Functions

Definition Let f be a relation defined from a set A to set B, then f is called a function "denote by $f: A \to B$" if the following holds:-

1) $\forall a \in A, \exists b \in B$ such that $(a,b) \in f$

2) If $(a,b_1) \in f$ and $(a,b_2) \in f$ then $b_1 = b_2$

The set A is called the domain of the function f, and B is called the co-domain of f. Further, if $a \in A$ then the element in B which is assigned to a is called the image of a and is denoted by $f(a)$.

Example:-

1) Let $A = \{a,b,c,d\}$ and $B = \{a,b,c\}$. Define a function f of A into B by the correspondence $f(a) = b$, $f(b) = c$, $f(c) = c$ and $f(d) = b$.

By this definition, the image, for example, of b is c.

2) Let $A = \{1,2,3\}$

ⓐ $g: A \to A$ define by $g = \{(1,2),(3,1)\}$ is not function because not for each $a \in A$, i.e. $2 \in A$ $\nexists b \in B$ s.t $(2,b) \in g$.

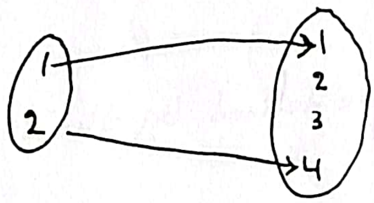
ⓑ $h: A \to A$ define by $h = \{(1,3),(2,1),(1,2),(3,1)\}$ is not function since $(1,2) \in h$ & $(1,3) \in h$ but $2 \neq 3$

Equal function

If f and g are functions defined on the same domain D and if $f(a) = g(a)$ for every $a \in D$, then the functions f and g are equal and we write $f = g$.

Example Let $f(x) = x^2$ where x is a real number. Let $g(x) = x^2$ where x is a complex number. Then the function f is not equal to g since they have different domains.

Example Let the function f be defined by the diagram



Let a function g be defined by the formula $g(x) = x^2$ where the domain of g is the set $\{1, 2\}$. Then $f = g$ since they both have the same domain and since f and g assign the same image to each element in the domain.

Range of a function Let f be a mapping of A into B, that is, $f: A \to B$ we define the range of f to consist of those elements in B which appear as the image of at least one element in A.

we denote the range of $f: A \to B$ by $f(A)$; i.e.

$f(A) = \{b \in B ; (a,b) \in f\}$

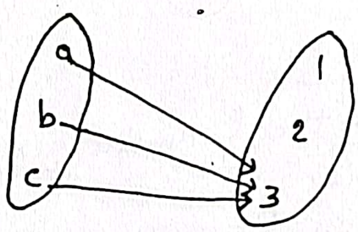
Identity function Let A be any set. Let the function $f: A \longrightarrow A$ be defined by the formula $f(x) = x$ for each element in A.

Then f is called the identity function and denote by I or I_A.

Constant function A function f of A into B is called a constant function if the same element $b \in B$ is assigned to every element in f

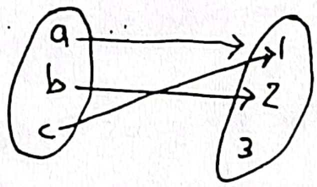
In other words, $f: A \longrightarrow B$ is a constant function if the range of f consists of only one element.

Example Let the function f be defined by the diagram:



Then f is a constant function since 3 is assigned to every element in A

Example Let the function f be defined by the diagram:



Then f is not a constant function since the range of f consists of both 1 and 2.

Inclusion function

دالة الاحتواء

Let A, B be two sets such that $A \subseteq B$, then the function

$i : A \longrightarrow B$ define by $i(a) = a$ for every $a \in A$ is called

inclusion function.

