Equivalence classes

صفوف التكافؤ

Let R be an equivalence relation defined on A. For each $a \in A$ the set

$$[a] = \{x \in A : (x,a) \in R\} = \{x \in A : xRa\}$$

is called the equivalence class of a and a is called the representation of the equivalence class.

Example:- Let $A = \{0,1,2,3\}$, and Let R be a relation defined on A by

$$R = \{(0,0),(1,1),(2,2),(3,3),(0,2),(1,3),(2,0),(3,1)\}$$

find the equivalence classes of A.

Solution:- It is clear that R is an equivalence relation ?

$[0] = \{x \in A : (x,0) \in R\} = \{0,2\}$

$[1] = \{x \in A : (x,1) \in R\} = \{1,3\}$

$[2] = \{x \in A ; (x,2) \in R\} = \{2,0\} = [0]$

$[3] = \{x \in A : (x,3) \in R\} = \{1,3\} = [1]$

so that, the equivalence classes are $[0]$, $[1]$

Example:- Let R be a relation defined on $\mathbb{Z}$ as follows

$$aRb \text{ iff } a-b=2k, k\in\mathbb{Z}.$$

Prove that R is an equivalence relation and find equivalence classes.

Solution:- R is an equivalence relation ?

Now, we found the equivalence classes

$[0]=\{x\in\mathbb{Z};\ xR0\}=\{x\in\mathbb{Z};\ x-0=2k, k\in\mathbb{Z}\}$

$=\{x\in\mathbb{Z};\ x=2k, k\in\mathbb{Z}\}=\{----, -6, -4, -2, 0, 2, 4, 6, ---\}$

$[1]=\{x\in\mathbb{Z};\ x-1=2k, k\in\mathbb{Z}\}$

$=\{x\in\mathbb{Z},\ x=2k+1, k\in\mathbb{Z}\}$

$=\{---, -5, -3, -1, 1, 3, 5, ---\}$

$[2]=\{x\in\mathbb{Z};\ xR2, k\in\mathbb{Z}\}$

$=\{x\in\mathbb{Z};\ x-2=2k, k\in\mathbb{Z}\}$

$=\{---, -6, -4, -2, 0, 2, 4, 6, ---\}$

$=[0]$

In general we can show that

$[0]=[2]=[4]=[6]=---$

$[1]=[3]=[5]=---$

Hence we have only two equivalence classes $[0]$, $[1]$

Remark:- Let R be an equivalence relation defined on A. Then For each $a \in A$

1- $[a] \subseteq A \quad \forall a \in A$

2- $[a] \neq \phi \quad \forall a \in A$

proof

② By definition of equivalence class

$$[a] = \{x \in A \,;\, xRa\} \quad \text{i.e} \quad x \in [a] \longleftrightarrow xRa$$

Since R is equivalence relation so that R is reflexive relation

hence $aRa \quad \forall a \in A$

$\rightarrow \quad a \in [a]$

So that, $[a] \neq \phi$

Proposition:- Let R be an equivalence relation defined on A, Let $a, b \in A$ Then

1- aRb iff $[a] = [b]$

2- $b \in [a]$ iff $[a] = [b]$

proof:- ① $\Rightarrow$) Let $x \in [a]$

$$x \in [a] \longleftrightarrow xRa$$

Now, aRb, xRa

Since R is an equivalence relation

so that $x R a \wedge a R b$ then $x R b$ since R is transitive relation

i.e $x R b \longleftrightarrow x \in [b]$

hence $[a] \subseteq [b]$ - - - ①

Let $y \in [b] \longrightarrow y R b$

Since $a R b$ and R is symmetric relation, hence $b R a$

$\rightarrow y R b \wedge b R a$ then $y R a$ since R is transitive relation

i.e $y R a \longleftrightarrow y \in [a]$

hence $[b] \subseteq [a]$ ——②

from ① & ② we get $[a] = [b]$

$\Leftarrow$) we know that $a \in [a]$

Since $[a] = [b] \longrightarrow a \in [b] \longrightarrow a R b$

② $\Rightarrow$)

Let $b \in [a] \longleftrightarrow b R a$

Since R is equivalence relation we get $a R b$

So that by proposition ① we get

$a R b \longrightarrow [a] = [b]$

$\Leftarrow$) $b \in [b]$

Since $[a] = [b]$

hence, $b \in [a]$

Definition Let A be a non-empty set and Let $P = \{A_i\}_{i \in J}$ be a family of subsets of A such that

1- $A_i \neq \phi \quad \forall i \in J$

2- For any A_i, A_j, either $A_i = A_j$ or $A_i \cap A_j = \phi$

3- $A = \bigcup_{i \in J} A_i$

Then P is called a partition of A.

Example

① Let $N = \{1, 2, 3, \dots\}$, $E = \{2, 4, 6, \dots\}$ and $O = \{1, 3, 5, \dots\}$.

Then $\{E, O\}$ is a partition of N.

