f is $(1,-1)$ and onto then f is isomorphism. .3

Definition: if $f: R \longrightarrow R'$ and f is isomorphism then we say that R is isomorphic to $R', R \simeq R'$.

Remark: if $f: R \longrightarrow R', f$ is homomorphism, then:

$f(O_R) = O_{\dot{R}}$. .1

$f(-a) = -f(a)$. .2

$f(l_R) = 1_{\dot{R}}$ when R and R' is a ring with identity. .3

Theorem: Any ring can be imbedded in a ring with identity.

Proof: Let $R \times Z = \{(r,n), r \in R, n \in Z\}$

Define + and . on $R \times Z$ as follows

$$(r,n) + (t,m) = (r+t, n+m)$$
$$(r,n).(t,m) = (rt + nt + mr, nm)$$

then $R \times Z$ is a ring with identity (0,1).

$$(r,n) \cdot (0,1) = (r,n)$$
$$R \times \{0\} \subseteq R \times Z$$

Now we must show that $R \times \{0\}$ is subring of $R \times Z$

$$(a,0)\{\in R \times \{0\}\} - (b,0)\{\in R \times \{0\}f = (a-b,0) \in R \times \{0\}$$
$$(a,0).(b,0) = (ab,0) \in R \times \{0\}$$

Now we define a map $\emptyset: R \rightarrow R \times \{0\}, \emptyset(r) = (r,0)$

(1) Let $\emptyset(r_1) = \emptyset(r_2)$

$$(r_1,0) = (r_2,0) \Rightarrow r_1 = r_2 \therefore \emptyset \text{ is } (1-1)$$

(2) let $(w,0) \in R \times \{0\}, \therefore \emptyset(w) = (w,0) \therefore \emptyset$ is onto, $\emptyset$ is homo.

(3) $\emptyset(r_1 + r_2) = (r_1 + r_2, 0) = (r_1, 0) + (r_2, 0) = \emptyset(r_1) + \emptyset(r_2)$

$$\emptyset(r_1 \cdot r_2) = (r_1 r_2, 0)$$
$$\emptyset(r_1) \cdot \emptyset(r_2) = (r_1, 0) \cdot (r_2, 0) = (r_1 r_2, 0)$$
$$\therefore R \cong R \times \{0\}$$

$\therefore$ R is imbedded in a ring $R \times Z$.

Definition: Let R be a ring an element $a \in R$ is said to be idempotent element if $a^2 = a$. And a is nilpotent if there exists an integer n such that $a^n = 0$.

Example: (1) $Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$

Solution: $\overline{0}, \overline{1}, \overline{3}, \overline{4}$ are idempotent. $\overline{0}$ is nilpotent only.

(2) $Z_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$

Solution: $\overline{0}, \overline{2}, \overline{4}, \overline{6}$ are nilpotent.

(3) Z_5 : the idempotent $\overline{0}, \overline{1}$ and nilpotent is $\overline{0}$.

(4) $(p(x), \Delta, \cap)$

Solution: $A \cap A = A, \forall A$ is idempotent $A \cap \ldots \cap A = \emptyset$, just when $A = \emptyset$

Definition: Let R be a ring such that every element of R is idempotent then R is Boolean ring.

Example : in $Z_2 = \{0,1\}, (\overline{0})^2 = 0, (\overline{1})^2 = 1$.

Theorem: Let R be a ring such that every element in R is idempotent (R is Boolean ring), then R is commutative.

Proof: $(a+b)=(a+b)^2=(a+b)(a+b)=a\cdot a+a\cdot b+b\cdot a+b\cdot b$

$$a+b=a^2+a\cdot b+b\cdot a+b^2$$
$$a+b=a+b+a\cdot b+b\cdot a$$
$$0=ab+ba\Rightarrow ab=-ba$$

$$\mathrm{ab}=(-\mathrm{ba})=(-\mathrm{ba})^2=\mathrm{b}^2\mathrm{a}^2=\mathrm{ba}$$

∴ R is commutative.

Remark: Let R be a ring commutative if there exists element $a\in R$, such that:

(1) a is idempotent.

(2) a is not zero divisor. Then a must be the identity of the ring.

Proof: (2) Let $b\in$ R

$$a\cdot b=a^2b\Rightarrow(a^2\cdot b)-a\cdot b=0$$
$$a(ab-b)=0[a \text{ is not zero divisof }]$$

$\therefore \mathrm{ab}-\mathrm{b}=0\Rightarrow \mathrm{ab}=\mathrm{b}$

∴ a is identity.

Example: Consider the ring $(\mathrm{p(x)},\Delta,\cap)\mathrm{p(x)}=\{\mathrm{A}:\mathrm{A}\subseteq \mathrm{X}\}$, for a fixed subset $S\subseteq X, S\in p(x)$, define $f:p(x)\longrightarrow p(x)$

$$\mathrm{f(\ A)}=\mathrm{A}\cap\ \mathrm{S}$$

(1) $\mathrm{A}=\mathrm{B}\Rightarrow \mathrm{A}\cap \mathrm{S}=\mathrm{B}\cap \mathrm{S}$

$$\therefore \mathrm{f(\ A)}=\mathrm{f(\ B)}\ \therefore \text{f is well define}$$

(2) $(A\Delta B)=f(A)\Delta f(B)$?

$$
\begin{aligned}
f(A\Delta B) &= (A\Delta B)\cap S\\
&= [(A-B)\cup(B-A)]\cap S\\
&= [(A-B)\cap S]\cup[(B-A)\cap S]\\
&= (A\cap S - B\cap S)\cup(B\cap S - A\cap S)\\
&= (A\cap S)\Delta(B\cap S) = f(A)\Delta f(B)
\end{aligned}
$$

(2) $f(A\cap B) = (A\cap B)\cap S = (A\cap S)\cap(B\cap S) = f(A)\cap f(B)$

∴ f is homo.