Definition Let f be a function of A into B. Then f is called

1) One - One [injective]

if $x_1 \neq x_2 \longrightarrow f(x_1) \neq f(x_2)$

or

if $f(x_1) = f(x_2) \longrightarrow x_1 = x_2$

2) onto [surjective]

if for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

i.e. $f(A) = B$

3) bijective

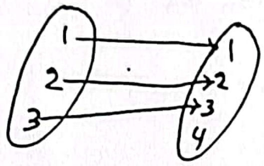
if f is both injective and surjective

Example

1) Both of the identity and inclusion functions are injective

Since $i(x_1) = i(x_2) \longrightarrow x_1 = x_2$

2) Identity function is surjective, but the inclusion function is not surjective, for example



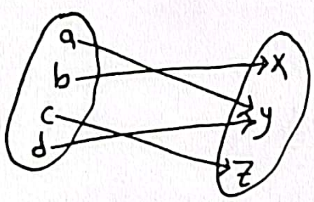
3) Let $A = \{a, b, c, d\}$, $B = \{a, b, c\}$. Define by

$f(a) = b$, $f(b) = c$, $f(c) = c$ & $f(d) = b$

then $f(A) = \{b, c\} \neq B$

So that f is not onto

4) Let $A = \{a, b, c, d\}$, $B = \{x, y, z\}$, Let $f: A \longrightarrow B$ define by



f is not injective because $f(a) = f(d)$ but $a \neq d$.

f is surjective, Notice that $f(A) = \{x, y, z\} = B$, thus f is onto

Composition function

تركيب الدوال

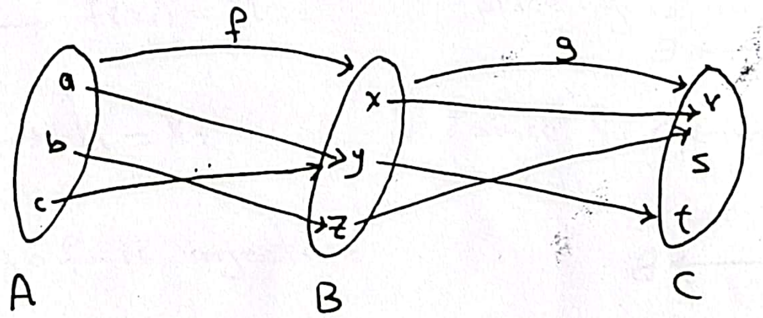
Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions then we define a function $(g \circ f) : A \longrightarrow C$ by

$$(g \circ f)(a) = g(f(a)) \qquad \forall\, a \in A$$

This new function is called the composition function of f and g.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "(g \circ f)"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Example Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be defined by the diagrams.



We compute $(g \circ f) : A \longrightarrow C$ by its definition

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$

$$(g \circ f)(b) = g(f(b)) = g(z) = r$$

$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Example Let $f: \mathbb{N} \longrightarrow \mathbb{N}$, $g: \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$f(x) = 3x^2 \quad \& \quad g(x) = 4x^2 + 2$$

Now,

$$(g\circ f)_{(x)} = g(f(x)) = g(3x^2) = 4(3x^2)^2 + 2 = 36x^4 + 2$$

$$(f\circ g)_{(x)} = f(g(x)) = f(4x^2+2) = 3(4x^2+2)^2 = 3(16x^4 + 16x^2 + 4)$$

It is clear that $\quad g\circ f \neq f\circ g$

Theorem Let $f: A \longrightarrow B$. Then

$$f\circ I_A = f \quad \text{and} \quad I_B \circ f = f$$

Proof

$$f: A \longrightarrow B$$

$$I_A: A \longrightarrow A$$

$$f\circ I_A : A \longrightarrow B$$

$$(f\circ I_A)(x) = f(I_A(x))$$

$$= f(x)$$

So that $f\circ I_A = f$

Similar $I_B \circ f = f$

Theorem Let $f: A \longrightarrow B$, $g: B \longrightarrow C$ be two functions. Then

1- If each f, g are injective then $g \circ f$ is injective.

2- If each f, g are surjective then $g \circ f$ is surjective.

