The converse of Lagrange Theorem is not true in general: Let G be a group with
$0(G) = n$, then it's not necessary there's subgroup of G with order k such that k divide n .

Example

The group A_4 is the set of all even permutation of S_4 has 12 element but A_4 has no subgroup of order 6 .

Remark: Let G be a group with $0(G) = n$,then the order of any element in G divide n .

Proof: Let $a \in G, < a >$ is the subgroup generated by a, then by "Lagrange" theorem $0 < a > \backslash\ 0(G)$.

GROUPS THEORY

Theorem: Every group of prime order is cyclic and hence is abelian.
Proof: Let G be a group with $0(G) = p$, let $a \in G, a \neq e$, let $< a >$ be a subgroup generated by a.
By Lagrange theorem $0(< a >) \backslash 0(G) = p$
$0 < a >= 1C! [a \neq e]$
$0 < a >= p = 0(G)$
So that $G =< a >$ and hence is cyclic
cyclic $\Rightarrow$ abelian.
Theorem: Every group of order less than $6 (< 6)$ is abelian.
Proof: If $0(G) = 1$, then $G = \{e\}$, then is abelian
If $0(G) = 2,3,5$, then G is abelian [since $0(G)$ is prime]
Suppose that $0(G) = 4$
Case 1: If G is cyclic $\Rightarrow G$ is abelian
Case 2: If G is not cyclic every element $e \neq a \in G$ has order 2
Then $a^2 = e \forall a \in G$, hence G is abelian
Theorem: Every group of composite order, has a nontrivial proper subgroups
.

Proof: Let G be a group whose order is n
$0(G) = st$, such that $1 < s \leq t < n$
Case 1: If G is cyclic, then $\exists a \in G$ such that
$G = \langle a \rangle, 0\langle a \rangle = n$

Dr. Enas Mustafa

$a^n = e \Rightarrow a^{st} = e \Rightarrow (a^s)^t = e$
$0 < a^s\rangle | n, 0 < a^s\rangle = t$
then $\langle a^{s'}\rangle \neq G$, so $< a^a\rangle$ is a nontrivial cyclic subgroup of G of ordert

Case 2: If G is not cyclic, let $e \neq a \in G, < a >$ is a subgroup generated by a.
$0 < a >\backslash\ (G)$
If $0 < a >= 1C!\, [e \neq a]$

If $0 < a >= n$, then $G =< a >$, i.e. G is cyclic C !
$0 < a > \backslash n$
$0 < \mathbf{a} >= s$
$0 < a >= t$
Thus $< a >$ is non trivial subgroup.
Definition: Let H be subgroup of a group G, H is called normal subgroup denoted by $H\Delta G$ if $aH = Ha \forall a \in G$.

Example: $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}, H = \{f_1, f_2\}$ is normal.
Solution: $f_1 o H = \{f_1, f_2\} = H o f_1 = \{f_1, f_2\}$
$f_2 \circ H = \{f_2, f_1\} = H \circ f_2 = \{f_2, f_1\}$
$f_3 \circ H = \{f_3, f_5\} \neq H \circ f_3 = \{f_3, f_4\}$
$\therefore$ Not normal.
Remark: If G is abelian, then every subgroup is normal.
Remark: Let G be a group then:
$\{e\}\Delta G$ and $G\Delta G$

Theorem: Let H be a subgroup of a group G , then $H\Delta G$ if and only if $aHa^{-1} \subseteq H \forall a \in G$.
Proof: $\Rightarrow$) let $x \in G$, we must show that $xHx^{-1} \subseteq H$
let $w \in xHx^{-1}$, then $w = xhx^{-1}; h \in H$
but $H\Delta G$, so $xH = Hx$, and if $xh \in xH = Hx$
then $xh = h_1 x; h_1 \in H \ \Rightarrow xhx^{-1} = h_1 x x^{-1} = h_1$
then $xhx^{-1} = h_1 \in H$, hence $w \in H$
$\Leftrightarrow) aHa^{-1} \subseteq H \overset{?}{\Rightarrow} aH = Ha$
let $x \in aH \Rightarrow x = ah; h \in H$
$xa^{-1} = aha^{-1} \in aHa^{-1} = H$
then $xa^{-1} \in H \Rightarrow [xa^{-1} = h_1]; h_1 \in H$, then
$x = h_1 a \in Ha$, hence $aH \subseteq Ha$
let $y \in Ha \Rightarrow y = ha; h \in H$
$a^{-1}y = a^{-1}ha \in a^{-1}Ha \subseteq H$

Dr. Enas Mustafa

put $a^{-1}ha = h_2; h_2 \in H$
$a^{-1}y = h_2 \Rightarrow y = ah_2 \in aH$
then $\mathrm{Ha} \subseteq \mathrm{aH}$
Thus $\mathrm{aH} = \mathrm{Ha}$, which implies that $\mathrm{H}\Delta\mathrm{G}$.
Theorem: Let H be a subgroup of a group G , and $\mathrm{H} \subseteq \mathrm{CentG}$, then $\mathrm{H}\Delta\mathrm{G}$.

