Chapter Two

Relations and Functions

العلاقات والدوال

تتعامل الرياضيات بنوع معين من المجموعات تسمى علاقات ولكن قبل أن ندخل في تعريف مفهوم العلاقة نوضح بعض الامثلة عن هذا المفهوم.

Example:-

1) " x is less than y"

2) " x divides y "

3) " x is the wife of y"

4) " The square of x plus the square of y is sixteen", i.e " $x^2 + y^2 = 16$".

Ordered Pairs

الزوج المرتب

An ordered pair consists of two elements, say a and b, in which one of them, say a, is designated (يشير ، يرمز الى) as the first element and the other as the second element. An ordered pair is denoted by (a,b)

Remark:-

1) An ordered pair (a,b) can be defined by

$$(a,b) \equiv \{\{a\},\{a,b\}\}$$

2) Ordered pairs can have the same first and second elements such as $(1,1)$, $(4,4)$ and $(5,5)$.

3) If $a=b$ then $(a,a)=\{\{a\}\}$

4) If $a\neq b$ then $(a,b)\neq(b,a)$

Theorem:- Two ordered pairs (a,b) and (c,d) are equal if and only if

$a=c$ and $b=d$

proof:- $\Rightarrow$) Let $(a,b)=(c,d)$

$\rightarrow \{\{a\},\{a,b\}\}=\{\{c\},\{c,d\}\}$

case -1- If $a=b$

$(a,b)=(a,a)=\{\{a\},\{a,a\}\}=\{\{a\}\}.$

$=(c,d)=\{\{c\},\{c,d\}\}$

$\rightarrow \{a\}=\{c\}=\{c,d\}$

$\rightarrow a=b=c=d$

hence, $a=c$ & $b=d$

case -2- If $a\neq b$.

$\{c\}\neq\{a,b\}$

$\{a\}\neq\{a,b\}$

$\left[\begin{array}{l} \text{if } \{c\}=\{a,b\} \rightarrow a=c=b \rightarrow a=b \text{ C!} \\ \text{if } \{a\}=\{a,b\} \rightarrow a=b \text{ C!} \end{array}\right]$

Now,

$\{c\}\in(c,d)=(a,b)=\{\{a\},\{a,b\}\}$

but $\{c\}\neq\{a,b\}$

$\rightarrow \{c\} = \{a\} \rightarrow a = c$

also, $\{a,b\} \in (a,b) = (c,d) = \{\{c\}, \{c,d\}\}$

but $\{a,b\} \neq \{c\}$

$\rightarrow \{a,b\} = \{c,d\}$

$\rightarrow b \in \{c,d\}$

$\rightarrow$ If $b = c = a$ C! , hence $b = d$

$\Leftarrow$) Let $a = c$ and $b = d$

$\rightarrow \{a\} = \{c\}$ & $\{a,b\} = \{c,d\}$

$\rightarrow \{\{a\}, \{a,b\}\} = \{\{c\}, \{c,d\}\}$

$\rightarrow (a,b) = (c,d)$

Product set

Let A and B be two sets. The product set of A and B consists of all ordered pairs (a,b) where $a \in A$ and $b \in B$. It is denoted by $A \times B$ which reads "A cross B".

i.e $A \times B = \{(a,b) \,;\, a \in A, b \in B\}$

The product set $A \times B$ is also called the Cartesian product (الجداء الديكارتي) of A and B.

Example:-

① Let $A=\{1,2,3\}$ and $B=\{a,b\}$. Then the product set

$$A\times B=\{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$

② Let $W=\{s,t\}$. Then

$$W\times W=\{(s,s),(s,t),(t,s),(t,t)\}$$

③ $\mathbb{R}\times\mathbb{R}=\{(a,b);\ a\in\mathbb{R}\}=\mathbb{R}^2$ the plane real number

Remarks:-

1) If set A has n elements and set B has m elements then the product set $A\times B$ has $n.m$ elements.