3- If each f, g are bijective then $g \circ f$ is bijective.

4- If $g \circ f$ is injective then f is injective.

5- If $g \circ f$ is surjective then g is surjective.

6- If $g \circ f$ is bijective then f is injective and g is surjective.

Proof

1- Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$

$\longrightarrow g(f(x_1)) = g(f(x_2))$

$\longrightarrow f(x_1) = f(x_2)$ since g is injective

$\longrightarrow x_1 = x_2$ since f is injective

hence $g \circ f$ is injective

2- $g \circ f : A \longrightarrow C$

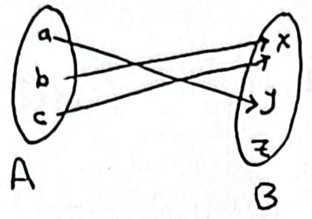
Let $y \in C$, since g is onto, $\exists\, b \in B$ s.t $g(b) = y$

Now $b \in B$, since f is onto, $\exists\, a \in A$ s.t $f(a) = b$

$(g \circ f)(a) = g(f(a)) = g(b) = y$

hence $g \circ f$ is onto

Example Let the function $f: A \longrightarrow B$ be defined by the diagram



Then $f^{-1}(x) = \{b, c\}$, $f^{-1}(y) = \{a\}$, $f^{-1}(z) = \phi$.

Definition

Let $f: X \longrightarrow Y$ be a function, Let $A \subseteq X, B \subseteq Y$. Then the set

$\{f(x) ; x \in A\}$ is called "the direct image" of the set A under the function f and denoted by $f(A)$ i.e. $f(A) = \{f(x) ; x \in A\}$

$$f(x) \in f(A) \longleftrightarrow x \in A$$

Definition

Let $f: X \longrightarrow Y$ be a function, Let $A \subseteq X$; $B \subseteq Y$. The set

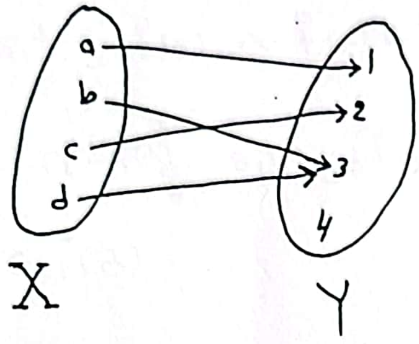
$\{x \in X ; f(x) \in B\}$ is called the opposite image of the set B under the function f and denoted by $f^{-1}(B)$ i.e.

$$f^{-1}(B) = \{x \in X ; f(x) \in B\}$$

$$x \in f^{-1}(B) \longleftrightarrow f(x) \in B$$

Remark $y \in f(A) \longleftrightarrow \exists x \in A$ s.t $y = f(x)$

Example Let the function $f: X \longrightarrow Y$ be defined by the diagram



Let $A = \{a, b, c\}$, $B = \{1, 2\}$

Then $f(A) = \{f(a), f(b), f(c)\} = \{1, 3, 2\}$

$f^{-1}(B) = \{f^{-1}(1), f^{-1}(2)\} = \{a, c\}$

Example Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ define by $f(x) = x^2 + 3$

Let $A = \{1, 0, 5\}$, $B = \{3, 4, 28\}$

$f(A) = \{f(1), f(0), f(5)\}$

$= \{4, 3, 28\}$

$f^{-1}(B) = \{f^{-1}(3), f^{-1}(4), f^{-1}(28)\}$

$= \{x^2 + 3 = 3 , x^2 + 3 = 4 , x^2 + 3 = 28\}$

$= \{x^2 = 0 , x^2 = 1 , x^2 = 25\}$

$= \{0, \mp 1, \mp 5\}$

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Theorem. Let $f: X \longrightarrow Y$ be a function, Let $A, B \subseteq X$. Then

