since n is Char R .

$$\Leftrightarrow \text{ Let } a \in R, \text{na} = \text{n.1}. a = 0 \cdot a = 0$$

$\therefore$ Char R $=$ n since n is the smallest positive integer, $\text{n.l} = 0$.

Corollary: Let R be an integral domain then Char R is either zero or prime integer.

Proof: Suppose Char R > 0 suppose $\text{n} = \text{n}_1 \cdot \text{n}_2, \text{l} < \text{n}_1 \leq \text{n}_2 < \text{n}$

$\text{n. a} = 0$ (n is the smallest positive int.)

$(\text{n}_1 \cdot \text{n}_2)\text{a} = (\text{n}_1 \cdot 1) \cdot (\text{n}_2 \cdot 1) \cdot \text{a}$[R integral domain]

but R is integral domain then either $\text{n}_1.1 = 0$ or $\text{n}_2.1 = 0$C ! by theorem

) since $\text{n}_1, \text{n}_2 < \text{n}$ and n is the smallest integer such that $\text{n.I} = 0$.السابقه(

$\therefore$ n is a prime integer.

Definition: Let R and R′ be rings, f: R → R′, then f is a ring homomorphism if

(1) $f(a + b) = f(a) + f(b)$.

(2) $f(a \cdot b) = f(a) \cdot f(b)$.

Example:

(1) Let $\emptyset: R \longrightarrow R', \emptyset(r) = ($ is a ring homo. called zero homo.

(2) $\text{I}: \text{R} \longrightarrow \text{R}, \text{I}(\text{r}) = \text{r}$ the identity homo.

(3) $h: Z \longrightarrow Z_n, h(n) = \overline{\mathbf{n}}$

Definition: Let $\text{f}: \text{R} \longrightarrow \text{R}'$ be a ring homomorphism.

if f is one to one then fis monomorphism. .1

if f is onto then f is epimorphism. .2

f is $(1,-1)$ and onto then f is isomorphism. .3

Definition: if $\mathrm{f}: \mathrm{R} \longrightarrow \mathrm{R}'$ and f is isomorphism then we say that R is isomorphic to $R', R \simeq R'$.

Remark: if $\mathrm{f}: \mathrm{R} \longrightarrow \mathrm{R}', \mathrm{f}$ is homomorphism, then:

$f(O_R) = O_{\dot{R}}$. .1

$f(-a) = -f(a)$. .2

$f(l_R) = 1_{\dot{R}}$ when R and R' is a ring with identity. .3

Theorem: Any ring can be imbedded in a ring with identity.

Proof: Let $\mathrm{R} \times \mathrm{Z} = \{(\mathrm{r}, \mathrm{n}), \mathrm{r} \in \mathrm{R}, \mathrm{n} \in \mathrm{Z}\}$

Define + and . on $\mathrm{R} \times \mathrm{Z}$ as follows

$$(\mathrm{r}, \mathrm{n}) + (\mathrm{t}, \mathrm{m}) = (\mathrm{r} + \mathrm{t}, \mathrm{n} + \mathrm{m})$$
$$(\mathrm{r}, \mathrm{n}).(\mathrm{t}, \mathrm{m}) = (\mathrm{rt} + \mathrm{nt} + \mathrm{mr}, \mathrm{nm})$$

then $\mathrm{R} \times \mathrm{Z}$ is a ring with identity $(0,1)$.

$$(\mathrm{r}, \mathrm{n}) \cdot (0,1) = (\mathrm{r}, \mathrm{n})$$
$$\mathrm{R} \times \{0\} \subseteq \mathrm{R} \times \mathrm{Z}$$

Now we must show that $R \times \{0\}$ is subring of $R \times Z$

$$(\mathrm{a}, 0)\{\in \mathrm{R} \times \{0\}\} - (\mathrm{b}, 0)\{\in \mathrm{R} \times \{0\}\mathrm{f} = (\mathrm{a} - \mathrm{b}, 0) \in \mathrm{R} \times \{0\}$$
$$(\mathrm{a}, 0).(\mathrm{b}, 0) = (\mathrm{ab}, 0) \in \mathrm{R} \times \{0\}$$