2) In general $A\times B\neq B\times A$

3) If A is any set and $B=\emptyset$ then $A\times B=\emptyset$

4) If $A\neq\emptyset$ and $B\neq\emptyset$ then $A\times B=B\times A$ iff $A=B$

Proof:-

4) $\Rightarrow$) Let $A\times B=B\times A$

and Let $a\in A$, $b\in B$

$\rightarrow (a,b)\in A\times B=B\times A$

$\rightarrow (a,b)\in B\times A$

$\rightarrow a\in B \wedge b\in A$

$\rightarrow A \subseteq B \wedge B \subseteq A$

$\rightarrow A = B$

$\Leftarrow$) Let $A = B$

$\rightarrow A \times B = A \times A = B \times A$

Remark:-

1- $(a,b) \in A \times B \longleftrightarrow a \in A \wedge b \in B$

2- $(a,b) \notin A \times B \longleftrightarrow a \notin A \vee b \notin B$

3- $a \in A \wedge b \notin B \longrightarrow (a,b) \notin A \times B$

4- $a \notin A \wedge b \in B \longrightarrow (a,b) \notin A \times B$

5- $a \notin A \wedge b \notin B \longrightarrow (a,b) \notin A \times B$

Proposition :- Let A, B, C and D be a set. Then

1- $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2- $A \times (B \cup C) = (A \times B) \cup (A \times C)$

3- $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Example :- Consider the relation R that is define on $\mathbb{Z}$ by

$$aRb \longleftrightarrow \exists k \in \mathbb{Z} \; ; \; a-b = 3k$$

$(3,0) \in R$ since $3-0 = 3 = 3(1)$, $k=1$

$(0,3) \in R$ since $0-3 = -3 = 3(-1)$, $k=-1$

$(15,9) \in R$ since $15-9 = 6 = 3(2)$, $k=2$

$(-1,-2) \notin R$ since $-1-(-2) = 1 \neq 3k$

Now,

1- R is reflexive ?

Let $a \in \mathbb{Z}$

$\rightarrow a-a = 0 = 3(0)$, $k=0 \longrightarrow aRa$

So that, R is reflexive

2- R is symmetric ?

Let aRb, we have to show bRa

$\because aRb \longrightarrow \exists k \in \mathbb{Z} \; ; \; a-b = 3k$

Now,

$b-a = -(a-b) = -3k = 3(-k) = 3k_1$ s.t $k_1 = -k$

$\therefore b-a = 3k_1 \longrightarrow bRa$

So that R is symmetric

3 R is transitive ?

Let $aRb \wedge bRc$, we have to show aRc

$a-b = 3k_1 \wedge b-c = 3k_2$; $k_1, k_2 \in \mathbb{Z}$

$a-c = a-b+b-c = 3k_1 + 3k_2 = 3(k_1+k_2) = 3k$; $k = k_1+k_2$

$\therefore aRc$

So that from 1 , 2 & 3 R is equivalence relation

Example:- Let $A = \mathbb{Z} \times \mathbb{Z} - \{0\}$, we define the relation R on A by

$$(a,b)\, R\, (c,d) \longleftrightarrow ad = bc \quad \left[\frac{a}{b} = \frac{c}{d} \right]$$

$(0,1)\, R\, (0,2)$

$(2,4)\, R\, (10,20)$

$(5,15) \not R\, (25,10)$

① R is reflexive relation ?

Let $(a,b) \in A$

It is clear that $(a,b)\, R\, (a,b)$ since $ab = ba$

② R is symmetric relation ?

Let $(a,b)\, R\, (c,d)$

$\rightarrow ad = bc$

$\rightarrow bc = ad$

$\rightarrow cb = da$

$\rightarrow (c,d)\, R\, (a,b)$

③ R is transitive relation

Let $(a,b)\, R\, (c,d) \wedge (c,d)\, R\, (e,f)$

We have to show that $(a,b)\, R\, (e,f)$ i.e $af = be$

$\because ad = bc \wedge cf = de$

$\rightarrow c = \frac{ad}{b}$

$\rightarrow cf = \frac{ad}{b} f = de \longrightarrow a\cancel{d}f = b\cancel{d}e \rightarrow af = be$

$\therefore (a,b)\, R\, (e,f)$

So that by ①, ② & ③ R is equivalence relation

③. Let A be the set of triangles in the Euclidean plane, and Let R be the relation in A which is defined by "x is similar to y". Then R is symmetric, since if triangle a is similar to triangle b then b is also similar to a.

Transitive relation

علاقة متعدية

A relation R in a set A is called a transitive relation if

$$(a,b) \in R \text{ and } (b,c) \in R \text{ implies } (a,c) \in R$$

Example:-

① Let R be the relation in the real numbers defined by "$x < y$". Then

if $a < b$ and $b < c$ implies $a < c$

Thus R is a transitive relation.