1) $f(A \cup B) = f(A) \cup f(B)$

2) $f(A \cap B) \subseteq f(A) \cap f(B)$

3) $f(A) - f(B) \subseteq f(A - B)$

proof

1) Let $y \in f(A \cup B)$

$\rightarrow \exists x \in (A \cup B)$ s.t $y = f(x)$

$\rightarrow \exists (x \in A \vee x \in B)$ s.t $y = f(x)$

$\rightarrow \exists x \in A$ s.t $y = f(x)$ $\vee$ $\exists x \in B$ s.t $y = f(x)$

$\rightarrow f(x) \in f(A)$ s.t $y = f(x)$ $\vee$ $f(x) \in f(B)$ s.t $y = f(x)$

$\rightarrow y \in f(A) \vee y \in f(B) \longrightarrow y \in f(A) \cup f(B)$

$\therefore f(A \cup B) \subseteq f(A) \cup f(B)$ ---- (1)

Let $y \in f(A) \cup f(B)$

$\rightarrow y \in f(A) \vee y \in f(B)$

$\rightarrow \exists x \in A$ s.t $y = f(x)$ $\vee$ $\exists x_1 \in B$ s.t $y = f(x_1)$

$\rightarrow \exists x \in A \cup B$ s.t $y = f(x)$ $\vee$ $\exists x_1 \in A \cup B$ s.t $y = f(x_1)$

$\rightarrow f(x) \in f(A \cup B)$ s.t $y = f(x)$ $\vee$ $f(x_1) \in f(A \cup B)$ s.t $y = f(x_1)$

$\rightarrow y \in f(A \cup B) \vee y \in f(A \cup B)$

$\rightarrow y \in f(A \cup B) \longrightarrow f(A) \cup f(B) \subseteq f(A \cup B)$ -- (2)

from (1) & (2) we get the result.

2). Let $y \in f(A \cap B)$

$\rightarrow \exists\, x \in A \cap B$ s.t $y = f(x)$

$\rightarrow \exists (x \in A \wedge x \in B)$ s.t $y = f(x)$

$\rightarrow \exists\, x \in A$ s.t $y = f(x) \wedge \exists\, x \in B$ s.t $y = f(x)$

$\rightarrow f(x) \in f(A)$ s.t $y = f(x) \wedge f(x) \in f(B)$ s.t $y = f(x)$

$\rightarrow f(x) \in f(A) \cap f(B)$ s.t $y = f(x)$

$\rightarrow y \in f(A) \cap f(B)$

$\therefore f(A \cap B) \subseteq f(A) \cap f(B)$

3) Let $y \in f(A) - f(B)$

$\rightarrow y \in f(A) \wedge y \notin f(B)$

$\rightarrow \exists\, x \in A$ s.t $y = f(x) \wedge f(x) \notin f(B)$

$\rightarrow \exists\, x \in A$ s.t $y = f(x) \wedge x \notin B$

$\rightarrow \exists\, x \in A \wedge x \notin B$ s.t $y = f(x)$

$\rightarrow x \in A - B$ s.t $y = f(x)$

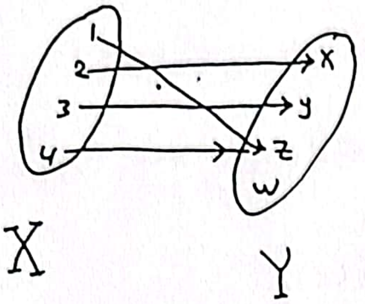
$\rightarrow f(x) \in f(A - B)$

$\rightarrow y \in f(A - B)$

$\therefore f(A) - f(B) \subseteq f(A - B)$

Remark In general the inverse of ② and ③ is not true.

for example



Let $A = \{1, 3\}$, $B = \{2, 4\}$

$A \cap B = \phi \longrightarrow f(A \cap B) = \phi$

$f(A) = \{y, z\}$

$f(B) = \{x, z\}$

$f(A) \cap f(B) = \{z\}$

$\therefore f(A) \cap f(B) \not\subseteq f(A \cap B)$

Also

$A - B = \{1, 3\}$

$f(A - B) = \{z, y\}$

$f(A) - f(B) = \{y\}$

It is clear that $f(A) - f(B) \subseteq f(A - B)$ and

$$f(A - B) \not\subseteq f(A) - f(B)$$

Theorem Let $f: X \longrightarrow Y$ be a function, Let A and B are subsets of X.

