frac{1}{2} [1 + 2 + 3 + 4 + 6 + 9] = \begin{cases} \frac{25}{2} & \text{for } Y_2 = 1, 2, 3, 4, 6, 9 \\ 0 & \text{o.w.} \end{cases}$$

Q₂: Let X_1 & X_2 have a J.P.d. of X_1, X_2 :

$$f(X_1, X_2) = \begin{cases} e^{-(X_1 + X_2)} & \text{for } 0 < X_1 < \infty, 0 < X_2 < \infty \\ 0 & \text{o.w.} \end{cases}$$

$Y_1 = X_1 + X_2$, $Y_2 = \frac{X_1}{X_1 + X_2}$; Find J.P.d.f. of Y_1 & Y_2 .

Sol.

(Transformations)

$$A = \{ (x_1, x_2) ; x_1 > 0, x_2 > 0 \}$$

$$B = \{ (y_1, y_2) ; y_1 > 0, y_2 > 0 \}$$

$$y_1 = x_1 + x_2 = u_1(x_1, x_2)$$

$$y_2 = x_1 / (x_1 + x_2) = u_2(x_1, x_2)$$

To find x_1 & x_2 :

$$y_1 = x_1 + x_2 \Rightarrow x_2 = y_1 - x_1$$

$$y_2 = x_1 / (x_1 + x_2) \Rightarrow y_2 = \frac{x_1}{x_1 + y_1 - x_1} = \frac{x_1}{y_1}$$

$$\Rightarrow y_2 = \frac{x_1}{y_1} \Rightarrow \boxed{x_1 = y_1 y_2} \text{ \& \& } \boxed{x_2 = y_1 - y_1 y_2}$$

$$\therefore x_1 = u_1^{-1}(y_1, y_2) = y_1 y_2$$

$$x_2 = u_2^{-1}(y_1, y_2) = y_1 - y_1 y_2$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = y_1$$

$$= -y_1 y_2 - y_1(1 - y_2)$$

$$= -(y_1 y_2 + y_1(1 - y_2))$$

$$= -(\cancel{y_1 y_2} + y_1 - \cancel{y_1 y_2}) = -y_1$$

$$f[u_1^{-1}(y_1, y_2), u_2^{-1}(y_1, y_2)] = e^{-(\cancel{y_1 y_2} + y_1 - \cancel{y_1 y_2})} \\ = e^{-y_1}$$

$$\therefore g(y_1, y_2) = f[u_1^{-1}(y_1, y_2), u_2^{-1}(y_1, y_2)] |J| \\ = \begin{cases} y_1 e^{-y_1} & \text{for } y_1 > 0, y_2 > 0 \\ 0 & \text{o.w.} \end{cases}$$

To find $g(y_2)$:

$$g(y_2) = \int_0^{\infty} y_1 e^{-y_1} dy_1 = [-y_1 e^{-y_1}]_0^{\infty} + \int_0^{\infty} e^{-y_1} dy_1 \\ = -e^{-y_1} \Big|_0^{\infty} = -[\bar{e}^{-\infty} - \bar{e}^{-0}] \\ = -(-1) = 1$$

(Integral by partition)

(Transformations)

$$Q_3: f(x_1, x_2) = \begin{cases} 4x_1x_2 & \text{for } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$Y_1 = \frac{x_1}{x_2}, Y_2 = x_1x_2; \text{ Find J.P.d.f. of } Y_1 \& Y_2?$$

Sol.

$$Y_1 = \frac{x_1}{x_2} = u_1(x_1, x_2)$$

$$Y_2 = x_1x_2 = u_2(x_1, x_2)$$

$$A = \{ (x_1, x_2); 0 < x_1 < 1 \& 0 < x_2 < 1 \}$$

$$B = \{ (y_1, y_2); 0 < y_1 < 1 \& 0 < y_2 < 1 \}$$

To find $x_1 \& x_2$

$$x_1 = Y_1 Y_2, x_2 = \frac{Y_2}{Y_1}$$

$$\Rightarrow x_1 = y_1 \left(\frac{y_2}{x_1} \right) \Rightarrow x_1^2 = y_1 y_2 \Rightarrow x_1 = \sqrt{y_1 y_2}$$

