$\exists n \in z^{+}$ s.t $na > b$
$n \in > 1 \leftrightarrow \frac{1}{n} < \epsilon$

The densely of rational numbers كثافة الاعداد النسبية

Theorem: Let $a, b \in \mathbb{R}$ such that $a < b$, then there exists $r \in Q$ such that

$a < r < b$

بين اي عددين حقيقيين يوجد عدد نسبي واحد على الاقل

Corollary:

Let $a, b \in \mathbb{R}$ such that $a < b$, then there exists a countable infinite set of rational numbers between a and b.

Proof: since $a < b$

by the densely of rational numbers theorem, there exist r_1 s.t $a < r_1 < b$ similarly $a, r \in R$ and $a < r_1$, then also by the densely, there exists $r_2 \in Q$ s.t $a < r_2 < r_1$

In general between a & r_{n-1}, there exists $r_n \in Q$ s.t $a < r_n < r_1$

Hence, we have the countable infinite set $\{r_1, r_2, \dots, r_n, \dots\}$ of rational numbers between a and b.

Lemma: If $r \in Q$ and $s \in Q'$, then $r + s \in Q'$

Proof: suppose that $r + s \notin Q'$, so that $r + s \in Q'$

$\rightarrow (r + s) - r \in Q$
$\longrightarrow (r + s) + (-r) \in Q$

but $(r + s) - r = s \in Q\ C$!

Hence, $r + s \in Q'$

The density of irrational numbers كثافة الاعداد الغير نسبية

Let $a, b \in \mathbb{R}$ and $a < b$, then the exists $s \in \varphi'$ such that $a < s < b$.

Proof: Suppose that theorem is not true.

so that by the density of rational numbers theorem, there exists $s \in Q$ such that $a < s < b$

Since $\sqrt{2} \in Q'$ and $s \in Q$, thus by preceding Lemma we get

$s + \sqrt{2} \in Q'$

Note that

$$\underset{\mathbb{R}}{\underset{\in}{a + \sqrt{2}}} < \underset{Q'}{\underset{\in}{s + \sqrt{2}}} < \underset{\mathbb{R}}{\underset{\in}{b + \sqrt{2}}}$$

$\therefore \nexists$ rational number between $a + \sqrt{2}$ and $b + \sqrt{2}$ $\quad C$!

(with densely of rational numbers)

Chapter Three

The complex numbers

Definition: The set of complex numbers is denoted by

$$\mathbb{C} = \{(a, b); a, b \in \mathbb{R}\}$$

Define (+) and (.) on $\mathbb{C}$ as follows

Let $Z_1 = (a_1, b_1), Z_2 = (a_2, b_2)$, then

1. $z_1 + z_2 = (a_1 + a_2, b_1 + b_2)$
2. $z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$

For example

Let $z_1 = (3,4)$ & $z_2 = (-1,2)$

$z_1 + z_2 = (3 - 1, 4 + 2) = (2,6)$
$z_1 \cdot z_2 = (-3 - 8, 6 - 4) = (-11,2)$

Proposition:

Let $z_1, z_2, z_3 \in \mathbb{C}$, then

1. $z_1 + z_2 = z_2 + z_1$

2. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

3. If $0 = (0,0)$, then $Z + 0 = 0 + Z = Z$

4. $\forall z \in \mathbb{C}$, there exists element denoted by $(-z)$ such that $z + (-z) = 0$

In fact, if $z = (a, b)$ then $-z = (-a, -b)$

5. $z_1 \cdot z_2 = z_2 \cdot z_1$

6. $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$

7. If $1 = (1,0)$, then $1 \cdot z = z \cdot 1 = z$.

8- If $z \in \mathbb{C}, z \neq 0$, then there exists element denoted by z^{-1} such that

$z \cdot z^{-1} = z^{-1} \cdot z = 1$

z^{-1} is called multiplication inverse.

9. $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 z_3$

Proof:

8. Let $z \in \mathbb{C}$ s.t $z = (a, b) \neq (0,0) \longrightarrow a^2 + b^2 > 0$

we define $z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

$$z \cdot z^{-1} = \left(a\left(\frac{a}{a^2+b^2}\right) + b\left(\frac{b}{a^2+b^2}\right), a\left(\frac{-b}{a^2+b^2}\right) + b\left(\frac{a}{a^2+b^2}\right)\right)$$
$$= \left(\frac{a^2+b^2}{a^2+b^2}, 0\right) = (1,0) = 1$$

For example

1. $z = (3/4) \longrightarrow z^{-1} = \left(\frac{3}{25}, \frac{-4}{25}\right)$
2. $z = (2, -1) \longrightarrow z^{-1} = \left(\frac{2}{5}, \frac{1}{5}\right)$
3. $z = (1,0) \longrightarrow z^{-1} = (1,0)$

Another definition to complex numbers

If $a, b \in \mathbb{R}$, then we can define the complex number z as follows:

$z = a + ib$, where $i = (0,1)$.

ie

$$z = (a, b) = (a, 0) + (0, b) = a(1,0) + (0,1)b$$
$$= a.1 + ib$$
$$= a + ib$$

Example:

1. Let $z_1 = 3 + 2i$, $z_2 = 6 + 8i$, then

a)

$$z_1 + z_2 = (3 + 2i) + (6 + 8i) = (3 + 6) + (2 + 8)i$$
$$= 9 + 10i$$

b)

$$z_1 \cdot z_2 = (3 + 2i) \cdot (6 + 8i) = [(3)(6) - (2)(8)] + i[(2)(6) + (3)(8)]$$
$$= (18 - 16) + (12 + 24)i$$
$$= 2 + 36i$$

2. $i^2 = i \cdot i = (0,1).(0,1) = (0 - 1,0) = (-1,0) = -1(1,0) = -1$

3. Find

$3(3+2i)-2(2-3i)+(6+8i)$
$=9+6i-4+6i+6+8i$
$=11+20i$

Remark: If $z=a+ib$ be a complex number, then a is called real part of z, and b is called imaginary part of z. i.e

$a=\mathrm{Re}(z), b=\mathrm{Im}(z)$

Example:

1. If $z=3+2i$, then $\mathrm{Re}(z)=3, \mathrm{Im}(z)=2$

2. $z=10, \mathrm{Re}(z)=10, \mathrm{Im}(z)=0$

3. $z=2i, \mathrm{Re}(z)=0, \mathrm{Im}(z)=2$

Proposition: Let $z_1, z_2 \in \mathbb{C}$, then

(1) $\mathrm{Re}(z_1+z_2)=\mathrm{Re}(z_1)+\mathrm{Re}(z_2)$

(2) $\mathrm{Im}(Z_1+Z_2)=\mathrm{Im}_m(Z_1)+\mathrm{Im}_m(Z_2)$.

Proof:

Let $z_1=a_1+ib_1$ & $z_2=a_2+ib_2$

$z_1+z_2=(a_1+a_2)+i(b_1+b_2)$
$\therefore \mathrm{Re}(z_1+z_2)=a_1+a_2=\mathrm{Re}(z_1)+\mathrm{Re}(z_2)$.

$\mathrm{Im}(z_1+z_2)=b_1+b_2=\mathrm{Im}(z_1)+\mathrm{Im}_m(z_2)$

Definition: Let $z \in \mathbb{C}, z=a+ib$, we define $|z|$ by

$|z|=\sqrt{a^2+b^2}$ is called absolute value of z.

Example:

1. $z=3+4i,\ |z|=\sqrt{(3)^2+(4)^2}=\sqrt{25}=5$

2. $z=-2i,\ |z|=\sqrt{(-2)^2}=\sqrt{4}=2$

3. $z=i, |z|=\sqrt{(1)^2}=1$

Proposition: Let $z_1, z_2 \in \mathbb{C}$

1. $|z| \geqslant 0$ and $|z| = 0$ if $z = 0$

2. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
3. $|z_1 + z_2| \leqslant |z_1| + |z_2|$
4. $||z_1| - |z_2|| \leqslant |z_1 - z_2|$

Proof:

2. Let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$

$$|z_1 \cdot z_2| = |(a_1 + ib_1) \cdot (a_2 + ib_2)| = |(a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)|$$
$$= \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2}$$
$$|z_1 \cdot z_2|^2 = (a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2$$
$$= a_1^2a_2^2 - 2a_1a_2b_1b_2 + b_1^2b_2^2 + a_1^2b_2^2 + 2a_1b_2a_2b_1 + a_2^2b_1^2$$

$$|z_1 \cdot z_2|^2 \doteq (a_1^2 + b_1^2)(a_2^2 + b_2^2) = |z_1|^2|z_2|^2 = (|z_1||z_2|)^2$$

$$\therefore |z_1 \cdot z_2| = |z_1||z_2|$$

4. Note that $z_1 = z_2 + (z_1 - z_2)$

$|z_1| = |z_2 + (z_1 - z_2)| \leqslant |z_2| + |z_1 - z_2|$
$\to |z_1| \leqslant |z_2| + |z_1 - z_2|$
$\to |z_1 - z_2| \geqslant |z_1| - |z_2|. \cdots (1)$
$z_2 = z_1 + (z_2 - z_1)$
$|z_2| = |z_1 + (z_2 - z_1)|$
$|z_2| \leqslant |z_1| + |z_2 - z_1|$
$\to |z_2| - |z_1| \leqslant |z_2 - z_1|$
$\to -(|z_1| - |z_2|) \leqslant |z_1 - z_2|. \cdots (2)$
from (1)&(2) we get
$||z_1| - |z_2|| \leqslant |z_1 - z_2|$

Corollary: For each finite $z_1, z_2, \dots, z_n$ of complex number. Then

1. $|z_1 \cdot z_2 \cdots \cdots z_n| = |z_1||z_2| \cdots |z_n|$

2- If $z_1, z_2 \in \mathbb{C}, z_2 \neq 0$, then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Proof:

1. We know that $|z_1 \cdot z_2| = |z_1||z_2|$

$$|z_1 \cdot z_2 \cdot z_3| = |z_1||z_2||z_3|$$
$$\vdots$$
$$|z_1 \cdots\cdots z_n| = |z_1||z_2| \ldots - |z_n|$$

2. Let $w = \frac{z_1}{z_2} \longrightarrow z_1 = wz_2$.

$$|z_1| = |wz_2| = |w||z_2| \longrightarrow |w| = \frac{|z_1|}{|z_2|}$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Definition: Let $z = a + ib$ be a complex number then the complex number $a - ib$ is called conjugate of z and denoted by $\bar{z}$**.**

i.e $\bar{z} = a - ib$

Proposition:

1- For each complex number z.

$z \cdot \bar{z} = |z|^2$, $\bar{\bar{z}} = z$

2. $\forall z_1, z_2 \in \mathbb{C}, \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

3. $\forall z_1, z_2 \in \mathbb{C}, \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

4. $\forall z_1, z_2 \in \mathbb{C}, z_2 \neq 0$, then $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

5. $\forall z \in \mathbb{C}, |z| = |\bar{z}|$

6. If $z \neq 0$, then $z^{-1} = \frac{\bar{z}}{|z|^2}$

Proof:

1-Let $Z = a + ib \longrightarrow \bar{Z} = a - ib$

$$z \cdot \bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2.$$
$$\bar{\bar{z}} = \overline{(a - ib)} = a + ib = z.$$

2. Let $Z_1 = a_1 + ib_1, Z_2 = a_2 + ib_2$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$
$$\overline{z_1 + z_2} = (a_1 + a_2) - i(b_1 + b_2) \cdots (1)$$
$$\bar{z}_1 + \bar{z}_2 = (a_1 - ib_1) + (a_2 - ib_2) = (a_1 + a_2) - i(b_1 + b_2)$$

From (1) &(2) we get $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$