f_5	f_5	f_3	f_6	f_3	f_1	f_2
f_6	f_6	f_4	f_5	f_2	f_4	f_1

Sol:

$$(f_1^\circ f_1)(x) = f_1(f_1(x)) = f_1(x) = x$$
$$(f_2^\circ f_2)(x) = f_2(f_2(x)) = f_2\left(\frac{1}{x}\right) = x$$

$$(f_2\ ^\circ f_3)(x) = f_2(f_3(x)) = f_2(1-x) = \frac{1}{1-x}$$
$$(f_3\ ^\circ f_2)(x) = f_3(f_2(x)) = f_3\left(\frac{1}{x}\right) = 1 - \frac{1}{3x}$$

Thus ($G,^\circ$) is not abelian group.

Remark:

Let G be a group, then the following are equivalent

1. G is abelian.
2. $(a*b)^{-1} = a^{-1}*b^{-1}$
3. $(a*b)^2 = a^2*b^2$

Proof: (1) → (2)

$(\mathrm{a}*\ \mathrm{b})^{-1} = \mathrm{b}^{-1}*\mathrm{a}^{-1} = \mathrm{a}^{-1}*\ \mathrm{b}^{-1}$

by(2) → (3)$(\mathrm{a}*\ \mathrm{b})^2 = (\mathrm{a}*\ \mathrm{b})(\mathrm{a}*\ \mathrm{b}) = \mathrm{a}*(\ \mathrm{b}*\mathrm{a})*\ \mathrm{b} = \mathrm{a}*((\ \mathrm{b}*\mathrm{a})^{-1})^{-1}*(2)$

$= \mathrm{a}*(\mathrm{a}^{-1}*\ \mathrm{b}^{-1})^{-1}*\ \mathrm{b} = \mathrm{a}*(\mathrm{a}^{-1})^{-1}*(\ \mathrm{b}^{-1})^{-1}*\ \mathrm{b} = \mathrm{a}*\mathrm{a}*\ \mathrm{b}*\ \mathrm{b}$

$= \mathrm{a}^2*\ \mathrm{b}^2$

(3) → (1)

$(a*b)^2 = a^2*b^2$

$(a*b)*(a*bl = a*a*b*b$ (Cancellation law)

$\therefore \mathrm{b}*\mathrm{a} = \mathrm{a}*\ \mathrm{b}$

$\therefore$ G is abelian.

Remark:

Let G be a group, if $a^2 = e, \forall a \in G$, then G is abelian.
Proof: Since $a^2 = e$, then $a * a = e$

$$a^{-1} * a * a = a^{-1} * e$$
$$a = a^{-1}$$
$$(a * b)^{-1} = b^{-1} * a^{-1}$$
$$a * b = b * a$$

$\therefore G$ is abelian.

Definition:

$a \equiv b(\text{mod}n) \Leftrightarrow a - b = nk.$
This relation ≡ used to construct a new group with exactly n element

$$[a] = \{x \in Z: x \equiv a(\text{mod}n)\}$$
$$= \{x \in Z: x = a + kn; k \in Z\}$$

If we deal with module 3

$$\overline{0} = [0] = \{x \in Z: x = 3k\} = \{0, \mp 3, \mp 6, \mp 9, \dots\}$$
$$\overline{1} = [1] = \{x \in Z: x = 3k + 1\} = \{\mp 4, \mp 7, +10, \dots\}$$
$$\overline{2} = [2] = \{x \in Z: x = 3k + 2\} = \{\dots, -4, -1, 0, 5, 8, \dots\}$$
$$Z_n = \{[0], [1], \dots, [n-1]\}$$
$$Z_n = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{\text{n} - 1}\}$$

Example: $(Z_n, +_n)$

$\bar{a} + t_n \bar{b} = \overline{a + b} = \bar{c}$ s.t $a + b = c + kin; a, b, c, k \in \mathbb{Z}$ In $(Z_3, +_3); Z_3 = \{\overline{0}, \overline{1}, \overline{2}\}$

$\overline{1} +\ _3\overline{\text{I}} = \overline{2}, \overline{2} +\ _3\overline{\text{I}} = \overline{2 + 1} = \overline{0}; \overline{0}$
Example:

$$Z_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$$

Solution: $\overline{\overline{5}} + \overline{6} = \overline{11} = \overline{3}$

Example:

$(Z_n, +_n)$ Is anabelian group.
(1) $+_n$ Is a binary operation.
(2) The identity is $\overline{0}$.
(3) The inverse is $\bar{n} - a$.

Definition:

Let G be a non empty set any one to one and onto function from G into G is called permutation on G.

Example:

Let $f: Z \longrightarrow Z$ defined by $f(n) = n + 1$
f Is a permutation on Z, but $g: Z \longrightarrow Z$ defined by $g(n) = n^2$ (not onto) is not permutation.