$$\& x_2 = y_2 / \sqrt{y_1 y_2}$$

$$\therefore x_1 = u_1^{-1}(y_1, y_2) = \sqrt{y_1 y_2} = \sqrt{y_1} \sqrt{y_2}$$

$$x_2 = u_2^{-1}(y_1, y_2) = y_2 / \sqrt{y_1 y_2} = \sqrt{y_2} / \sqrt{y_1}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y_2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1}}{2\sqrt{y_2}} \\ \frac{y_2}{2y_1} & \frac{1}{2\sqrt{y_1 y_2}} \end{vmatrix}$$

$$= \left(\frac{\sqrt{y_2}}{2\sqrt{y_1}} \right) \left(\frac{1}{2\sqrt{y_1 y_2}} \right) - \left(\frac{\sqrt{y_1}}{2\sqrt{y_2}} \right) \left(\frac{y_2}{2y_1} \right)$$

$$= \frac{1}{4y_1} - \frac{\sqrt{y_2}}{4\sqrt{y_1 y_2}} = \frac{1}{4y_1} \left(1 - \frac{\sqrt{y_2}}{\sqrt{y_1}} \right)$$

$$f[u_1^{-1}, u_2^{-1}] = 4(\sqrt{y_1} \sqrt{y_2}) \frac{\sqrt{y_2}}{\sqrt{y_1}} = 4y_2$$

$$\therefore g(y_1, y_2) = f[u_1^{-1}, u_2^{-1}] |J|$$

$$= [4y_2] \frac{1}{4y_1} \left[1 - \frac{\sqrt{y_2}}{\sqrt{y_1}} \right]$$

$$g(y_1, y_2) = \begin{cases} \frac{y_2}{y_1} \left(1 - \frac{\sqrt{y_2}}{\sqrt{y_1}} \right) & \text{for } 0 < y_1 < 1 \& 0 < y_2 < 1 \\ 0 & \text{o.w.} \end{cases}$$

(J.P.m.f of $Y_1 \& Y_2$)

Exercises About ch. 5

Q₁: Given a J.P. f. :

$$f(x, y) = \begin{cases} cy^2 & \text{for } 0 \leq x \leq 2 \text{ \& } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find :

① The value of (c)

② $P(X+Y > 2)$

③ $P(Y < \frac{1}{2})$, ④ $P(X \leq 1)$

Q₂: Let the J.P.m.f of X & Y be:

$$f(x, y) = \begin{cases} \frac{x+y}{30} & \text{for } x=1, 2, 3 \text{ \& } y=1, 2 \\ 0 & \text{o.w.} \end{cases}$$

(i) Find the marginal p.f. of X?

(ii) Find the marginal p.f. of Y?

(iii) Are X & Y independent?

Q₃: Let X & Y have the J.P.m.f. be:

$$f(x, y) = \begin{cases} \frac{x+2y}{18} & , \quad x=1, 2, y=1, 2 \\ 0 & \text{o.w.} \end{cases}$$

Find: $\mu_x = E(X)$, $\mu_y = E(Y)$, & $\text{Cov}(X, Y)$.

Q₄: Suppose that X, Y are C.R.V.s with J.P.d. f. :

$$f(x, y) = \begin{cases} e^{-(x+y)} & \text{for } x > 0, y > 0 \\ 0 & \text{o.w.} \end{cases}$$

Find: $M_{x,y}(t_1, t_2)$, $M_x(t_1)$, $M_y(t_2)$.

