Chapter Two

Rational and Real numbers

The construction of rational numbers

Let Z be a set of integer numbers and let $A = Z/\{0\}$

$\mathrm{Z} \times A = \{(m,n); m \in Z, n \in A \text{ i.e } n \neq 0\}$

Let $\sim$ be a relation defined on $Z \times A$ as follows

$(m,n),(p,q) \in Z \times A$
$(m,n) \sim (p,q)$ if $mq = np$

H.W: Prove this is equivalence relation.

Definition: The set of all equivalence classes

$\overline{(m,n)} = \{(p,q) \in Z \times A; (p,q) \sim (m,n)\}$

is called the rational numbers and denoted by Q

For example

$\overline{(0,1)} = \{(p,q) \in Z \times A; (p,q) \sim (0,1)\} = \{(p,q); p.1 = 0.q\}$
$= \{(p,q); p = 0\} = \{(0,1),(0,2),(0,3),\dots.\}$

Remark: $\overline{(m,n)} \equiv \frac{m}{n}$

ينظر الى العدد النسبي $\overline{(m,n)}$ على انه $\frac{m}{n}$

Summation and multiplication on Q.

Definition: Let $x, y \in Q$ such that $x = \overline{(m,n)}, y = \overline{(r,s)}$, then

1. $x + y = \overline{(m,n)} + \overline{(r,s)} = \overline{(ms + nr, ns)}$

i.e

$$\left(\frac{m}{n} + \frac{r}{s}\right) = \frac{ms + nr}{ns} = \overline{(ms + nr, ns)}$$

2. $x \cdot y = \overline{(m,n)} \cdot \overline{(r,s)} = \overline{(mr, ns)}$

i.e

$$\frac{m}{n} \cdot \frac{r}{s} = \frac{mr}{ns} = \overline{(mr, ns)}$$

Proposition: Let $x, y \in$ Q, then

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z), \forall z \in Q$
3. If $0 = \overline{(0,n)}$ then $x + 0 = 0 + x = x$

4. For each $x \in Q, \exists(-x) \in Q$ s.t $x + (-x) = 0$

In fact if $x = \overline{(m,n)}$ then $-x = \overline{(-m,n)}$

$(-x)$ is called summation inverse of x

5. $x \cdot y = y \cdot x$

6. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

7. If $1 = \overline{(1,1)}$ then $x \cdot 1 = 1 \cdot x = x \ \forall x \in Q$.

8. If $x \in Q$ and $x \neq 0$, then $\exists x^{-1} \in Q$ s.t

$x \cdot x^{-1} = x^{-1} \cdot x = 1$

In fact if $x = \overline{(m,n)}$ then $x^{-1} = \overline{(n,m)}$

x^{-1} is called multiplication inverse of x such that $m \neq 0$

Definition: Let $0 \neq x \in Q$ such that $x = \overline{(m,n)}, x$ is called positive if $m \cdot n$ is positive (i.e $m.n > 0$) and x is negative if $m \cdot n$ is negative.

Definition: Let $x, y \in Q$, then $x - y$ is defined as follows

$$x - y = x + (-y)$$

and if $y \neq 0$ then $\frac{x}{y}$ is defined by:

$$\frac{x}{y} = x \cdot y^{-1}$$

The order: Let $x, y \in Q$, than we say x greater than $y (x > y)$ if $x - y$ is positive and we say $(x < y)$ if $x - y$ is negative.

Remark: Let $x, y \in Q$, then either $x = y$ or $x > y$ or $x < y$.

Theorem: Let $x, y \in Q$, then:

1- If x, y are positive then $x + y$ and $x \cdot y$ are positive.

2- If x, y are negative then $x + y$ is negative and $x \cdot y$ is positive.

3- If x is negative, y is positive then $x \cdot y$ is negative.

4- If $x \leqslant y$ and $y \leqslant x$ then $x = y$.

5- If $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.

Proof:

1- Let $x = (\overline{a, b}), y = \overline{(c, b)}$ and $(ab > 0) \wedge (cd > 0)$ s.t $a, b, c, d \in Z$

$x + y = \overline{(a, b)} + \overline{(c, d)} = \overline{(ad + cb, bd)}$

Note that, $(ad + cb) \cdot bd = abdd + cdbb > 0$

$\therefore x + y$ is positive

$x \cdot y = \overline{(a, b)} \cdot \overline{(c, d)} = (ab)(cd) > 0$

$\therefore x . y$ is positive

Real Numbers:

Let $(\mathbb{F}, +, \cdot)$ be atriple consist, of a nonempty set $\mathbb{F}$ with two operations $(+), (\cdot), (+)$ is called plus operation and (.) is called time operation.

$(\mathbb{F}, +, \cdot)$ is called field if it is satisfy the following properties:-

1. $\forall a, b, c \in \mathbb{F}$, then $(a + b) + c = a + (b + c)$.

2. $\exists$ an element $o \in \mathbb{F}$ s.t $a + 0 = 0 + a = a,\ \forall a \in \mathbb{F}$.

0 is called additive identity. العنصر المحايد الجمعي

3. $\forall a \in \mathbb{F}, \exists - a \in \mathbb{F}$ s.t $a + (-a) = (-a) + a = 0$

-a is called summation inverse

4. $\forall a, b \in \mathbb{F}$, then $a + b = b + a$

5. $\forall a, b, c \in \mathbb{F}$, then $a \cdot (b . c) = (a \cdot b) \cdot c$

6. $\exists 1 \in \mathbb{F}$ s.t $1 . a = a . 1 = a,\ \forall a \in \mathbb{F}$

1 is called multiplicative identity

7. $\forall a, b \in \mathbb{F}$, then $a \cdot b = b.a$

8. $\forall 0 \neq a \in \mathbb{F}, \exists a^{-1} \in F$ sit $a \cdot a^{-1} = a^{-1} \cdot a = 1$

a^{-1} is called multiplication inverse

9. $\forall\ a, b, c \in \mathbb{F}$, then $(a + b) \cdot c = a \cdot c + b \cdot c$

10. $1 \neq 0$

Definition: Let $(\mathbb{F}, +, .)$ Be a field and let $(\leqslant)$ be order relation on $\mathbb{F}$.

i.e. $(\mathbb{F}, \leqslant)$ is ordered set

Then $(\mathbb{F}, +, ., \leqslant)$ is ordered field if:-

1. For each $, b, c, d \in \mathbb{F}$, if $a \leqslant b$ and $c \leqslant d$ then $a + c \leqslant b + d$.
2. If $0 < a \leqslant b$ and or $0 < c \leqslant d$ then $a.c \leqslant b.d$.

Definition:

$\mathbb{F}^+ = \{x \in \mathbb{F}; x > 0\}$ represent the positive elements (numbers) in $\mathbb{F}$.

$\mathbb{F}^- = \{x \in \mathbb{F}; x < 0\}$ represent the negative elements (numbers) in $\mathbb{F}$.

Note that $\mathbb{F}^+ \cap \mathbb{F}^- = \phi$

Remark:

1- If $x, y \in \mathbb{F}^+$, then $x + y \in \mathbb{F}^+, x \cdot y \in \mathbb{F}^+$

2- If $x, y \in \mathbb{F}^-$, then $x + y \in \mathbb{F}^-, x \cdot y \in \mathbb{F}^+$

3- If $x \in \mathbb{F}^+, y \in \mathbb{F}^-$then $x \cdot y \in \mathbb{F}^-$

4- 1 is greater than 0.

Definition: Let $(S, \leqslant)$ be ordered set and $A \subseteq S$

1. $a \in S$ is called "upper bound" of the set A if $x \leqslant a,\ \forall x \in A$.

2. $b \in S$ is called "lower bound" of the set A if $b \leqslant x,\ \forall x \in A$.

3. $c \in S$ is called "least upper bound" of the set A if

$c \leqslant a$, for each upper bound a of A. (denoted by L.U.b).

4. $d \in S$ is called "greatest lower bound" of the set A if

b $\leqslant d$, for each lower bound b of A. (denoted by g.L.b)

Definition:

Let $(S, \leqslant)$ be ordered set and $A \subseteq S$

1- A is called "upper bounded set" if A has upper bound.

2- A is called "lower bounded set" if A has lower bound.

3-A is called "bounded set" if it has upper bound and bower bound.

Remark:

$$c = L \cdot u \cdot b(A) = \sup(A)$$
$$d = g \cdot L \cdot b(A) = \inf(A)$$

c, d are unique

The completeness property of real numbers خاصية الكمال للاعداد الحقيقية

Every nonempty subset S of real numbers has an upper bound, then it has a least upper bound.

Proposition:

Each ordered field consists of a subfield similar to the rational number field,

Question: Can the rational number equal to the real number ie $\mathrm{Q} = \mathbb{R}$?

To answer this question, we begin the following proposition.

Proposition: The equation $x^2 = 2$ has no solution in Q.

Proof: Let $y \in Q$ that satisfy the equation

i.e $y = \frac{a}{b}, a, b \in Z, b \neq 0$ and $\left(\frac{a}{b}\right)^2 = 2 \rightarrow a^2 = 2b^2$

Case1: If a, b are odd $\rightarrow a^2$ is odd

but $a^2 = 2b^2 \longrightarrow a^2$ is even $C!$

Case2: If a is odd and b is even

$\rightarrow b = 2d$
$\rightarrow a^2 = 2(2d)^2 = 8d^2$

but a^2 is odd "since a is odd". & $8d^2$ is even C!

Case3: If a, is even and b is odd

$\rightarrow a = 2c$
$\rightarrow 4c^2 = 2b^2$
$\rightarrow 2c^2 = b^2$

but b^2 is odd "since b is odd" and $2c^2$ is even C!

So that, there is no $y \in Q$ satisfy the equation.

Theorem: The equation $x^2 = 2$ has a unique positive real solution.

Corollary: Q is not complete field.

Theorem: For each positive real number a and positive integer number n, there is a unique real number satisfy the equation $x^n = a$.

The solution is denoted by $\sqrt[n]{a}$

Corollary: show that $Q \subset \mathbb{R}$. H.W.

Definition: Let Q' be the set of complement of Q in $\mathbb{R}$.

ie $Q' = \mathbb{R} \setminus Q$

Note that $\sqrt{2} \notin Q$ while $\sqrt{2} \in Q'$, hence $Q \neq \varphi'$

Q' is called the set of irrational number

Archimedes property theorem:

For each real numbers a, b such that $a > 0$, then there exists a positive integer number n such that $na > b$.

Corollary: For each positive real number $\in$, there exists a positive integer number n such that $\frac{1}{n} < \epsilon$.

Proof:

put $b = 1, a = \epsilon > 0$

Thus by Archimedes property theorem