ogical Equivalence   التكافؤ المنطقي

Two statements are said to be logically equivalent if their truth tables are identical. We denote the logical equivalent of p and q by " $p \equiv q$ ".

Example :- The truth tables of $(p \to q) \wedge (q \to p)$ and $p \leftrightarrow q$ are as follows

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p	q	$p \leftrightarrow$
T	T	T
T	F	F
F	T	F
F	F	T

Hence $(p \to q) \wedge (q \to p) \equiv p \leftrightarrow q$

Example :- The truth tables below ~~sho~~ show that $p \to q$ and $\sim p \vee q$ are logically equivalent, i.e., $p \to q \equiv \sim p \vee q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Theorem:- The following propositions are logically equivalent

1- $p \vee p \equiv p$, $p \wedge p \equiv p$

2- $(p \vee q) \vee s \equiv p \vee (q \vee s)$, $(p \wedge q) \wedge s \equiv p \wedge (q \wedge s)$

3- $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$

4- $p \vee (q \wedge s) \equiv (p \vee q) \wedge (p \vee s)$, $p \wedge (q \vee s) \equiv (p \wedge q) \vee (p \wedge s)$

5- $\sim(\sim p) \equiv p$, $(p \rightarrow q) \wedge (q \rightarrow p) \equiv p \leftrightarrow q$

6- $\sim(p \vee q) \equiv \sim p \wedge \sim q$, $\sim(p \wedge q) \equiv \sim p \vee \sim q$

7- $p \rightarrow q \equiv \sim q \rightarrow \sim p$, $q \rightarrow p \equiv \sim p \rightarrow \sim q$

8- $p \rightarrow q \equiv \sim p \vee q$, $p \rightarrow \sim q \equiv \sim p \vee \sim q$

9- $\sim p \rightarrow q \equiv p \vee q$, $\sim q \rightarrow p \equiv p \vee q$

10- $\sim(p \rightarrow q) \equiv p \wedge \sim q$

11- $(p \vee s) \rightarrow q \equiv (p \rightarrow q) \wedge (s \rightarrow q)$

$p \rightarrow (q \wedge s) \equiv (p \rightarrow q) \wedge (p \rightarrow s)$

$p \rightarrow (q \vee s) \equiv (p \rightarrow q) \vee (p \rightarrow s)$

Tautologies and Contradiction

التحصيل حاصل والتناقض

Some propositions $P(p,q,---)$ contain only T in the last column of their truth tables. In other words the proposition $P(p,q,---)$ will always become a true statement no matter which statements $p_0, q_0, ---$, true or false, are substituted for the variables. Such propositions are called tautologies.

Definition:- A proposition $P(p,q,---)$ is a tautology if $P(p_0,q_0,--)$ is true for any statements $p_0, q_0, ----$

Definition:- A proposition $P(p,q,---)$ is a contradication if $P(p_0,q_0,---)$ is fals for any statements $p_0, q_0, ----$ In other words, a contradiction will contain only F in the last column of its truth table.

Example:- The proposition "p or not p", i.e. $p \vee \sim p$, is a tautology. while the proposition "p and not p", i.e $p \wedge \sim p$, is a contradiction.

This fact is verified by the following table

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

Arguments

A statement that a set of assumptions $S_1, S_2, ---, S_n$ yields another assumption (S is called result), denoted by $S_1 \wedge S_2 \wedge --- \wedge S_n \longrightarrow S$ is called a argument.

Remark:- An argument on propositions $S_1 \wedge S_2 --- \wedge S_n \longrightarrow S$ is said to be valid if S is true.

i.e $S_1 \wedge S_2 --- \wedge S_n \longrightarrow S$ is a tautology تحصيل حاصل

Example:- consider the argument,

S_1: If a man is not married, he is unhappy.

S_2: If a man is unhappy, he dies young.

S: A man is not married die young.

Solution:-

Let p:- he is not married.

q:- he is unhappy.

r:- he dies young

Then $S_1, S_2 \longrightarrow S$ can be written

$$p \to q,\ q \to r \longrightarrow (p \to r)$$

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	Argument
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

So that, the given argument is valid.

Example:- If the sky is not rain then Ahmed will feel very well. But it is rain, so that Ahmed won't feel very well.