Q5: Suppose X and Y are two r.v.s having the following J.p.d.f.:

$$f(x,y) = \begin{cases} x+y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

- (i) Find the conditional p.d.f $f(x|y)$ and $f(y|x)$.
 (ii) Are these Conditional function $f(x|y)$ and $f(y|x)$ Pr. d.f.? (prove that).
-

Q6: Consider the following J.P.M.F. of X and Y whose values are given by:

$y \backslash x$	2	3	4	5	6	$f_2(y)$
0	$1/9$	0	$4/27$	0	$2/27$	$1/3$
1	$2/27$	0	$8/81$	0	$4/81$	$2/9$
2	$4/27$	0	$16/81$	0	$8/81$	$4/9$
$f_1(x)$	$1/3$	0	$4/9$	0	$2/9$	1

- Find: ① $f(x|y)$ & $f(y|x)$
 ② $P(X|Y=1)$ & $P(Y|X=2)$
-

Q7: If J.P.m.f.:

$$\begin{aligned} P(1,1) &= \frac{1}{9}, & P(2,1) &= \frac{1}{9}, & P(3,1) &= \frac{1}{9} \\ P(1,2) &= \frac{1}{9}, & P(2,2) &= 0, & P(3,2) &= \frac{1}{18} \\ P(1,3) &= 0, & P(2,3) &= \frac{1}{6}, & P(3,3) &= \frac{1}{9} \end{aligned}$$

Are X & Y independent?

Q8: If X and Y are independent with m.p.d.f.:

$$f_1(x) = \begin{cases} 3x^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}, \quad f_2(y) = \begin{cases} 2y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(XY)$.

Q₉: Suppose that J.P.d.f of X & Y is :

$$f(x,y) = \begin{cases} C \sin x & \text{for } 0 \leq x \leq \pi, 0 \leq y \leq 3 \\ 0 & \text{o.w.} \end{cases}$$

Find the value of C .

Q₁₀: Let X and Y have the joint p.m.f. be :

(x,y)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
$P(X,Y)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

Find the Correlation Coefficient $\rho_{x,y}$.

Q₁₁: If $f(x,y) = \begin{cases} e^{-x-y} & \text{for } 0 \leq x < \infty \text{ \& } 0 \leq y < \infty \\ 0 & \text{o.w.} \end{cases}$

is the J.P.d.f of two r.v.s X & Y .

Show that X & Y are stochastically independent.

Q₁₂: Let X & Y be independent r.v.'s with the following distribution :

X	1	2
$f_1(x)$	.6	.4

Y	10	15	5
$f_2(y)$	.5	.3	.2

① Find the Joint pr. f of X & Y

② also, find $E(XY)$.

Q₁₃: Let $f(x|y) = \begin{cases} \frac{C_1 x}{y^2} & \text{for } 0 < x < y, 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$

& $f_2(y) = \begin{cases} C_2 y^4 & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$; then :

Find:

- (1) Determine the constants C_1 and C_2 .
 - (2) the $f_1(x)$ p.d.f. of X .
 - (3) the J.P.d.f. of X & Y ; $f(x,y)$
 - (4) $P(-\frac{1}{2} < X < \frac{1}{2} \mid Y = \frac{5}{8})$
 - (5) $P(\frac{1}{4} < X < \frac{1}{2})$.
-

Q14: Let $f(x,y) = \begin{cases} \frac{1}{16} & \text{for } x=1,2,3,4 \\ & y=1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

be the J.P.f of X & Y .

Show that X & Y are stochastically independent.

Q15: Given a J.P.f of X & Y :

$$f(x,y) = \begin{cases} x+y & \text{for } 0 < x < 1 \text{ \& } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the Correlation Coefficient of X & Y ($\rho_{x,y}$).

Solution of Q₁:

① by cond. (2) $\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$

$C \int_0^2 \int_0^1 y^2 dy dx = 1 \Rightarrow \boxed{C = \frac{3}{2}}$

② $P(X+Y > 2) = 1 - P(X+Y \leq 2)$

Let $x+y = 2 \Rightarrow y = 2-x$

$x+y \leq 2 \Rightarrow y \leq 2-x$

$R = \left\{ \begin{array}{l} 0 \leq y \leq 2-x \\ 0 \leq x \leq 2 \end{array} \right\}$

$P(X+Y \leq 2) = \int_0^2 \int_0^{2-x} \frac{3}{2} y^2 dy dx = \dots = \frac{1}{2}$

③ $P(Y < \frac{1}{2}) = ?$

To find $f_2(y)$ in the first.