Now we define a map $\emptyset: \mathrm{R} \rightarrow \mathrm{R} \times \{0\}, \emptyset(\mathrm{r}) = (\mathrm{r}, 0)$

(1) Let $\emptyset(\mathrm{r}_1) = \emptyset(\mathrm{r}_2)$

$$(\mathrm{r}_1, 0) = (\mathrm{r}_2, 0) \Rightarrow \mathrm{r}_1 = \mathrm{r}_2 \therefore \emptyset \text{ is } (1-1)$$

(2) let $(w,0) \in R \times \{0\}, \therefore \emptyset(w) = (w,0) \therefore \emptyset$ is onto, $\emptyset$ is homo.

(3) $\emptyset(r_1 + r_2) = (r_1 + r_2, 0) = (r_1, 0) + (r_2, 0) = \emptyset(r_1) + \emptyset(r_2)$

$$\emptyset(r_1 \cdot r_2) = (r_1 r_2, 0)$$
$$\emptyset(r_1) \cdot \emptyset(r_2) = (r_1, 0) \cdot (r_2, 0) = (r_1 r_2, 0)$$
$$\therefore R \cong R \times \{0\}$$

$\therefore$ R is imbedded in a ring $R \times Z$.

Definition: Let R be a ring an element $a \in R$ is said to be idempotent element if $a^2 = a$. And a is nilpotent if there exists an integer n such that $a^n = 0$.

Example: (1) $Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$

Solution: $\overline{0}, \overline{1}, \overline{3}, \overline{4}$ are idempotent. $\overline{0}$ is nilpotent only.

(2) $Z_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$

Solution: $\overline{0}, \overline{2}, \overline{4}, \overline{6}$ are nilpotent.

(3) Z_5 : the idempotent $\overline{0}, \overline{1}$ and nilpotent is $\overline{0}$.

(4) $(p(x), \Delta, \cap)$

Solution: $A \cap A = A, \forall A$ is idempotent $A \cap \ldots \cap A = \emptyset$, just when $A = \emptyset$

Definition: Let R be a ring such that every element of R is idempotent then R is Boolean ring.

Example : in $Z_2 = \{0,1\}, (\overline{0})^2 = 0, (\overline{1})^2 = 1$.

Theorem: Let R be a ring such that every element in R is idempotent (R is Boolean ring), then R is commutative.

Proof: $(a+b)=(a+b)^2=(a+b)(a+b)=a\cdot a+a\cdot b+b\cdot a+b\cdot b$

$$a+b=a^2+a\cdot b+b\cdot a+b^2$$
$$a+b=a+b+a\cdot b+b\cdot a$$
$$0=ab+ba\Rightarrow ab=-ba$$

$$\mathrm{ab}=(-\mathrm{ba})=(-\mathrm{ba})^2=\mathrm{b}^2\mathrm{a}^2=\mathrm{ba}$$

∴ R is commutative.

Remark: Let R be a ring commutative if there exists element $a\in R$, such that:

(1) a is idempotent.

(2) a is not zero divisor. Then a must be the identity of the ring.

Proof: (2) Let $b\in$ R

$$a\cdot b=a^2b\Rightarrow(a^2\cdot b)-a\cdot b=0$$
$$a(ab-b)=0[a\text{ is not zero divisof }]$$

$\therefore \mathrm{ab}-\mathrm{b}=0\Rightarrow \mathrm{ab}=\mathrm{b}$

∴ a is identity.

Example: Consider the ring $(\mathrm{p(x)},\Delta,\cap)\mathrm{p(x)}=\{\mathrm{A}:\mathrm{A}\subseteq \mathrm{X}\}$, for a fixed subset $S\subseteq X, S\in p(x)$, define $f:p(x)\longrightarrow p(x)$

$$\mathrm{f(A)}=\mathrm{A}\cap \mathrm{S}$$

(1) $\mathrm{A}=\mathrm{B}\Rightarrow \mathrm{A}\cap\mathrm{S}=\mathrm{B}\cap\mathrm{S}$

$$\therefore \mathrm{f(A)}=\mathrm{f(B)}\ \therefore \mathrm{f}\text{ is well define}$$

(2) $(A\Delta B)=f(A)\Delta f(B)$?