$nt = (m+k) \cdot t$ (بضرب t)
$= mt + kt$
$\rightarrow mt < nt$

5. $m + t < n + t$

By (1) either $m = n$ or $m < n$ or $n < m$

if $m = n \rightarrow m + t = n + t$ C!

if $n < m \longrightarrow n + t < m + t$ by (3) C! [since $m + t < n + t$]

so that $m < n$

Construction of the integer numbers.

Let ℕ be the natural numbers, and Let $\mathbb{N} \times \mathbb{N} = \{(m,n); m,n \in W\}$

First, we define relation (N) on the set $\mathbb{N} \times \mathbb{N}$ as follows

$(m,n) \sim (p,q)$ iff $m + q = n + p$

claim that $(\sim)$ is equivalence relation

1. Reflexive

Let $(m,n) \in \mathbb{N} \times| N$

we know that $m + n = n + m \longrightarrow (m,n) \sim (m,n)$

2. Symmetric

Let $(m,n),(p,q) \in \mathbb{N} \times \mathbb{N}$

we have to show that if $(m,n) \sim (p,q)$ then $(p,q) \sim (m,n)$

$(m,n) \sim (p,q) \longrightarrow m + q = n + p \longrightarrow p + n = q + m$

So that, $(p,q) \sim (m,n)$

3. Transitive

Let $(m,n),(p(q),(r,s) \in \mathbb{N} \times \mathbb{N}$

we have to show that

if $(m,n) \sim (p,q)$ and $(p,q) \sim (r,s)$, then $(m,n) \sim (r,s)$,

$m + q = n + p$
$p + s = q + r$
$m + q + p + s = n + p + q + r \longrightarrow m + s = n + r$

So that $(m,n) \sim (r,s)$

Hence $(\sim)$ is equivalence relation on $\mathbb{N} \times \mathbb{N}$

Second,

Since $(\sim)$ is the equivalence relation on $\mathbb{N} \times \mathbb{N}$, then the equivalence classes of (m,n) is called integer number and dented by $(\overline{m,n})$.

(m,n) is called the representation of the integer number. ie

$\overline{(m,n)} = \{(p,q) \in \mathbb{N} \times \mathbb{N}; (p,q) \sim (m,n)\}$

The set of all equivalence classes is called "set of integer numbers" and is denoted by Z.

Remark: $x = \overline{(m,n)}$ denoted by $x = m - n$

ينظر الى صف التكافؤ $\overline{(m,n)}$على انه العدد m-n

Example:

1. $0 =: \overline{(m,m)}$
$= \{(p,q) \in \mathbb{N} \times \mathbb{N}; (p,q) \sim (m,m)\}$
$= \{(p,q) \in \mathbb{N} \times \mathbb{N}; p + m = q + m\}$
$= \{(p,q) \in \mathbb{N} \times \mathbb{N}; p = q\}$
$= \{(1,1), (2,2), (3,3), \dots\}$

2.
$\overline{(2,1)} = \{(p,q) \in \mathbb{N} \times \mathbb{N}; (p,q) \sim (2,1)\}$
$= \{(p,q); p + 1 = q + 2\}$
$= \{(p,q); p = q + 1\}$
$= \{(2,1), (3,2), (4,3), \dots\}$

3.
$\overline{(1,2)} = \{(p,q) \in \mathbb{N} \times \mathbb{N}; (p,q) \sim (1,2)\}$
$= \{(p,q); p + 2 = q + 1\}$
$= \{(p,q); p + 1 = q\}$
$= \{(1,2), (2,3); (3,4); \dots\}$

4. What is the integer number (+7)

$+7 = \{(8,1), (9,2), (10,3), \dots\}$

Summation and multiplication on $\boldsymbol{Z}$.