$f_2(y) = \int_0^2 f(x,y) dx = \frac{3}{2} \int_0^2 y^2 dx = \dots = \begin{cases} 3y^2 & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$

$P(Y < \frac{1}{2}) = 3 \int_0^{\frac{1}{2}} y^2 dy = \dots = \frac{1}{8}$

④ $P(X \leq 1) = ?$

To find $f_1(x)$ in the beginning.

$f_1(x) = \int_0^1 f(x,y) dy = \frac{3}{2} \int_0^2 y^2 dy = \dots = \begin{cases} \frac{1}{2} & \text{for } 0 \leq x \leq 2 \\ 0 & \text{o.w.} \end{cases}$

$P(X \leq 1) = \int_0^1 f_1(x) dx$

$= \int_0^1 \frac{1}{2} dx = \dots = \frac{1}{2}$

⑤ $P(X=3Y) = \int_0^2 \int_0^{\frac{1}{3}x} f(x,y) dy dx = \int_0^2 \int_0^{\frac{1}{3}x} \frac{2}{3} y^2 dy dx$
 $\Rightarrow \boxed{y = \frac{1}{3}x}$
 $= \dots = \frac{2}{27}$

Q2: If $f_1(x) \cdot f_2(y) = f(x,y)$, then X & Y are indep.

$$f_1(x) = \sum_{y=0}^3 f(x,y) = \sum_{y=0}^3 \frac{1}{30} (x+y)$$

$$= \frac{1}{30} [x + (x+1) + (x+2) + (x+3)]$$

$$= \frac{1}{30} (4x+6) = \begin{cases} \frac{1}{15} (2x+3) & \text{for } x=0,1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \sum_{x=0}^2 f(x,y) = \sum_{x=0}^2 \frac{1}{30} (x+y)$$

$$= \frac{1}{30} [y + (1+y) + (2+y)]$$

$$= \frac{1}{30} (3+3y) = \frac{1}{10} (y+1)$$

$$f_2(y) = \begin{cases} \frac{1}{10} (y+1) & \text{for } y=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$f_1(x) f_2(y) = \frac{1}{15} (2x+3) \cdot \frac{1}{10} (y+1)$$

$$\neq \frac{1}{30} (x+y) = f(x,y)$$

$$\therefore f_1(x) \cdot f_2(y) \neq f(x,y)$$

Q3: To find f_1 & f_2 in the first time.

$$f_1(x) = \sum_{x_2=1}^2 \frac{1}{18} (x_1 + 2x_2)$$

$$= \frac{1}{18} \sum_{x_2=1}^2 x_1 + \frac{1}{18} \sum_{x_2=1}^2 2x_2$$

$$= \dots = \begin{cases} \frac{x_1+3}{9} & \text{for } x_1=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_{x_1} = E(x_1) = \sum_{x_1=1}^2 x_1 f_1(x_1) = \sum_{x_1=1}^2 x_1 \left(\frac{x_1+3}{9} \right) = \dots = \frac{14}{9}$$

$$f_2(x_2) = \sum_{x_1=1}^2 \frac{1}{18} (x_1 + 2x_2) = \begin{cases} \frac{1}{18} (3+4x_2) & \text{for } x_2=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \mu_{x_2} = E(x_2) &= \sum_{x_2=1}^2 x_2 f_2(x_2) = \sum_{x_2=1}^2 x_2 \left(\frac{3+4x_2}{18} \right) = \dots \\ &= \frac{29}{18} \end{aligned}$$

$$\text{Cov}(X_1, Y) = E(X_1 X_2) - E(X_1)E(X_2)$$

$$= \mu_{X_1 X_2} - \mu_{X_1} \mu_{X_2}$$

$$\mu_{X_1 X_2} = \sum_{X_1=1}^2 \sum_{X_2=1}^2 f(X_1, X_2) \cdot X_1 X_2$$

$$= \frac{1}{18} \sum_{X_1=1}^2 \sum_{X_2=1}^2 X_1 X_2 (X_1 + 2X_2)$$

$$= \frac{1}{18} [(1)(1)(1+2(1)) + (1)(2)(1+2(2)) + (2)(1)(2+2(1)) + (2)(2)(2+2(2))] = \frac{9}{2}$$