Then

1) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

2) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

3) $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$

Proof

2) Let $x \in f^{-1}(A \cap B)$

$\longleftrightarrow f(x) \in (A \cap B)$

$\longleftrightarrow f(x) \in A \wedge f(x) \in B$

$\longleftrightarrow x \in f^{-1}(A) \wedge x \in f^{-1}(B)$

$\longleftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)$

so that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

3) Let $x \in f^{-1}(A - B)$

$\longleftrightarrow f(x) \in A - B$

$\longleftrightarrow f(x) \in A \wedge f(x) \notin B$

$\longleftrightarrow x \in f^{-1}(A) \wedge x \notin f^{-1}(B)$

$\longleftrightarrow x \in f^{-1}(A) - f^{-1}(B)$

So that $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$

Theorem Let $f: X \longrightarrow Y$ be a map and Let $A \subseteq X$. Then

1) $A \subseteq f^{-1}(f(A))$

2) If f is injective then $A = f^{-1}(f(A))$

proof

1) Let $x \in A$

$\rightarrow f(x) \in f(A) \equiv B$

(Since $w \in f^{-1}(B) \longleftrightarrow f(w) \in B$)

$\rightarrow x \in f^{-1}(B) = f^{-1}(f(A))$

$\therefore x \in f^{-1}(f(A))$

So that $A \subseteq f^{-1}(f(A))$

2) suppose that f is injective and prove $f^{-1}(f(A)) \subseteq A$

Let $w \in f^{-1}(f(A))$

$\rightarrow f(w) \in f(A)$

[by remark $y \in f(A) \longleftrightarrow \exists x \in A$ s.t $y = f(x)$]

So that there exist $x \in A$ s.t $f(w) = f(x)$

but f injective, we get $w = x$

$\therefore w \in A$

from part 1 & 2 we get $f^{-1}(f(A)) = A$

Theorem Let $f: X \longrightarrow Y$ be a map, and Let $B \subseteq Y$. Then

1) $f(f^{-1}(B)) \subseteq B$

2) If f is surjective then $f(f^{-1}(B)) = B$

Proof

1) Let $y \in f(f^{-1}(B))$

$\rightarrow \exists x \in f^{-1}(B)$ s.t $y = f(x)$

$\rightarrow f(x) \in B$ s.t $y = f(x)$

$\rightarrow y \in B$

$\therefore f(f^{-1}(B)) \subseteq B$

2) suppose that f is surjective and prove $B \subseteq f(f^{-1}(B))$

Let $y \in B$

Since $B \subseteq Y \rightarrow y \in Y$

but f is surjective, so that there exist $x \in X$ s.t $f(x) = y$

$\therefore f(x) \in B$

$\rightarrow x \in f^{-1}(B)$

$\rightarrow f(x) \in f(f^{-1}(B))$

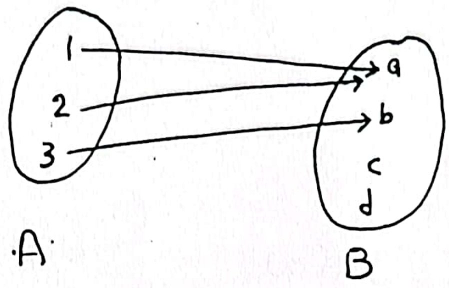
$\rightarrow y \in f(f^{-1}(B))$

$\therefore B \subseteq f(f^{-1}(B))$

from part ① & ② we get $f(f^{-1}(B)) = B$

Inverse function   دالة المعكوس

Let $f: A \longrightarrow B$ define by



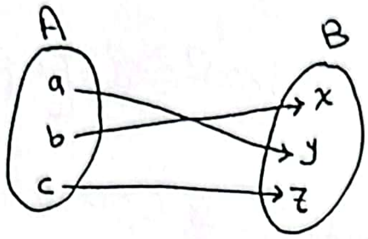
$f^{-1}(a) = \{1, 2\}$, $f^{-1}(b) = \{3\}$

$f^{-1}(c) = \phi$, $f^{-1}(d) = \phi$

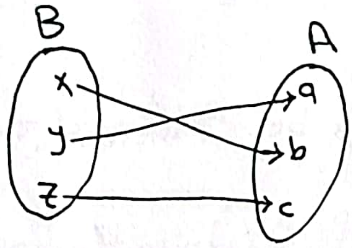
In the above example, we can show that if $x \in B$ then $f^{-1}(x)$ possible consist of two elements [if the function is not injective], OR $f^{-1}(x)$ equal to the empty set [if the function is not surjective]. But if the function is bijective then $f^{-1}(x)$ consist of only one element. In this case we can define function from B to A denote by $f^{-1}: B \longrightarrow A$, f^{-1} is called inverse function of the function f.

Example

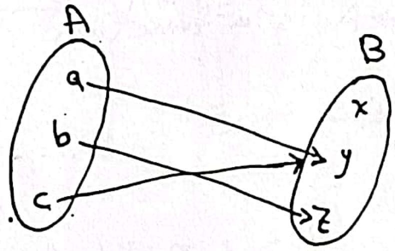
1) Let $f : A \longrightarrow B$ be a function define by



It is clear that f is bijective, so that f^{-1} exists define in the form

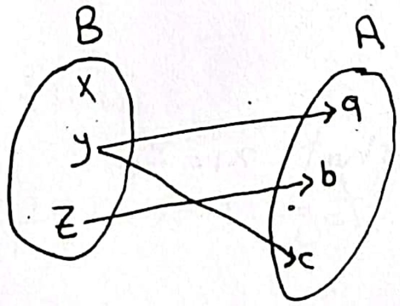


2) Let $f : A \longrightarrow B$ be a function define by



It is clear that f is not injective $[f(a) = f(c) = y$ but $a \neq c]$

So that we cannot find $f^{-1} : B \longrightarrow A$



f^{-1} is not function

Remark

1) Let $f: A \longrightarrow B$ be a function. Then

$$(x,y) \in f \longleftrightarrow (y,x) \in f^{-1} \; [\, f(x)=y \longleftrightarrow x=f^{-1}(y) \,]$$

2) Let $f: A \longrightarrow B$ be a function. When we say that f has inverse we mean $f^{-1}: B \longrightarrow A$ is a function (i.e. satisfy condition of function).

Proposition Let $f: A \longrightarrow B$ be a function. Then

f has inverse iff f is bijective

Proof $\Longrightarrow$) suppose that f has inverse

i.e $f^{-1}: B \longrightarrow A$ is a function

1) f is injective

Let $f(x_1) = f(x_2) = y$

$\rightarrow (x_1, y) \in f \wedge (x_2, y) \in f$

$\rightarrow (y, x_1) \in f^{-1} \wedge (y, x_2) \in f^{-1}$

but f^{-1} is a function, so that $x_1 = x_2$

Hence f is 1-1

2) f is surjective

Let $y \in B$, since f^{-1} is a function so that there exist $x \in A$ s.t

$f^{-1}(y) = x$ i.e. $(y, x) \in f^{-1}$

$\rightarrow (x, y) \in f \longrightarrow f(x) = y$

So that f is onto

from ① & ② we get f is bijective

$\Leftarrow$ suppose that f is bijective

1) Let $y \in B$, since f is onto

$\rightarrow \exists x \in A$ s.t $f(x) = y \longrightarrow (x, y) \in f$

$\rightarrow (y, x) \in f^{-1} \longrightarrow f^{-1}(y) = x$

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2) Let $(y, x_1) \in f^{-1} \wedge (y, x_2) \in f^{-1}$