② Let $W = \{a, b, c\}$, and Let $R = \{(a,b), (c,b), (b,a), (a,c)\}$

Then R is not a transitive relation since

$$(c,b) \in R \text{ and } (b,a) \in R \text{ but } (c,a) \notin R$$

③ Let $\mathcal{T}$ be a family of sets, and Let R be the relation in $\mathcal{T}$ defined by "x is a subset of y". Then R is a transitive relation since

$$A \subset B \text{ and } B \subset C \text{ implies } A \subset C.$$

Anti-symmetric relation

علاقة ضد متناظرة :

A relation R in a set A, is called an anti-symmetric relation if

$(a,b) \in R$ and $(b,a) \in R$ implies $a=b$

Example :-

① Let $W=\{1,2,3,4\}$, and Let $R=\{(1,3),(4,2),(4,4),(2,4)\}$

Then R is not an anti-symmetric relation in W since

$(4,2) \in R$ and $(2,4) \in R$ but $2 \neq 4$

② Let $\mathcal{T}$ be a family of sets, and Let R be the relation in $\mathcal{T}$ defined by "x is a subset of y". Then R is anti-symmetric since

$A \subset B$ and $B \subset A$ implies $A=B$

Equivalence relation

علاقة تكافؤ

A relation R in a set A is an equivalence relation if

1- R is reflexive, that is, for every $a \in A$, $(a,a) \in R$.

2- R is symmetric, that is, $(a,b) \in R$ implies $(b,a) \in R$

3- R is transitive, that is, $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$.

② Let R be the relation in the real numbers defined by the open sentence "$x < y$". Then R is not reflexive since $a \not< a$ for any real number a.

③ Let $\mathcal{T}$ be a family of sets, and Let R be the relation in $\mathcal{T}$ defined by "x is a subset of y". Then R is a reflexive relation since every set is a subset of itself.

Symmetric relation

علاقة متناظرة

Let R be a relation in a set A. Then R is called a symmetric relation if

$$(a,b) \in R \text{ implies } (b,a) \in R$$

that is, if a is related to b then b is also related to a.

Example:-

① Let $S = \{1, 2, 3, 4\}$, and Let $R = \{(1,3), (4,2), (2,4), (2,3), (3,1)\}$

Then R is not a symmetric relation since

$$(2,3) \in R \text{ but } (3,2) \notin R$$

② Let R be the relation in the natural numbers $\mathbb{N}$ which is defined by "x divides y". Then R is not symmetric since 2 divides 4 but 4 does not divide 2.

i.e $(2,4) \in R$ but $(4,2) \notin R$

Example :- ① Let $A=\{1,2,3\}$ and $B=\{a,b\}$. Then

$$R=\{(1,a),(1,b),(3,a)\}$$

is a relation from A to B. Furthermore,

$$1Ra,\ 2\not R b,\ 3Ra,\ 3\not R b$$

② Let $W=\{a,b,c\}$. Then

$$R=\{(a,b),(a,c),(c,c),(c,b)\}$$

is a relation in W. Moreover

$$a\not R a,\ b\not R a,\ cRc,\ aRb$$

Reflexive relation

علاقة انعكاسية

Let R be a relation in a set A, i.e. let R be a subset of $A\times A$. Then R is called a reflexive relation if, for every $a\in A$,

$$(a,a)\in R$$

In other words, R is reflexive if every element in A is related to itsel

Example :- ① Let $V=\{1,2,3,4\}$ and

$$R=\{(1,1),(2,4),(3,3),(4,1),(4,4)\}$$

Then R is not reflexive relation since (2,2) does not belong to R. Notice that all ordered pairs (a,a) must belong to R in order for R to be reflexive.

3) Let $(a,b) \in (A \times B) \cap (C \times D)$

$\longleftrightarrow (a,b) \in (A \times B) \wedge (a,b) \in (C \times D)$

$\longleftrightarrow (a \in A \wedge b \in B) \wedge (a \in C \wedge b \in D)$

$\longleftrightarrow (a \in A \wedge a \in C) \wedge (b \in B \wedge b \in D)$

$\longleftrightarrow a \in (A \cap C) \wedge b \in (B \cap D)$

$\longleftrightarrow (a,b) \in (A \cap C) \times (B \cap D)$

$\therefore (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Relations

Let A and B are two sets, every subset of $A \times B$ is called relation from A to B, denoted by R.