Definition: Let $x = \overline{(m,n)}$ and $y = \overline{(r,s)}$ are two integer numbers. Then

1. $x + y = \overline{(m + r, n + s)}$

2. $x \cdot y = \overline{(m \cdot r + n \cdot s, m \cdot s + n \cdot r)}$

Example: Find

1. $(-3) + 7$
2. $(-3).7$

Solution

$-3 = \overline{(1,4)}, \quad 7 = \overline{(8,1)}$

1. $(-3) + 7 = \overline{(1,4)} + \overline{(8,1)} = \overline{(1 + 8, 4 + 1)} = \overline{(9,5)} = 4$

2. $(-3) \cdot 7 = \overline{(1,4)} \cdot \overline{(8,1)} = \overline{(8 + 4, 1 + 32)} = \overline{(12,33)} = -21$

Proposition: Let $x, y, z \in Z$. Then

1. $x + y = y + x, \qquad x \cdot y = y \cdot x$

2. $(x + y) + z = x + (y + z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$

3. $x + 0 = 0 + x = x, \qquad x \cdot 1 = 1 \cdot x = x$

4. $x \cdot (y + z) = x \cdot y + x \cdot z$

5. $x + (-x) = 0$, $(-x)$ is called inverse of x

Proof:

1. Let $x = \overline{(m,n)}, \quad y = \overline{(r,s)}$

$$x + y = \overline{(m,n)} + \overline{(r,s)} = \overline{(m + r, n + s)} = \overline{(r + m, s + n)} = \overline{(r,s)} + \overline{(m,n)} = y + x$$

$$x \cdot y = \overline{(m,n)} \cdot \overline{(r,s)} = \overline{(mr + ns, ms + nr)} = \overline{(rm + sn, sm + rn)} = \overline{(r,s)} \cdot \overline{(m,n)} = y \cdot x$$

3. Let $x = \overline{(r,s)}, 0 = \overline{(m,m)}$

$$x + 0 = \overline{(r,s)} + \overline{(m,m)} = \overline{(r + m, s + m)} = \overline{(r,s)} = x$$

Since $r + m + s = s + m + r \longrightarrow (r + m, s + m) \sim (r,s) \rightarrow \overline{(r + m, s + m)} = \overline{(r,s)}$

Similarly $0 + x = x$

Let $1 = \overline{(m+1, m)}$

$\therefore x \colon 1 = \overline{(r,s)} \cdot \overline{(m+1, m)} = \overline{(r \cdot (m+1) + sm, rm + s(m+1))}$

$= \overline{(rm + r + sm, rm + sm + s)} = \overline{(r,s)} = x$

Since

$r\mathrm{m} + r + sm + \mathrm{s} = rm + sm + \mathrm{s} + r$

$\rightarrow (rm + r + sm, rm + sm + s) \sim (r,s)$

$\rightarrow \overline{(rm + r + sm, rm + sm + \mathrm{s})} = \overline{(r,s)}$

5. Let $x = \overline{(m,n)}$, put $-x = \overline{(n,m)}$

$x + (-x) = \overline{(m,n)} + \overline{(n,m)} = \overline{(m+n, n+m)} = \overline{(m,m)} = 0$

Since

$m + n + m = n + m + m \longrightarrow (m+n, n+m) \sim (m,m)$

$\longrightarrow (m+n, n+m) = \overline{(m,m)}$.

Definition: Let $x, y \in Z$; the difference between x and y denoted by

$x - y = x + (-y)$

Definition: Let $x = \overline{(m,n)}$, then x is said to be positive integer number if $m > n$ and x is called negative integer number if $n > m$, Where " $>$ " is order relation on $\mathbb{N}$.

Proposition: Let $x, y \in Z$

1- If each of x, y are positive then $x + y, x \cdot y$ are positive.

2- If each of x, y are negative then $x + y$ is negative and $x \cdot y$ is positive.

3- If x is positive and y is negative then $x . y$ is negative.

Proof:

1. Let $x = \overline{(m,n)}, y = \overline{(r,s)}$

Since each of x, y are positive

So that

$m > n$

$r > s$

$\rightarrow m + r > n + \mathrm{s}$

Hence $x + y = (m + r, n + \mathrm{s})$ is positive.