② Let $T = \{1, 2, \dots, 9, 10\}$, and Let $A = \{1, 3, 5\}$, $B = \{2, 6, 10\}$ and $C = \{4, 8, 9\}$. Then $\{A, B, C\}$ is not a partition of T since

$$T \neq A \cup B \cup C$$

i.e since $7 \in T$ but $7 \notin (A \cup B \cup C)$

③ Let $T = \{1, 2, \dots, 9, 10\}$, and Let $F = \{1, 3, 5, 7, 9\}$, $G = \{2, 4, 10\}$ and $H = \{3, 5, 6, 8\}$. Then $\{F, G, H\}$ is not a partition of T

Since $F \cap H \neq \phi$, $F \neq H$

Theorem: Let R be an equivalence relation defined on a set A. Then

① For every $a, b \in A$, then either $[a]=[b]$ or $[a] \cap [b] = \phi$

② If $P = \{[a] ; a \in A\}$ be a family of equivalence classes then

$$A = \bigcup_{[a]\in P} [a] .$$

Proof

① suppose that $[a] \cap [b] \neq \phi$

$\rightarrow \exists\, x \in [a] \cap [b]$

$\rightarrow x R a \wedge x R b$

$\rightarrow a R x \wedge x R b$ (since R is symmetric relation)

$\rightarrow a R b$ (since R is transitive relation)

$\rightarrow [a] = [b]$ (by proposition ?)

② Let $x \in A \rightarrow x \in [x] \subseteq A$

$\rightarrow [x] \in P$

$\rightarrow [x] \subseteq \bigcup_{[a]\in P} [a] \longrightarrow A \subseteq \bigcup_{[a]\in P} [a]$ ---①

if $[a] \in \bigcup_{[a]\in P} [a] \longrightarrow [a] \subseteq A$

$\rightarrow \bigcup_{[a]\in P} [a] \subseteq A$ ---②

from ① & ② we get $A = \bigcup_{[a]\in P} [a]$

Example :- Let $\mathbb{Z}$ be an integer number and Let R be a relation defined on $\mathbb{Z}$ by aRb iff $a-b=3k$, $k\in\mathbb{Z}$

prove that R is equivalence relation and find the partition.

Solution

R is equivalence relation ?

Now, we find the equivalence classes

$[0]=\{x\in\mathbb{Z}, xR0\}=\{x\in\mathbb{Z}, x-0=3k, k\in\mathbb{Z}\}$

$=\{\dots,-6,-3,0,3,6,\dots\}$

$[1]=\{x\in\mathbb{Z}; xR1\}=\{x\in\mathbb{Z}; x-1=3k, k\in\mathbb{Z}\}$

$=\{x\in\mathbb{Z}; x=3k+1, k\in\mathbb{Z}\}$

$=\{\dots,-5,-2,1,4,7,\dots\}$

$[2]=\{x\in\mathbb{Z}; xR2\}=\{x\in\mathbb{Z}; x-2=3k, k\in\mathbb{Z}\}$

$=\{x\in\mathbb{Z}; x=3k+2; k\in\mathbb{Z}\}$

$=\{\dots,-4,-1,2,5,8,\dots\}$

So that

$[0]=[3]=[-6]=\dots$

$[1]=[-5]=[7]=\dots$

$[2]=[-4]=[8]=\dots$

(by proposition $b\in[a]$ iff $[a]=[b]$)

Hence

$\mathbb{Z}/R=\{[0],[1],[2]\}$

لاحظنا سابقاً انه اذا كان لدينا علاقة تكافؤ على مجموعة ما فان هذه العلاقة تعطينا تجزئة للمجموعة .

سؤال :- اذا كان لدينا تجزئة ما. فهل من الممكن الحصول على علاقة تكافؤ على تلك المجموعة من خلال التجزئة المعطاة ؟ ولماذا ؟

الجواب :- نعم وحسب القضية التالية .

Proposition Let $P = \{A_i\}_{i \in J}$ be a partion of a set A, define a relation R on A as follows

$a R b$ iff each of a, b belongs to the same set A_i

Then R is an equivalence relation defined on A

proof

① R is reflexive

$\forall a \in A \rightarrow a R a$ since $a, a \in A$

② R is symmetric

Let $a R b$, we have to show $b R a$

since $a R b$ so that $\{a, b\} \in A_i$

hence $\{b, a\} \in A_i$

$\rightarrow b R a$

③ R is transitive

Let $aRb \wedge bRc$

$\rightarrow \{a,b\} \in A_i \wedge \{b,c\} \in A_j$

$b \in A_i \cap A_j \longrightarrow A_i \cap A_j \neq \phi$

so that $A_i = A_j$ since P is a partion

$\rightarrow aRc$

Example Let $A = \{1,2,3,5\}$

① Let $P = \{\{1,2\},\{3\},\{5\}\}$ be a partion of A, hence by the preceding proposition we have the following equivalence relation

$R = \{(1,1),(2,2),(1,2),(2,1),(3,3),(5,5)\}$

$[1] = \{1,2\} = [2]$

$[3] = \{3\}$

$[5] = \{5\}$

$\therefore A/R = \{[1],[3],[5]\}$

② Let $P = \{\{1,3\},\{2,5\}\}$ be apartion of A, so that

$R = \{(1,1),(3,3),(1,3),(3,1),(2,2),(5,5),(2,5),(5,2)\}$

$[1] = \{1,3\} = [3]$

$[2] = \{2,5\} = [5]$

$\therefore A/R = \{[1],[2]\}$