$\rightarrow (x_1, y) \in f \wedge (x_2, y) \in f$

$\rightarrow f(x_1) = y \wedge f(x_2) = y$

$\rightarrow f(x_1) = f(x_2)$

but f is 1-1, we get $x_1 = x_2$

$\therefore$ By ① & ② f has inverse.

Proposition Let $f: A \longrightarrow B$ be a function. If f has inverse then

1) $f^{-1} \circ f = I_A$

2) $f \circ f^{-1} = I_B$

Proof

1)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "f^{-1} \circ f = I_A"'] & B \arrow[r, "f^{-1}"] & A \end{tikzcd}$$

$f: A \longrightarrow B$

$f^{-1}: B \longrightarrow A$

$I_A: A \longrightarrow A$ s.t $I_A(x) = x$

$$(f^{-1} \circ f)_{(x)} = f^{-1}(f(x))$$

$$= f^{-1}(y) \qquad (f(x) = y \text{ iff } f^{-1}(y) = x)$$

$$= x = I_A(x)$$

2) $f \circ f^{-1}(y) = f(f^{-1}(y))$

$$= f(x)$$

$$= y$$

$$= I_B(y)$$

$\therefore f \circ f^{-1} = I_B$

proposition

Let $f: A \longrightarrow B$ be a function. If f has inverse then each of f, f^{-1} are bijective.

proof Since f has inverse, then by preceding proposition we have

$$f^{-1} \circ f = I_A \quad , \quad f \circ f^{-1} = I_B$$

Since I_A and I_B are bijective then $f^{-1} \circ f$ is bijective

so that we get f is 1-1 and f^{-1} is onto [?]

Also $f \circ f^{-1}$ is bijective

$\rightarrow f^{-1}$ is 1-1 and f is onto

Hence f & f^{-1} are bijective.

proposition

Let $f: A \longrightarrow B$ be a function. If f has inverse then it is unique.

proof suppose that $g: B \longrightarrow A$ be another inverse of f

So that $f \circ f^{-1} = I_B \quad , \quad f^{-1} \circ f = I_A$ [?]

$$f \circ g = I_B \quad , \quad g \circ f = I_A$$

Now,

$$f \circ g = f \circ f^{-1}$$

$$f^{-1} \circ (f \circ g) = f^{-1} \circ (f \circ f^{-1})$$

$$(f^{-1} \circ f) \circ g = (f^{-1} \circ f) \circ f^{-1}$$

$$I_A \circ g = I_A \circ f^{-1}$$

$$g = f^{-1}$$

Hence the inverse is unique.

Proposition If each of $f : A \longrightarrow B$ and $g : B \longrightarrow C$ have inverse then the composition function $g \circ f : A \longrightarrow C$ has inverse and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof Since each of f and g have inverse, then each of f and g are bijective [ ]

So that $g \circ f$ is bijective [ ]

$\rightarrow$ $g \circ f$ has inverse [ ]

By preceding proposition we have

$$(g \circ f)^{-1} \circ (g \circ f) = I_A$$

$$(g \circ f) \circ (g \circ f)^{-1} = I_C$$

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f$$

$$= f^{-1} \circ (I_B) \circ f$$

$$= f^{-1} \circ f$$

$$= I_A$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1}$$

$$= g \circ (I_B) \circ g^{-1}$$

$$= g \circ g^{-1}$$

$$= I_C$$

Hence each of $(g \circ f)^{-1}$ and $(f^{-1} \circ g^{-1})$ are inverse of $(g \circ f)$

But the inverse is unique, we get [ ]

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$