Remark: 1. For each $x, y, \in Z$ then either $x = y$ or $x > y$ or $x < y$.

2. $x \geqslant y$ that is mean $x > y$ or $x = y$.

Proposition: Let $x, y, z \in Z$

1- $x \geqslant x$ -1

2- If $x \geqslant y$ and $y \geqslant x$ then $x = y$

3- If $x \geqslant y$ and $y \geqslant z$ then $x \geqslant z$

Proof:

$$\begin{aligned} &1- \quad x \geqslant x \longrightarrow x > x \text{ or } x = x \\ &2- \quad x \geqslant y \longrightarrow x > y \text{ or } x = y \\ &\qquad\; x \leqslant y \longrightarrow x < y \text{ or } x = y \end{aligned}$$

$$\begin{aligned} x \geqslant y \text{ and } x \leqslant y &\rightarrow (x > y \vee x = y) \wedge (x < y \vee x = y) \\ &\rightarrow (x = y \vee x > y) \wedge (x = y \vee x < y) \\ &\rightarrow (x = y) \vee (x > y \wedge x < y)\text{C!} \\ &\rightarrow x = y \end{aligned}$$

$3 - \ x \geqslant y \longrightarrow x > y$ or $x = y$.

$$\left.\begin{aligned} &y \geqslant z \rightarrow y > z \text{ ar } y = z \\ &(x > y \vee x = y) \wedge (y > z \vee y = z) \\ &x > y \wedge y > z \longrightarrow x > z \\ &x > y \wedge y = z \longrightarrow x > z \\ &x = y \wedge y > z \longrightarrow x > z \\ &x = y \wedge y = z \longrightarrow x = z \end{aligned}\right\} \Rightarrow x > z \vee x = z \rightarrow x \geqslant z$$

Since

$$\left.\begin{aligned} x > y \rightarrow x - y > 0 \\ y > z \rightarrow y - z > 0 \end{aligned}\right\} \rightarrow (x - y) + (y - z) > 0 \rightarrow x - z > 0 \rightarrow x > z$$

Theorem [Division Algorithm] خوارزمية القسمة

Let a, b are positive integer numbers and $a > b$, there exist positive integer number q and nonnegative integer number r such that

$a = bq + r,\ 0 \leqslant r < b$. In addition, q and r are unique elements satisfies this condition.

r is called Remainder and q is called quotient

For example

1. $a = 3, b = 2 \longrightarrow 3 = 2.1 + 1; q = 1, r = 1$, os r < 2

2. $a = 7, b = 3 \longrightarrow 7 = 3.2 + 1; q = 2, r = 1, 0 \leqslant r \leqslant 3$

3. $a = 8, b = 4 \longrightarrow 8 = 4.2 + 0,; q = 2, r = 0$

Example: Let a be positive number, if $b = 2$ then either

$$a = 2q \text{ or } a = 2q + 1;\ 0 \leqslant r < 2$$

(i.e either a is even number or a is add number)

Solution: By division algorithm, $\exists q, r$ s.t

$a = 2q + r\ ;\ 0 \leqslant r < 2$

So that

if $r = 0 \rightarrow a = 2q$, hence a is even number

if $r = 1 \longrightarrow a = 2q + 1$, hence a is odd number

Example: Prove that each odd integer number can be written as

$4k + 1$ or $4k + 3; k \in Z$

Solution:

We take a is odd number and take $b = 4$, hence by division algorithm $a = 4k + r$ wher s $0 \leqslant r < 4$

If $r = 0 \rightarrow a = 4k$ C! Since a is odd while $4k$ is even

If $r = 1 \longrightarrow a = 4k + 1$

If $r = 2 \longrightarrow a = 2k + 2$ C! Since a is odd while $4k$ is even

If $r = 3 \longrightarrow a = 4k + 3$

So that, any odd number is write as $4k + 1$ or $4k + 3$