Proof: We must show $\mathrm{aHa}^{-1} \subseteq \mathrm{H}\forall \mathrm{a} \in \mathrm{G}$
let $\mathrm{w} \in \mathrm{aHa}^{-1}$, then $\mathrm{w} = \mathrm{aha}^{-1}; \mathrm{h} \in \mathrm{H}$, then $\mathrm{w} = \mathrm{haa}^{-1}$, so $\mathrm{w} = \mathrm{h} \Rightarrow$
$\mathrm{w} \in \mathrm{H}$
hence $\mathrm{aHa}^{-1} \subseteq \mathrm{H}$.

Theorem: Let H be a subgroup of a group G , if $\mathrm{a}, \mathrm{b} \in \mathrm{G}$, then $\mathrm{aH} = \mathrm{bH}$ if and only if $\mathrm{a}^{-1}\,\mathrm{b} \in \mathrm{H}$. $[\mathrm{aH} = \mathrm{H} \Leftrightarrow \mathrm{a} \in \mathrm{H}]$
Proof: $\Rightarrow$ Since $\quad |'\mathrm{b} \in \mathrm{bH} = \mathrm{aH} \Rightarrow \mathrm{b} \in \mathrm{aH}$
then $\mathrm{b} = \mathrm{ah}; \mathrm{h} \in \mathrm{H}$

$$a^{-1}b = h \in H \Rightarrow a^{-1}b \in H$$
$$\Leftrightarrow) a^{-1}b \in H \Rightarrow a^{-1}b = h; h \in H$$

then $b = ah$ and $a = bh^{-1}$
To show $\mathrm{aH} = \mathrm{bH}$
let $\mathrm{x} \in \mathrm{aH} \Rightarrow \mathrm{x} = \mathrm{a}h_1; \mathrm{h}_1 \in \mathrm{H}$

$$x = bh^{-1}h_1 \Rightarrow x = bh_2 \in bH; h_2 \in H, h_2 = h^{-1}h_1 \#(1)$$

so $\mathrm{x} \in \mathrm{bH} \Rightarrow \mathrm{aH} \subseteq \mathrm{bH}$.

$$y \in bH \Rightarrow y = bh_3; h_3 \in H$$

so $\mathrm{y} \in \mathrm{aH} \Rightarrow \mathrm{bH} \subseteq \mathrm{aH}$
From (1) and (2) $\Rightarrow \mathrm{aH} = \mathrm{bH}$.

Theorem: Let H be a subgroup of a group G , then $\mathrm{H}\Delta\mathrm{G}$ if and only if:

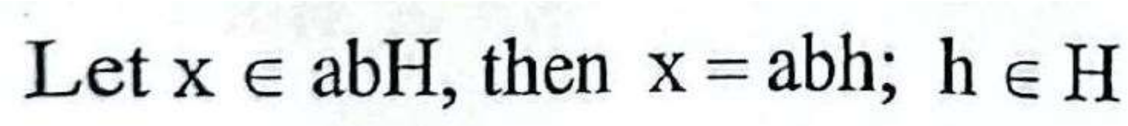
$$(\mathrm{aH})(\mathrm{bH}) = \mathrm{abH}\forall \mathrm{a}, \mathrm{b} \in \mathrm{G}$$

Proof: Let $\mathrm{a}, \mathrm{b} \in \mathrm{G}$, let $\mathrm{w} \in (\mathrm{aH})(\mathrm{bH})$
then $\mathrm{w} = (\mathrm{ah}_1)(\mathrm{bh}_2); \mathrm{h}_1,\ \mathrm{h}_2 \in \mathrm{H}$

$$= \mathrm{a}(\mathrm{bb}^{-1})\mathrm{h}_1\mathrm{bh}_2, \text{ since } \mathrm{aHa}^{-1} \subseteq \mathrm{H}, \text{ then}$$

Dr. Enas Mustafa

so $(aH)(bH) \subseteq abH$



Let x ∈ abH, then $x = abh;\ h \in H$

$$x = aebh \Rightarrow x \in (aH)(bH) \#(2)$$

so $abH \subseteq (aH)(bH)$
From (1) and (2)

$$(aH)(bH) = abH$$

⇔) To show $H \Delta G$, (i.e $aHa^{-1} \subseteq H \forall a \in G$)

Let $w \in aHa^{-1}$, then $w = aha^{-1}; a \in G, h \in H$

$$w = aha^{-1}e \in (aH)(a^{-1}H) = aa^{-1}H$$
$$w = aa^{-1}h$$
$$w = eh = h, \text{ then } w \in H$$

so $aHa^{-1} \subseteq H \Rightarrow H \Delta G$.