$$\rho_{xy} = \frac{\text{Cov}(X_1, Y)}{\sigma_x \sigma_y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)} \sqrt{V(Y)}}$$

$$\sigma_{X_1} = \sqrt{\sigma_{X_1}^2} ; \sigma_{X_1}^2 = E(X_1^2) - [E(X_1)]^2$$

$$\sigma_{X_2} = \sqrt{\sigma_{X_2}^2} ; \sigma_{X_2}^2 = E(X_2^2) - [E(X_2)]^2$$

$$E(X_1^2) = \sum_{X_1=1}^2 X_1^2 f_1(X_1) = \sum_{X_1=1}^2 X_1^2 \left(\frac{1+3}{9}\right) = \frac{24}{9}$$

$$E(X_2^2) = \sum_{X_2=1}^2 X_2^2 f_2(X_2) = \sum_{X_2=1}^2 X_2^2 \left(\frac{1}{18} (3+4X_2)\right) = \frac{51}{18}$$

$$\mu_{X_1}^2 = [E(X_1)]^2 \quad \& \quad \mu_{X_2}^2 = [E(X_2)]^2$$

$$\sigma_{X_1}^2 = E(X_1^2) - [E(X_1)]^2$$

$$= \frac{24}{9} - \left(\frac{14}{9}\right)^2 = -$$

$$\sigma_{X_2}^2 = \frac{51}{18} - \left(\frac{29}{18}\right)^2 = -$$

(المطلوب)

Q6 : ① $f(X|Y) = \frac{f(X,Y)}{f(Y)}$

② $f(X|Y=1) = \frac{f(X,Y=1)}{f(Y=1)} = \frac{f(X_i|Y=1)}{2/9}$, $f(Y|X) = \frac{f(X,Y)}{f(X)}$, $X_i = 2, 3, \dots, 6$.

$$= \begin{cases} \frac{2/27}{2/9} = \frac{1}{3} & \text{for } X=2 \\ \frac{0}{2/9} = 0 & \text{for } X=3 \\ \frac{8/81}{2/9} = \frac{4}{9} & \text{for } X=4 \\ \frac{0}{2/9} = 0 & \text{for } X=5 \\ \frac{4/81}{2/9} & \text{for } X=6 \end{cases}$$

$$f(y|X=2) = \frac{f(y|X=2)}{f(X=2)} = \frac{f(y|X=2)}{\frac{1}{3}}, y=0,1,2$$

$$= \begin{cases} \frac{1/9}{1/3} = \frac{1}{3} & \text{for } y=0 \\ \frac{2/27}{1/3} = \frac{2}{9} & \text{for } y=1 \\ \frac{4/27}{1/3} = \frac{4}{9} & \text{for } y=2 \end{cases}$$

Hence $f(y) = f(y|X=2)$; therefore X & Y are independent.

Q7:

$Y \backslash X$	1	2	3	$f_2(y)$
1	$1/9$	$1/3$	$1/9$	$5/9$
2	$1/9$	0	$1/18$	$3/18$
3	0	$1/6$	$1/9$	$15/54$
$f_1(x)$	$2/9$	$3/6$	$3/18$	1

$$f(1,1) = \frac{1}{9} \neq f_1(1)f_2(1) = \left(\frac{2}{9}\right)\left(\frac{5}{9}\right)$$

$$f(2,1) = \frac{1}{3} \neq f_1(2)f_2(1) = \left(\frac{3}{6}\right)\left(\frac{5}{9}\right)$$

$$f(3,1) = \frac{1}{9} \neq f_1(3)f_2(1) = \left(\frac{3}{18}\right)\left(\frac{5}{9}\right)$$

$$f(1,2) = \frac{1}{9} \neq f_1(1)f_2(2) = \left(\frac{2}{9}\right)\left(\frac{3}{18}\right)$$

$\therefore X$ & Y are dependent.

