$\therefore 2,3 \in 2Z \cup 3Z \therefore 3-2 \in 2Z \cup 3ZC!\ 1 \notin 2Z \cup 3Z$

$\therefore 2Z \cup 3Z$ is not ideal.

Definition: Let S be a nonempty subset of a ring R the set $< S >$, where:

$< S > = \cap$ {I: I is an ideal of R containing S}

is called the ideal generated by S .

Remark: $(*)$ $< S >$ is smallest ideal containing S, $(*)$ $< S > = S$ if and only if S is an ideal.

(*) if $S = \{a\}, < S > = < a >$ is called principle ideal.

Remark: If R is commutative ring with identity and $x \in R$, then $\langle x \rangle = \{rx, r \in R\} = Rx$ for example: $2Z, < 2 > = 2Z, 3Z, < 3 > = 3Z$

Definition: A ring R is called principle ideal ring if every ideal in R is principle ideal.

Theorem: (Z, +, .) is p.I. R.

Proof: suppose I be an ideal in Z if $I = \{0\}$, then $I = < 0 >$ if $I \neq \{0\}$, then $\exists$ an integer $0 \neq m \in I$, if it is negative then $-m \in I$ then I contains a positive integer

let n be the least positive integer such that $n \in I$, we claim that $I = \langle n \rangle$.

Clear that $< n > \subseteq I$ since $n \in I$.

Now, let $m \in I$ by division algorithm theorem $\exists q, r \in Z$, such that:

$$m = nq + r, 0 \leq r < n, r = m\{\in I\} - nq\{\in I\}$$

$\therefore r \in IC$! since n is the least positive int. $\in I$ and $r < n$.

$\therefore r = 0 \Rightarrow m = nq$

$\therefore m \in < n >$

$\therefore I = < n >$

The union is not ideal for example:

$$Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}, I_1 = \{\overline{0}, \overline{2}, \overline{4}\}, I_2 = \{\overline{0}, \overline{3}\},$$
$$I_i = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$$
$$3 - 2 = 1 \notin \cup\, I_i, i = 1,2$$

Definition: Let I and J be ideals of a ring R then the sum of I and J denoted by:

$$I + J = \{a + b, a \in I, b \in J\}.$$

Remark: If I and J ideals in R then $I + J$ is also ideal in R .

Proof: $I + J \neq \emptyset [0 \in I, 0 \in J \therefore 0 \in I + J]$

Let $w_1, w_2 \in I + J \Rightarrow w_1 = a_1 + b_1, a_1 \in I, b_1 \in J, w_2 = a_2 + b_2, a_2 \in I, b_2 \in J$

$$w_1 - w_2 = a_1 + b_1 - a_2 - b_2 = (a_1 - a_2)(\in I) + (b_1 - b_2)(\epsilon J)$$

$\therefore w_1 - w_2 \in I + J.$

Let $w \in I + J, r \in R, w = a + ba \in I, b \in J$

$$rw = r(a + b) = ra(\in I) + rb(\in J) \in I + J$$

$\therefore I - J$ is an ideal.

Example: $Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}, I = \{\overline{0}, \overline{1}, \overline{3}, \overline{5}\}, J = \{\overline{0}, \overline{2}, \overline{4}\}$

$I + J = \{\overline{0}, \overline{2}, \overline{4}, \overline{3}, \overline{5}, \overline{1}\} = Z_6$ is ideal

Example: in $(Z, +,)2Z + .3Z = \cdots$ ideal.

Definition: Let I and J be ideals in a ring R we say that R is internal direct sum of I and J if

(1) $R = I + J$

(2) $I \cap J = \{\emptyset\}$

we denote that by: $R = I \oplus J$.

Example: $Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}, I = \{\bar{0}, \bar{3}\}, J = \{\bar{0}, \bar{2}, \bar{4}\}$

$\therefore Z_6 = I \oplus J$ or $Z_6 = Z_6 \oplus \{0\}$

Theorem: Let I and J be ideal in R then $R = I \oplus J$ if and only if every element in R can be written in only one way.

Proof: $\Rightarrow$ Let $R = I \oplus J \Rightarrow R = I + J, I \cap J = \{0\}$ let $r \in R$

$\therefore \exists a \in I, b \in J$ such that $r = a + b$ if not $r = a_1 + b_1, a_1 \in I, b_1 \in J$

$$a_1 + b_1 = a + b \Rightarrow a_1 - a = b - b_1 \in I \cap J = \{0\}$$

$\therefore a_1 - a = 0 \Rightarrow a = a_1, b - b_1 = 0 \Rightarrow b = b_1$

$\Leftrightarrow I + J \subseteq R$, let $w \in R, w = w + 0 \in I + J$

$\therefore R \subseteq I + J \therefore R = I + J$

Let $w \in I \cap J \Rightarrow w \in I$ and $\in J w = w + 0 = 0 + wC$!

$\therefore w = 0$

Definition: Let R_1, R_2 be rings consider the set $R_1 \times R_2 = \{(x, y), x \in R_1, y \in R_2\}$, define + , on $R_1 \times R_2$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 \cdot y_2)$$

Then we can show that $R_1 \times R_2$ is a ring? Is the called external direct sum of R_1 and R_2

$R_1 \simeq R_1 \times \{0\}, R_2 \simeq \{0\} \times R_2$

Theorem: Let $f: R \longrightarrow R'$ be ring homomorphism.

(1) If k is an ideal in R' then $f^{-1}(k)$ is an ideal in R.

(2) If J is an ideal in R and f is onto then $f(J)$ is ideal in R'.