Solution:-

Let p:- The sky is not rain

q:- Ahmed will feel very well

Then the above argument can be written in the form

$$(p \to q) \wedge \sim p \longrightarrow \sim q$$

OR

Let P_1:- The sky is rain

P_2:- Ahmed will feel very well

So that

$$(\sim P_1 \longrightarrow P_2) \wedge P_1 \longrightarrow \sim P_2$$

P_1	P_2	$\sim P_1$	$\sim P_2$	$\sim P_1 \to P_2$	$(\sim P_1 \to P_2) \wedge P_1$	Argument
T	T	F	F	T	T	F
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	F	T	T	F	F	T

Hence the argument is not true.

ج :

في مثال سابق وجدنا هناك ثلاث فرضيات جعلت الجدول كبير لذلك فان الحاجه لا يمكن أن نعرف بأنها صائبه أو خاطئه بسهوله لهذا نحتاج الى الجدول.

Assumptions	Result
$p \wedge (p \to q)$	q
$(p \to q) \wedge \sim q$	$\sim p$
$p \wedge q$	p
$p \wedge q$	q
$(p \vee q) \wedge \sim q$	p
$(p \vee q) \wedge \sim p$	q
$(p \to q) \wedge (q \to r)$	$p \to r$

Example:- show the following argument is true.

If it is rain, then ALi could not play tennis.

ALi tell me if he could not play tennis, he will go to the library.

Conclusion (result)

It is rain then ALi is read in the library.

Solution:-

p:- It is rain

q:- ALi play tennis

r:- ALi go to the library

assumptions:-

① $p \longrightarrow \sim q$

② $\sim q \longrightarrow r$

③ p

So that the argument can be written as the form

$$(p \longrightarrow \sim q) \wedge (\sim q \longrightarrow r) \wedge p \xrightarrow{?} r$$

$(p \to \sim q) \wedge (\sim q \to r) \longrightarrow (p \to r)$ by

$p \wedge (p \to r) \longrightarrow r$ by

So that the argument is true.

OR

$p \wedge (p \to \sim q) \longrightarrow \sim q$ by

$\sim q \wedge (\sim q \to r) \longrightarrow r$ by

<u>Example:-</u> Is the following argument is true? why?

<u>assumptions</u>

① p ② $p \to q$ ③ $r \to \sim q$

<u>result</u> $\sim r$

<u>Solution:-</u> $p \wedge (p \to q) \wedge (r \to \sim q) \xrightarrow{?} \sim r$

$p \wedge (p \to q) \longrightarrow q$

$r \to \sim q \equiv q \to \sim r$

$q \wedge (q \to \sim r) \longrightarrow \sim r$

Hence, the argument is true

OR

$$r \longrightarrow \sim q \equiv q \longrightarrow \sim r$$

$$(p \longrightarrow q) \wedge (q \longrightarrow \sim r) \longrightarrow (p \longrightarrow \sim r)$$

$$p \wedge (p \longrightarrow \sim r) \longrightarrow \sim r$$

Example:- prove the following argument is true.

assumptions

① $\sim p \longrightarrow q$ ② $\sim r \vee \sim s$ ③ $q \longrightarrow s$ ④ r

result p

Solution:- $(\sim r \vee \sim s) \wedge r \longrightarrow \sim s$

$$q \longrightarrow s \equiv \sim s \longrightarrow \sim q$$

$$\sim s \wedge (\sim s \longrightarrow \sim q) \longrightarrow \sim q$$

$$\sim p \longrightarrow q \equiv \sim q \longrightarrow p$$

$$\sim q \wedge (\sim q \longrightarrow p) \longrightarrow p$$

OR

$$r \wedge (\sim r \vee \sim s) \longrightarrow \sim s$$

$$(q \longrightarrow s) \wedge \sim s \longrightarrow \sim q$$

$$(\sim p \longrightarrow q) \wedge \sim q \longrightarrow \sim(\sim p)$$

$$\sim(\sim p) \equiv p$$

OR $(\sim p \rightarrow q) \wedge (q \rightarrow s) \longrightarrow (\sim p \rightarrow s)$

$$(\sim r \vee \sim s) \wedge r \longrightarrow \sim s$$

$$\sim p \rightarrow s \equiv \sim s \rightarrow p$$

$$(\sim s) \wedge (\sim s \rightarrow p) \longrightarrow p$$

so that the argument is true.

ملاحظة :- أن عملية تجزئة الحاجيات الى حاجيات جزئية أصغر تدعى بالبرهان المباشر Direct proof

لكن هناك كثيراً من المبرهنات في الرياضيات بالصيغة " اذا كان p فان q " ولغرض برهان هذا النوع من المبرهنات نفرض أن p معطاه ونكون سلسله من العبارات بالصيغه

$$p \rightarrow p_1 ,\ p_1 \rightarrow p_2 ,\ ---- ,\ p_{n-1} \rightarrow p_n ,\ p_n \rightarrow q$$

حيث أن كل من هذه العبارات هي نتائج سابقه أو مبرهنات سبق التأكد من صحتها أو حقائق علميه معروفه . لذلك عندما نريد معرفه كون الحاجه صائبه نعمل على استخدام البرهان المباشر وهو نفس العمليه التي تمت في الامثله الســابقه أي الحصول على النتيجه من خلال ربط الفرضيات مع بعضها البعض واستخدام بعض الصيغ التي تسهل البرهان

Example:- Use the Direct proof to show that the following argument is true.

assumptions

① $p \rightarrow q$ ② $q \rightarrow \sim s$ ③ $\sim t \rightarrow s$ ④ $m \rightarrow \sim t$

result

$p \rightarrow \sim m$

Solution:-

$$(p \rightarrow q) \wedge (q \rightarrow \sim s) \longrightarrow (p \rightarrow \sim s)$$

$$(m \rightarrow \sim t) \wedge (\sim t \rightarrow s) \longrightarrow (m \rightarrow s)$$

$$m \rightarrow s \equiv \sim s \rightarrow \sim m$$

$$(p \rightarrow \sim s) \wedge (\sim s \rightarrow \sim m) \longrightarrow (p \rightarrow \sim m)$$

OR

$$(p \rightarrow q) \wedge (q \rightarrow \sim s) \longrightarrow (p \rightarrow \sim s)$$

$$\sim t \rightarrow s \equiv \sim s \rightarrow t$$

$$(p \rightarrow \sim s) \wedge (\sim s \rightarrow t) \longrightarrow (p \rightarrow t)$$

$$m \rightarrow \sim t \equiv t \rightarrow \sim m$$

$$(p \rightarrow t) \wedge (t \rightarrow \sim m) \longrightarrow (p \rightarrow \sim m)$$

Hence, the argument is true

Example:- prove that if a is even then a^2 is even.

Solution:- since a is even then $a = 2n \; ; n \in \mathbb{Z}$

$$a^2 = (2n)^2 = 4n^2 = 2(2n^2) \quad ; \; 2n^2 \in \mathbb{Z}$$

so that a^2 is even number

How to proof this example ?

P_1 :- a is even number

P_2 :- a is written by $2n \; ; n \in \mathbb{Z}$

P_3 :- take square, we get $a^2 = 2(2n^2)$

q :- a^2 is even number

$$P_1 \wedge P_2 \wedge P_3 \longrightarrow q$$

Indirect proof

البرهان غير المباشر

ات يصعب أو يتعذر أثبات صحتها بأستخدام البرهان المباشر لذلك

يدة للبرهان تدعى بالبرهان غير المباشر .

هان غير المباشر نفرض أن النتيجه هي عبارة كاذبه ثم نقوم بربطها مع

نحاول الحصول على تناقض (عبارة مع نفيها في آن واحد).

Example:- By use the indirect proof show that the following argument is true.

Assumptions

① $\sim a \longrightarrow d$ ② $\sim c \vee d$ ③ $b \longrightarrow c$ ④ $a \longleftrightarrow b$

result d

Solution:- By use indirect proof, suppose that d is false i.e $\sim d$

so that we have the following argument

$$(\sim a \longrightarrow d) \wedge (\sim c \vee d) \wedge (b \longrightarrow c) \wedge (a \longleftrightarrow b) \wedge (\sim d) \xrightarrow{?} \text{contradiction}$$

$\sim d \wedge (\sim c \vee d) \longrightarrow \sim c$

$(a \longleftrightarrow b) \wedge (b \longrightarrow c) \longrightarrow (a \longrightarrow c)$

$(\sim a \longrightarrow d) \wedge (\sim d) \longrightarrow \sim(\sim a) \equiv a$

$a \wedge (a \longrightarrow c) \longrightarrow c$

$\sim c \wedge c \longrightarrow c!$

OR

$(\sim a \longrightarrow d) \wedge \sim d \longrightarrow \sim(\sim a) \equiv a$

$(a \longleftrightarrow b) \wedge (b \longrightarrow c) \longrightarrow (a \longrightarrow c)$

$(\sim c \vee d) \wedge \sim d \longrightarrow \sim c$

$(a \longrightarrow c) \wedge \sim c \longrightarrow \sim a$

$a \wedge \sim a \longrightarrow c!$

so that d is true, hence the argument is true

Example:- prove that if a^2 is even number then a is even number.

By indirect proof:

Suppose that a is odd number, hence a can be written as the form

$a = 2n+1 \quad ; n \in \mathbb{Z}$

$a^2 = (2n+1)^2 = 4n^2+4n+1 = 2(2n^2+2n)+1 = 2m+1 \quad ; m = 2n^2+2n \in \mathbb{Z}$

So that a^2 is odd number C!

Hence, a is even number

Example:- For each set A prove that $\phi \subseteq A$.

Solution:- Using indirect proof

Suppose that ϕ is not subset of A (i.e $\phi \not\subseteq A$)

So that there exist element $x \in \phi$ such that $x \notin A$

But ϕ empty set, that is contradiction

Hence $\phi \subseteq A$

Example :- Use indirect proof to show that the following argument is true.

assumptions

① $p \to q$ ② $q \to \sim s$ ③ $\sim t \to s$ ④ $m \to \sim t$

result $p \to \sim m$

Solution:- suppose that $p \to \sim m$ is false statement

hence $\sim(p \to \sim m) \equiv p \wedge m$

$$\left[\begin{array}{l} p \to \sim m \equiv \sim p \vee \sim m \\ \sim(p \to \sim m) \equiv \sim(\sim p \vee \sim m) \equiv p \wedge m \end{array}\right]$$

we must to prove that

$(p \to q) \wedge (q \to \sim s) \wedge (\sim t \to s) \wedge (m \to \sim t) \wedge (p \wedge m) \to C!$

$(p \to q) \wedge (q \to \sim s) \to (p \to \sim s)$

$(m \to \sim t) \wedge (\sim t \to s) \to (m \to s)$

$m \to s \equiv \sim s \to \sim m$

$(p \to \sim s) \wedge (\sim s \to \sim m) \to (p \to \sim m)$

$p \wedge m \to p$

$p \wedge (p \to \sim m) \to \sim m$

$p \wedge m \to m$

$m \wedge \sim m \to C!$, hence the argument is true

Let a set A be given. An open-sentence on A is an expression denoted by p(x) which has the property that p(a) is true or false for each $a \in A$.

In other words, p(x) is an open sentence on A if p(x) becomes a statement whenever any element $a \in A$ is substituted for the variable x.

Example: ① Let p(x) be "$x+2 > 7$". Then p(x) is an open sentence on N; the set of natural numbers.

② Let p(x) be "$x+1 < 3$". Then p(x) is an open sentence on A; where $A = \{0, 1, 2, 3\}$

$0+1 < 3$	true statement
$1+1 < 3$	" "
$2+1 < 3$	false statement
$3+1 < 3$	" "

Moreover, if p(x) is an open sentence on a set A, then the set of elements $a \in A$ with the property that p(a) is true is called the truth set T_p of p(x). In other words

$$T_p = \{x ; x \in A, p(x) \text{ is true}\}$$

or, simply,

$$T_p = \{x, p(x)\}$$

Example ① Consider the open sentence " $x+2>7$ " defined on $\mathbb{N}$, the set of natural numbers. Then

$T_p = \{x \mid x \in \mathbb{N}, x+2>7\} = \{6, 7, 8, ---\}$

is its truth set.

② Let $p(x)$ be " $x+5<3$ ". Then the truth set of $p(x)$ on $\mathbb{N}$ is

$T_p = \{x \mid x \in \mathbb{N}, x+5<3\} = \phi$

the empty set.

③ Let $p(x)$ be " $x+5>1$ ". Then the truth set of $p(x)$ on $\mathbb{N}$ is

$T_p = \{x \mid x \in \mathbb{N}, x+5>1\} = \mathbb{N}$

Notice, by the preceding examples, that if $p(x)$ is an open sentence defined on a set A then $p(x)$ could be true for all $x \in A$, for some $x \in A$ or for no $x \in A$.