Q8: Since X & Y are independent; then:

$$f_1(x)f_2(y) = f(x,y)$$

$$f_1(x) \cdot f_2(y) = \begin{cases} 6yx^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(xy) = \int \int xy f(x,y) dx dy$$

$$= \int_0^1 \int_0^1 (6yx^2) xy dy dx = \dots = \frac{1}{2}$$

Sol.
Q10

$$f_1(x) = \sum_{y=1}^3 f(x,y)$$

$$= \begin{cases} 9/15 & \text{for } x=1 \\ 6/15 & \text{for } x=2 \\ 0 & \text{o.w.} \end{cases}$$

J.P.F.
 $f(x,y)$

$x \backslash y$	1	2	$f_2(y)$
1	$2/15$	$1/15$	$3/15$
2	$4/15$	$1/15$	$5/15$
3	$3/15$	$4/15$	$7/15$
$f_1(x)$	$9/15$	$6/15$	1

$$f_2(y) = \sum_{x=1}^2 f(x,y)$$

$$= \begin{cases} 3/15 & \text{for } y=1 \\ 5/15 & \text{for } y=2 \\ 7/15 & \text{for } y=3 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \sum_{x=1}^2 x f_1(x)$$

$$= (1)(9/15) + (2)(6/15)$$

$$= \frac{21}{15}$$

$$E(Y) = \sum_{y=1}^3 y f_2(y)$$

$$= (1)(3/15) + (2)(5/15) + (3)(7/15)$$

$$= \frac{34}{15}$$

$$\rho_{XY} = \text{Cov}(X,Y) / \sigma_X \sigma_Y = \dots$$

Q12: ①

$$f(1,5) = f_1(1) f_2(5) = (.6)(.2) = .12$$

$$f(1,10) = f_1(1) f_2(10) = (.6)(.5) = .30$$

$$f(1,15) = f_1(1) f_2(15) = (.6)(.3) = .18$$

$$f(2,10) = f_1(2) f_2(10) = (.4)(.5) = .20$$

$$f(2,5) = f_1(2) f_2(5) = (.4)(.2) = .08$$

$$f(2,15) = f_1(2) f_2(15) = (.4)(.3) = .12$$

Since X & Y are independent r.v.s
 $\Rightarrow f_1(x) f_2(y) = f(x,y)$

$$\therefore f(x,y) = \begin{cases} .12 & \text{for } x=1, y=5 \\ .08 & \text{for } x=2, y=5 \\ .30 & \text{for } x=1, y=10 \\ .20 & \text{for } x=2, y=10 \\ .18 & \text{for } x=1, y=15 \\ .12 & \text{for } x=2, y=15 \\ 0 & \text{o.w.} \end{cases}$$

$$② E(XY) = \sum_x \sum_y XY f(x,y)$$

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Q13: ① $\int_0^1 f(x|y) dx = 1$ (By cond.(2) of pr.f.)

$$C_1 \int_0^1 \frac{x}{y^2} dx = 1 \Rightarrow C_1 = 2y^2$$

$$\therefore f(x|y) = \begin{cases} 2x & \text{for } 0 < x < y \text{ or } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

② $\int_0^1 f_2(y) dy = 1$ (by cond.(2) of pr.f.)

$$C_2 \int_0^1 y^4 dy = 1 \Rightarrow C_2 = 5$$

$$\therefore f_2(y) = \begin{cases} 5y^4 & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \text{c) } f(x,y) &= \frac{f(x|y)}{f_2(y)} \Rightarrow f(x,y) = f(x|y) f_2(y) = (2x)(5y^4) \\ &= \int_0^1 10xy^4 \text{ for } 0 < x < y \text{ or } 0 < x < 1 \\ &\quad \text{o.w.} \end{aligned}$$

$$\text{b) } f_1(x) = \int_y f(x,y) dy = \int_0^1 xy^4 dy = \int_0^1 2x \text{ for } 0 < x < 1 \text{ or } 0 < x < y \text{ o.w.}$$

$$\text{d) } P(-\frac{1}{2} < x < \frac{1}{2} | y = \frac{5}{8}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x|y = \frac{5}{8}) dx = \int_0^{\frac{5}{8}} 10x(\frac{5}{8})^4 dx$$

$$\text{e) } P(\frac{1}{4} < x < \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} f_1(x) dx = \int_{\frac{1}{4}}^{\frac{1}{2}} 2x dx = \dots = \frac{3}{16}$$