Proof: $f^{-1}(k) = \{r \in R, f(r) \in k\} \neq \emptyset$ since $\left[0 \in f^{-1}(k), f(0) = \overline{0} \in k\right]$

Let $x, y \in f^{-1}(k) \Rightarrow f(x) \in k, f(y) \in k$

k is ideal $\Rightarrow f(x) - f(y) \in k$, f is ring homomorphism $\Rightarrow f(x-y) \in k$

$\therefore x - y \in f^{-1}(k)$

let $w \in f^{-1}(k), r \in R, f(w) \in k, f(r) \in R'$ and k is ideal

$\therefore f(w).f(r) \in k$[f is ring homomorphism], $f(w.r) \in k \Rightarrow w.r \in f^{-1}(k)$

$\therefore f^{-1}(k)$ is ideal.

(2) $f(J) \neq \emptyset$ since $[0_{\dot{R}} = f(0_R) \therefore 0_{\dot{R}} \in f(J)]$

Let $x, y \in f(J) \Rightarrow x = f(w_1), w_1 \in J, y = f(w_2), w_2 \in J$

$w_1 - w_2 \in J$ [since J is an ideal], $f(w_1 - w_2) \in f(J)$[f is homomorphism]

$f(w_1) - f(w_2) \in f(J), x - y \in f(J)$

let $a \in f(J), r' \in R', a = f(w), w \in J$

$r' \in R'$ since f is onto then $\exists r \in R$ such that $f(r) = r'$

$\therefore rw \in J$[J is ideal]

$f(rw) \in f(J), f(r)f(w) \in f(J)[f$ is homomorphism $], r'a \in f(J)$

$\therefore f(J)$ is an ideal.

Corollary: Let $f: R \rightarrow R'$ be a ring homo then kerf is ideal in R .

Proof: $\ker f = \{r \in R; f(r) = 0 = f^{-1}(0_{\dot{R}}), 0_{\dot{R}}$ is ideal by theorem $f^{-1}(0_{\dot{R}})$ is ideal

$\therefore$ ker f is ideal.

The quotient ring, let I ideal in a ring $R, \frac{R}{I} = \{x + I, x \in R\}$ define + , as

$$(x+I)+(y+I)=(x+y)+I\in\frac{R}{I}$$
$$(x+I)\cdot(y+I)=(x\cdot y)+I\in\frac{R}{I}$$

to show that + , is well define (1) is well defined

$$x+I=x_1+I\Leftrightarrow x-x_1\in I$$
$$y+I=y_1+I\Leftrightarrow y-y_1\in I$$
$$(x+I)(y+I)=(x_1+I)(y_1+I)$$
$$xy+I=x_1y_1+I\Leftrightarrow xy-x_1y_1\in I$$

$$xy-x_1y_1=xy-xy_1+xy_1-x_1y_1$$
$$=x(y-y_1)+(x-x_1)y_1\in I(I\text{ is ideal })$$

then $xy-x_1y_1\in I\Rightarrow$. is well define.

Theorem: Let I be an ideal of a ring R , then $\left(\frac{R}{I},+,.\right)$ is a ring which is called the quotient ring of R by I .

Proof: (1) well defined

$$a+I=a_1+I\Leftrightarrow a-a_1\in I,b+I=b_1+I\Leftrightarrow b-b_1\in I$$
$$a+I+b+I=a_1+I+b_1+I$$
$$a+b+I=a_1+b_1+I\Leftrightarrow a+b-(a_1+b_1)\in I$$
$$a+b-a_1-b_1=a-a_1(\in I)+b-b_1(\in I)\in I$$

$\therefore +$ is well define, is well define.

(2) associative

$\mathrm{r+I+(r_1+I+r_2+I)=(r+I+r_1+I)+r_2+I}$

$\mathrm{r+I+r_1+r_2+I=r+r_1+I+r_2+I}$

$\therefore \mathrm{r+r_1+r_2+I=r+r_1+r_2+I}$

(3) identity

$r + I + 0 + I = r + 0 + I = r + I$

$\therefore$ $0 + I = I$ is the identity.

(4) $r + I + (-r) + I = r - r + I = 0 + I = I$

$\therefore$ $(-r) + I$ is the inverse

(5) $r + I + r_1 + I = r_1 + I + r + I$

$$r + r_1 + I = r_1 + r + I$$

$r + r_1 + I = r + r_1 + I$, since $r + r_1 \in R$ and R is a ring $r + r_1 = r_1 + r$ [abelian group]

$\therefore$ $\left(\frac{R}{I}, +\right)$ is abelian group.

(6) $((a + I \cdot (b + I)) \cdot (c + I) = (a \cdot b + I)(c + I) = a \cdot b \cdot c + I$

$$(a + I) \cdot ((b + I) \cdot (c + I)) = (a + I) \cdot (b \cdot c + I) = a \cdot b \cdot c + I$$

(7) $(a + I) \cdot (b + I + c + I) = (a + I)(b + c + I)$

$$\begin{aligned} &= a \cdot (b + c) + I \\ &= a \cdot b + a \cdot c + I \\ &= ab + I + a \cdot c + I \\ &= (a + I)(b + I) + (a + I)(c + I) \end{aligned}$$

$\therefore$ is associative : $\left(\frac{R}{I}, + \cdots\right)$ is a ring.

Note: If R with 1 , then $\frac{R}{I}$ with $1 + I$.

Example: Let Z be a ring,

(1) $\frac{Z}{3Z} = \{3Z, 1 + 3Z, 2 + 3Z\}$.