Remark:-

1- We say R relation on A if $R \subseteq A \times A$

2- If $(a,b) \in R$, then denoted by aRb.

3- If $(a,b) \notin R$, then denoted by $a \not R b$.

Proof:- 2) Let $(x,y) \in A\times(B\cup C)$

$\rightarrow x \in A \wedge y \in (B\cup C)$

$\rightarrow x \in A \wedge (y \in B \vee y \in C)$

$\rightarrow (x\in A \wedge y \in B) \vee (x \in A \wedge y \in C)$

$\rightarrow (x,y) \in (A\times B) \vee (x,y) \in (A\times C)$

$\rightarrow (x,y) \in (A\times B) \cup (A\times C)$

$\therefore A\times(B\cup C) \subseteq (A\times B)\cup(A\times C)$ ---- 1

Let $(e,f) \in (A\times B)\cup(A\times C)$

$\rightarrow (e,f) \in (A\times B) \vee (e,f) \in (A\times C)$

$\rightarrow (e\in A \wedge f\in B) \vee (e \in A \wedge f \in C)$

$\rightarrow e \in A \wedge (f \in B \vee f \in C)$

$\rightarrow e \in A \wedge f \in (B\cup C)$

$\rightarrow (e,f) \in A\times(B\cup C)$

$\therefore (A\times B)\cup(A\times C) \subseteq A\times(B\cup C)$ ---- 2

From 1 & 2 we get

$A\times(B\cup C) = (A\times B)\cup(A\times C)$

Now,

Since $(a,b) \in R_1 \cap R_2 \quad \wedge (b,c) \in R_1 \cap R_2$

$\rightarrow (a,b) \in R_1 \wedge (a,b) \in R_2 \wedge (b,c) \in R_1 \wedge (b,c) \in R_2$

$\rightarrow ((a,b) \in R_1 \wedge (b,c) \in R_1) \wedge ((a,b) \in R_2 \wedge (b,c) \in R_2)$

$\rightarrow (a,c) \in R_1 \wedge (a,c) \in R_2$ [R_1, R_2 are transitive]

$\rightarrow (a,c) \in R_1 \cap R_2$

$\rightarrow R_1 \cap R_2$ is transitive

Hence, by ①, ② & ③ $R_1 \cap R_2$ is an equivalence relation.

② $R_1 \cup R_2$ not necessary

For example

Let $A = \{0,1,2,3\}$

$R_1 = \{(0,0),(1,1),(2,2),(3,3),(1,3),(3,1)\}$

$R_2 = \{(0,0),(1,1),(2,2),(3,3),(2,1),(1,2)\}$

$R_1 \cup R_2 = \{(0,0),(1,1),(2,2),(3,3),(1,3),(3,1),(2,1),(1,2)\}$

Note that $R_1 \cup R_2$ is not an equivalence relation

Since $(2,1) \in R_1 \cup R_2 \wedge (1,3) \in R_1 \cup R_2$

but $(2,3) \notin R_1 \cup R_2$

Proposition:- Let A be any set, and Let R_1 & R_2 are two equivalence relation on A. Then

1- $R_1 \cap R_2$ is an equivalence relation.

2- $R_1 \cup R_2$ not necessary equivalence relation.

Proof:-

① Since each of R_1 and R_2 are subsets of $A \times A$, so that $R_1 \cap R_2$ is subset of $A \times A$. Hence $R_1 \cap R_2$ is relation on A.

1- $R_1 \cap R_2$ is reflexive.

$\forall a \in A$

$$\left.\begin{array}{l} a R_1 a \rightarrow (a,a) \in R_1 \\ a R_2 a \rightarrow (a,a) \in R_2 \end{array}\right\} \rightarrow (a,a) \in R_1 \cap R_2$$

2- $R_1 \cap R_2$ is symmetric

Let $(a,b) \in R_1 \cap R_2$

$\rightarrow (a,b) \in R_1 \wedge (a,b) \in R_2$

$\rightarrow (b,a) \in R_1 \wedge (b,a) \in R_2$ [R_1, R_2 is symmetric]

$\rightarrow (b,a) \in R_1 \cap R_2$

$\rightarrow R_1 \cap R_2$ is symmetric

3- $R_1 \cap R_2$ is transitive

Let $(a,b) \in R_1 \cap R_2 \wedge (b,c) \in R_1 \cap R_2$

we have to show that $(a,c) \in R_1 \cap R_2$