Remark:

If $N = \{1,2, \dots, n\}$ a permutation on N is any function $N \to N$ which is one to one and onto

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 2 & 3 & 4 & \dots & \dots & n & 1 \end{pmatrix}$$

The number of permutation is $n!$. The set of allpermutation on N is denoted by $\mathrm{S_n}$.

$$S_6 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$$

If G is a group, then G and $\{e\}$ are subgroups of G is called trivial subgroups and H is called proper subgroup if $H \subset \exists$.

Theorem:

Let G be a group and $\emptyset \neq H \subseteq G$, then H is subgroup of G if and only if $a * b^{-1} \in H \; \forall a, b \in H$.
Proof: $\Rightarrow$) trivial
$\Leftrightarrow H \neq \emptyset, \exists y \in H$
Then $y * y^{-1} \in H$, but $H \subseteq G$,
Thus $e \in H$.

$e, y \in H$, then $e * y^{-1} \in H$, then $y^{-1} \in H$.
$a, b \in H$, then $a^{-1}, b^{-1} \in H$.
Thus $a * (b^{-1})^{-1} \in H$, which implies that $a * b \in H$ and H is subgroup.

Theorem:

Let H be a non empty finite subset of a group G, then H is a subgroup of G if and only if $a * b. \in H \forall a, b \in H$.

Proof: $\Rightarrow$) trivial
$\Leftrightarrow$)H is not empty then
So $\exists$a $\in$ H, a $\in$ H, scieh that $a * a = a^2 \in H$ and $a^3, a^4, ... \in H$.
But H is finite so $\exists i, j \in Z_*^+$such that $j > i$ and $a^j = a^i$, then
(a.) ${}^{i}a^j = e$ and $a^{-i} * a^j = e \Rightarrow a^{j-i} = e, a * \left(a^{j-i-1}\right) = e$,so $a^{-1} = a^{j-i-1}, \leftarrow H$ then a $*$ $a^{-1} \in$ H, so e $\in$ H and H is subgroup.

$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$,
$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$
$\alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$,
$\alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$
$\alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$,
$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$
(S_3,$\circ$) is a group, but not abelian.
The identity $= \alpha_1$
The inverse of $\alpha_1 = \alpha_1, \alpha_2 = \alpha_3, \alpha_3 = \alpha_2, \alpha_4 = \alpha_4, \alpha_5 = \alpha_5, \alpha_6 = \alpha_6$
0 is a binary operation is associative.

Definition:

Let $(G,*)$ be a group and $\emptyset \neq H \subseteq G$, then $(H,*)$ is called a subgroup of G if $(H,*)$ is a group itself.

For each $a, b \in H$, then $a * b \in H$.
For each $a \in H, \exists e \in H$ such that $a * e = e * a = a$.
For each $a \in H, \exists a^{-1} \in H$ such that $a * a^{-1} = e$.

Example:

$2Z \subsetneq Z \neq \emptyset$, is subgroun of Z
$Zodd \subseteq Z$,isnot subgroup

Remark:

Theorem:

Let $\{H_\alpha\}$ be a family of subgroups of a group G , then $\bigcap\limits_{\alpha\in\Lambda} \mathrm{H}_\alpha$ is also subgroups of G.

Proof: $\bigcap_{\alpha\in\Lambda} \mathrm{H}_\alpha \neq \emptyset$ {since $\forall H \in G, e \in H$}
Let $a, b \in \bigcap_{\alpha\in\Lambda} H_\alpha$, then $a \in H\alpha, \forall\alpha \in \Lambda, b \in H\alpha, \forall\alpha \in \Lambda$.
But $H\alpha$ $\forall\alpha \in \Lambda$ are subgroup, then $a * b^{-1} \in \bigcap_{\alpha\in\Lambda} \mathrm{H}_\alpha, \forall\alpha \in \Lambda$
Thus $\bigcap_{\alpha\in\Lambda} H_\alpha$ is subgroup.

Example:

$Z_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$, then $(Z_4, +_4)$ is a group
$H = \{\overline{0}, \overline{2}\}$ is a subgroup of Z_4
Q: (1) If H is subgroup of G, is the identity of H is the same as the identity of G .

Solution:Let $x \in H, H \subseteq G$, if a is the identity of H

$$x^{-1}(xa = x)$$
$$x^{-1}xa = x^{-1}x \Rightarrow a = e$$

(2) If H_1 and H_2 are subgroups of a group G. Is $H_1 \cup H_2$ a subgroup of G ?
Solution: If. $Z6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ is a group, $H_1 = \{\overline{0}, \overline{3}\}, H_2 = \{\overline{0}, \overline{2}, \overline{4}\}$ are subgroups of Z_6
$H_1 \cup H_2 = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$ is not subgroup.

Theorem: