Ordered relation علاقات الترتيب

Partially ordered sets المجموعات المرتبة جزئياً

A partial order in a set A is a relation R in A which is

1) reflexive, i.e. $(a,a) \in R$ for every $a \in A$

2) anti-symmetric, i.e. $(a,b) \in R$ and $(b,a) \in R$ implies $a=b$

3) transitive, i.e. $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$

Furthermore, if a relation R in A defines a partial order in A, then $(a,b) \in R$ is denoted by $a \lesssim b$

which reads "a precedes b"

Example ① Let A be a family of sets. Then the relation in A defined by "x is a subset of y" is a partial order in A

Example ② Let A be any subset of the real numbers. Then the relation in A defined by "$x \lesssim y$" is a partial order in A.

Example ③ Let R be the relation in the natural numbers N defined by "x is a multiple of y"; then R is a partial order in N.

Moreover, $6 \lesssim 2$, $15 \lesssim 3$ and $17 \lesssim 17$

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Definition Two elements a and b in a partially ordered set are said to be not comparable (لا يقبلان المقارنة)

if $a \not\preceq b$ and $b \not\preceq a$

In other words, a and b are comparable if $a \preceq b$ or $b \preceq a$.

In example 3, the numbers 3 and 5 are not comparable since neither number is a multiple of the other.

Example Let R be the relation in the natural numbers N defined by "x divides y".

Note that 4, 6 not comparable

while 5, 10 is comparable

Totally Ordered

A total order in a set A is a partial order in A with the additional property that

$a < b$ or $a = b$ or $a > b$

Example The partial order in any set A of real numbers (with the natural order) is a total order since any two numbers are comparable.

Example Let R be the partial order in $V = \{1, 2, 3, 4, 5, 6\}$ defined by "x divides y". Then R is not a total order in V since 3 and 5 are not comparable.

Remark The word "order" will be used instead of either partial order or total order.

First and Last elements العناصر الأولى والأخيرة

Let A be an ordered set. The element $a \in A$ is called a first element of A if, for every element $x \in A$, $a \lesssim x$

that is, if a precedes every element in A.

Also, an element $b \in A$ is called a last element of A if, for every $x \in A$

$$x \lesssim b$$

that is, if b dominates every element belonging to A.

Example Consider N, the natural numbers (with the natural order). Then 1 is a first element of N. There is no last element.

Example Let $A = \{a, b, c, d, e\}$

Let $R_1 = \{(a,a), (b,b), (c,c), (d,d), (e,e), (e,c), (c,a), (e,a), (d,c), (d,a), (d,b), (b,a)\}$

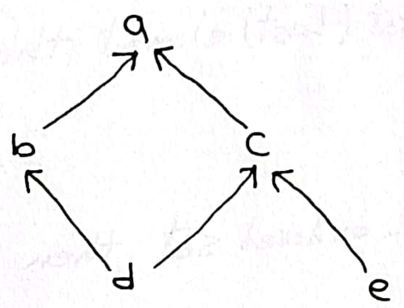
There is not exist first element.

The last element is a

Let $R_2 = \{(a,a), (b,b), (c,c), (d,d), (e,e), (c,e), (d,e), (e,a), (e,b), (c,a)\}$

R_2 contains no first element and no last element.

Example Let $W = \{a, b, c, d, e\}$ be ordered by the following diagram:



Then a is a last element in W since a dominates every element.

Note that W has no first element. The element d is not a first element since d does not precede e.

Definition Let A be an ordered set with the property that every subset of A contains a first element. Then A is called a well-ordered set.

Example

① $\mathbb{Z}$ is not a well-ordered set.

since $\mathbb{Z} \subseteq \mathbb{Z}$ and $\mathbb{Z}$ does not contains first element

② $\mathbb{R}$ and $\mathbb{Q}$ are not well-ordered sets.

Similar

Proposition

① If A is an well-ordered set then it is totally ordered.

② Every subset of a well-ordered set is a well-ordered.

③ If there exists first (Last) element then it is unique.

Proof

① since A is well-ordered set then A is partially ordered set

Now,

if $a, b \in A \longrightarrow \{a, b\} \subseteq A$

But A is well-ordered set, so that $\{a, b\}$ contains a first element which, therefore, must precede the other; hence

A is totally ordered set.

(2) Let A be a well-ordered set

Let $B \subseteq A$, we have to show that B is well-ordered set

Now,

Let $C \subseteq B$

since $B \subseteq A$, hence $C \subseteq A$

But A is a well-ordered set, i.e. every subset of A contains first element.

so that C has a first element

hence B is well-ordered set.

(3) suppose that both a and b are first element

$\therefore a R x \quad \forall x \in A$

$\rightarrow a R b$ --- (1)

Also $b R x \quad \forall x \in A$

$\rightarrow b R a$ --- (2)

since R is anti-symmetric relation

we get from (1) & (2) $a = b$

i.e the first element is unique

Maximal and Minimal elements

العناصر العظمى والصغرى

Let A be an ordered set. The element $a \in A$ is called a maximal element if $a \lesssim x$ implies $a = x$.

In other words, a is a maximal element if there is no element in A which strictly dominates a. Similarly

An element $b \in A$ is called a minimal element if

$$x \lesssim b \text{ implies } b = x$$

that is, if there is no element in A which strictly precedes b.

Example:- Let $A = \{a, b, c, d, e\}$

① $R_1 = \{(a,a), (b,b), (c,c), (d,d), (e,e), (e,c), (c,a), (e,a), (d,c), (d,a), (d,b), (b,a)\}$

Then a is maximal element and e and d are minimal elements in R_1

② $R_2 = \{(a,a), (b,b), (c,c), (d,d), (e,e), (d,e), (e,a), (e,d), (c,a)\}$

a and b are maximal elements

b and c are minimal elements

Proposition Let A be a partially ordered set. Then

1) If a is a first element in A, then a is a minimal element in A.

2) If b is Last element in A, then b is a maximal element in A.

Proof

1) suppose that a is a first element in A.

by definition we get $aRx \quad \forall\, x \in A$

Now,

Let $b \in A$ such that bRa

$\therefore aRb \wedge bRa$

Since R is anti-symmetric, so that $a = b$

hence a is a minimal element

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