(3) $\ker f = \{A\subseteq p(x): f(A)=\emptyset\} = \{A\subseteq p(x): A\cap S=\emptyset\} = S^{c} \neq$ identity

(4) $\forall A\subseteq X \Rightarrow X\cap A = A$, identity $= X$

∴ f is not $(1-1)$

Problems:

Let R be a ring and $a\in R$, If C (a) the set of all elemente with a, .1
$C(a)=\{r\in R, ra = \text{ar}\}$ show that $C(a)$ is subring of R. and Cent
$R = n_{a\in R}$ C(a).

Let $(G,+)$ be abelian group, R set of all groups hommo. of G in to .2
itself, $f+g(x) = f(x)+g(x), f^{\circ}g(x) = f(g(x))$, show that
$(R,+,^{\circ})$ form a ring, determine the invertible elements of R.

Given that fis homo. from the ring R in to the ring R \,prove that .3
A. $f(\text{cent}(R))\subseteq \text{cent}(f(R))$
B. If $a\in R$ is nilpotent then $f(a)$ is nilpotent in R.
C. If R has positive characteristic then char $f(R)\leq \text{char}R$.

Let R be a ring without zero divisors: .4
i. $a\cdot b = 1$ iff $b\cdot a = 1$
ii. If $a^2 = 1$ then either $a = 1$ or $a = -1$.

Sol (i):

if $a.b = 1$ then $b \neq 0$

[if $b = 0 \therefore a.0 = 0 \neq 1$]

$\therefore a \cdot b = l \Rightarrow b \cdot a \cdot b = b$

$bab - b = 0 \therefore (ba - 1)b = 0$, $b \neq 0 \therefore ba = 1$

Sol (ii):

$$a^2 = 1, a \cdot a = 1 - a + a$$
$$a \cdot a + a - a - 1 = 0$$
$$a(a + 1) - (a + 1) = 0$$
$$(a + 1)(a - 1) = 0$$

Either $a = 1$ or $a = -1$

Definition: Let I be a nonempty subset of ring R , then I is ideal of R if

(1) $a - b \in I \forall a, b \in I$.

(2) $ar \in I \forall a \in I, r \in R$.

(3) $I \neq \emptyset$.

Remark: Every ideal is subring.

Proof: Let I be an ideal, to show that I is subring

(1) $= I \neq \emptyset$

(2) $a, b \in I \Rightarrow a.b \in I, a - b \in I$

$\therefore$ I is subring

But the converse is not true for example:

($Q, +, .$) is a ring, $Z \subseteq Q, Z$ is subring

$3 \in Z, \frac{1}{2} \in Q, 3.\frac{1}{2} = \frac{3}{2} \notin Z$

$\therefore$ Z is not ideal

Example: In Z,

(1) $2Z$ is subring and ideal;

(2) $5Z, 3Z$ is ideal.

In general nZ is an ideal.

Remar$^{(*)}$ () Let I be an ideal of a ring with 1 . If $1 \in I$ then $I = R$.

Proof: $I \subseteq R$, let $r \in R, 1 \in I$ but I is ideal

$\therefore 1.r \in I \Rightarrow r \in I \Rightarrow R \subseteq I. \therefore I = R$

Remark: Let I be an ideal of a ring with I and I contains an invertible element then $I = R$.

Proof: $a \in I$ but a is invertible then $\exists b \in R$ such that $a \cdot b \in I \Rightarrow 1 \in I$

$\therefore I = R$, by remark (*)

Definition: An ideal I of a ring R is called a proper ideal if $I \neq R$ and I is called nontrivial ideal if $I \neq \{0\}$ and $I \neq R$.

Theorem: Let $\{I_\alpha, \alpha \in \Lambda\}$ be a family of ideals of a ring R , then $\bigcap_{\alpha\in\Lambda} I_\alpha$ is an ideal in R.

Proof: $\cap_{\alpha\in\Lambda} I_\alpha \neq \emptyset [0 \in I_\alpha \forall \alpha \in \Lambda]$

let $a, b \in \bigcap_{\alpha\in\Lambda} I_\alpha \Rightarrow a \in I_\alpha \forall \alpha \in \Lambda$ and $b \in I_\alpha \forall \alpha \in \Lambda$

$\therefore a - b \in I_\alpha \forall \alpha \in \Lambda$ [ideal def.] $\therefore a - b \in \bigcap_{\alpha\in\Lambda} I_\alpha$

Let $a \in \bigcap_{\alpha\in\Lambda} I_{\alpha,}, r \in R$

$\therefore a \in I_\alpha \forall \alpha \in \Lambda \Rightarrow ra \in I_\alpha \forall \alpha \therefore ra \in \bigcap_{\alpha\in\Lambda} I_\alpha \therefore \bigcap_{\alpha\in\Lambda} I_\alpha$ is ideal.

But the union is not ideal for example:

2 Z is ideal, 3 Z is ideal, $2 \in 2Z, 3 \in 3Z$

if $2Z \cup 3Z$ is ideal