The quotient group of group G by H :

Let $H \Delta G$

Let $\frac{G}{H} = \{aH : a \in G\}$ be set of all distinct left cosets

Define ⊗ on $\frac{G}{H}$ by $(aH) \otimes (bH) = abH$

$\left(\frac{G}{H}, \otimes\right)$ is a group ?

(1) ⊗ is well defined:

$$aH = a_1H, a_1 \neq a$$
$$bH = b_1H, b_1 \neq b$$
$$abH = a_1b_1H \Leftrightarrow (ab)^{-1}(a_1b_1) \in H \Leftrightarrow (b^{-1}a^{-1})(a_1b_1) \in H$$

To show ^{8} is well define, suppose $aH = a_1H$ and $bH = b_1H$

$$\sqrt{aH} = a_1H \Leftrightarrow a^{-1}a_1 \in H, \therefore a^{-1}a_1 = h_1; h_1 \in H$$
$$bH = b_1H \Leftrightarrow b^{-1}b_1 \in H, \therefore b^{-1}b_1 = h_2; h_2 \in H$$

Dr. Enas Mustafa

$$(ab)^{-1}(a_1b_1) = b^{-1}a^{-1}a_1b_1 = [b^{-1}(a^{-1}a_1)b](b^{-1}b_1), \text{ since } H\Delta$$

G, then $b^{-1}(a^{-1}a_1)b \in H)$, thus $(ab)^{-1}(a_1b_1) \in H$
so $\otimes$ is well defined.
$\left(\frac{\mathrm{G}}{\mathrm{H}}, \otimes\right)$ is a group ?
(1) $\otimes$ is a binary operation (well defined)
(2) $(\mathrm{aH} \otimes \mathrm{bH}) \otimes \mathrm{cH} = \mathrm{aH} \otimes (\mathrm{bH} \otimes \mathrm{cH})$
$\mathrm{abH} \otimes \mathrm{cH} = \mathrm{aH} \otimes \mathrm{bcH}$
$\mathrm{abcH} = \mathrm{abcH}$
$\otimes$ is associative
(3) $aH \otimes \mathrm{eH} = \mathrm{aH} \overset{?}{=} eH \otimes aH = \mathrm{eaH} = aH$
$\mathrm{aeH} = \mathrm{aH}$, hence $\mathrm{eH} = \mathrm{H}$ is the identity
(4) $\mathrm{aH} \otimes \mathrm{a}^{-1}\mathrm{H} = \mathrm{aa}^{-1}\mathrm{H} = \mathrm{eH} = \mathrm{H}$

$$\begin{aligned} a^{-1}H \otimes aH &= -a^{-1} \cdot H \\ &= e^{H}H \end{aligned}$$

$\therefore \frac{\mathrm{G}}{\mathrm{H}}$ is a group called a quotient group.
Example:

$$\begin{aligned} &\frac{\dot{z}}{4\mathrm{Z}} = \{4\mathrm{Z}, 1+4\mathrm{Z}, 2+4\mathrm{Z}, 3+4\mathrm{Z}, \dots, -1+4\mathrm{Z}, -2+4\mathrm{Z}, \dots\} \\ &4Z = \{0, \pm 4, \pm 8, \cdots\} \\ &1+4\mathrm{Z} = \{1,5,9, \dots\} \\ &2+4\mathrm{Z} = \{2,6,10, \dots\} \\ &3+4Z = \{3,7,11, \dots\} \end{aligned}$$

Example: $\mathrm{G_s} = \{\mathrm{e}, \mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_{3:}\mathrm{d}_1, \mathrm{d}_2, \mathrm{h}, \mathrm{v}\}$

$H = \{e, r_2\} =$ Cent $G_s \Rightarrow$ abelian $\Rightarrow$ normal
$AG, = \frac{\mathrm{G_s}}{\mathrm{H}} = \left\{\{\mathrm{e}, \mathrm{r}_2\}, \{\mathrm{r}_1, \mathrm{r}_3\}, \{\mathrm{d}_1, \mathrm{d}_2\}, \{\mathrm{h}, v\}\right\}$

Remark: (1) let H be normal in G , if G is abelian group, then $\frac{\mathrm{G}}{\mathrm{H}}$ is abelian

Proof: Let $\mathrm{aH}, \mathrm{bH} \in \frac{\mathrm{H}}{\mathrm{G}}$, we have to show that

$$\mathrm{aH} \otimes \mathrm{bH} = \mathrm{bH} \otimes \mathrm{aH}$$

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