جامعة بغداد

كلية العلوم للبنات

قسم الرياضيات

المرحلة الاولى

اسس الرياضيات II

Fundamental of Mathematics II

م.د. شيرين عودة دخيل

م.د. ايناس مصطفى كامل

2024-2023

Chapter One

The natural numbers and integers

Principle of mathematical induction مبدأ الاستقراء الرياضي

Let S be a subset of $\mathbb{N}$, the natural numbers, with the following properties:-

1. $1 \in S$
2. $n \in S$ implies $n + 1 \in S$

Then S is the set of natural numbers (ie. $\mathrm{S} = \mathbb{N}$).

The way of using principle of mathematical induction:-

*) suppose that $p(n)$ is a statement depend on natural number n.

*) Suppose that S be a solution set of natural numbers n such that the statement is true

i.e $S = \{n \in \mathbb{N}; p(n) \text{ is true }\}$

we shall prove that

1. $1 \in S$ (i.e $p(1)$ is true)

2. Suppose that $p(k)$ is true. (i.e $k \in S$)

3. $p(k + 1)$ is true. (i.e $k + 1 \in S$)

so that, $S = \mathbb{N}$

i.e $p(n)$ is true $\forall n$

Example: prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, $\forall\, n \geqslant i$

Solution: suppose that $p(n)$ is $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

and $S = \{n \in \mathbb{N}: P(n) \text{ is true }\}$

1. Since $\frac{1(1+1)}{2} = \frac{2}{2} = 1$

hence $p(1)$ is true (i.e $1 \in S$)

2. Suppose that $p(k)$ is true

i.e $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

We have to show that $1+2+3+\cdots+k+(k+1)=\frac{(k+1)[(k+1)+1]}{2}$

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)\ \left(\text{بأضافة } (k+1) \text{ للطرفين}\right)$$
$$=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$
$$=\frac{(k+1)[(k+1)+1]}{2}$$

So that $p(k+1)$ is true (i.e $k+1\in S$)

Hence $P(n)$ is true $\forall n\geqslant 1$.

Example: Prove that $\sum_{k=1}^{n}(2k-1)^2=\frac{n(2n-1)(2n+1)}{3}\ \forall n\in\mathbb{N}$

Solution: Suppose that $p(n)$ is $\sum_{k=1}^{n}(2k-1)^2=\frac{n(2n-1)(2n+1)}{3}$

$$\text{i.e } (1)^2+(3)^2+(5)^2+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Since $(1)^2=\frac{1(2-1)(2+1)}{3}=1$, hence $p(1)$ is true.

Suppose that $p(k)$ is true

$$\text{i.e } 1^2+3^2+5^2+--+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}\ \ldots\ldots(*)$$

Now, we prove that $\rho(k+1)$ is true

$$1^2+3^2+\cdots+(2k-1)^2+(2(k+1)-1)^2=\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

بأضافة $[2(k+1)-1]^2=(2k+1)^2$ لطرفي المعادلة (*) نحصل على:

$$1^2+3^2+\cdots+(2k-1)^2+(2(k+1)-1)^2=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$
$$=\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$
$$=\frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$$
$$=\frac{(2k+1)[2k^2+5k+3]}{3}$$
$$=\frac{(2k+1)[(k+1)(2k+3)]}{3}=\frac{(k+1)(2k+1)(2k+3)}{3}$$

$$= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

So that, $p(k+1)$ is true.

Example: Prove that $2^{3n}-1$ divide by 7 $\forall n \geqslant 1$

Solution: Suppose that $p(n)$ is $2^{3n}-1$ divide by 7 .

1. $p(1)$ is true. Since $2^{3(1)}-1=7$ divide by 7

2. Suppose that $p(k)$ is true. i.e $2^{3k}-1$ divide by 7

That is mean $\exists c \in Z$ s.t $2^{3k}-1 \ = 7c$

Now, we have to show that, $\exists d \in z$ sit $2^{3(k+1)}-1=7d$

$$\begin{aligned} 2^{3(k+1)}-1 &= 2^{3k+3}-1 = 2^3 2^{3k}-1 \\ &= 2^3(7c+1)-1 \ , \quad \text{since } 2^{3k}-1=7c \\ &= 2^3 7c+8-1 \\ &= 2^3 \cdot 7c+7 \\ &= 7(2^3 c+1) \\ &= 7d \ , \qquad d=2c+1 \in z \end{aligned}$$

So that $2^{3(k+1)}$ divide by 7

Hence $P(n)$ is true $\forall n$

Piano's axioms

فرضيات بيانو

ان مجموعة الاعداد الطبيعية هي مجموعة يرمز لها بالرمز ℕ تحقق الفرضيات الاتية:

(1) ℕ مجموعة غير خالية.

(2) لكل عنصر n في ℕ يوجد عنصر اخر يرمز له بالرمز n^+ يسمى تابع للعنصر n و هذا التابع وحيد فضلا عن هذا اذا كان $n^+ = m^+$ ، فأن n=m.

(3) يوجد عنصر واحد فقط في ℕ ليس تابعا لاي عنصر اخر يرمز له بالرمز 1 و يسمى العدد الاول في ℕ.

Theorem:

1- For each $n, m \in \mathbb{N}$, there exist unique element $n+m \in \mathbb{N}$ is called summation n, m such that

a) $n+1=n^+$

b) $m+1=m^+$

c) $n+m=m+n$ (Commutative Law)

d) $\forall n, m, k \in \mathbb{N}, (n+m)+k=n+(m+k)$ (associative Law)

e) If $m + n_1 = m + n_2$ then $n_1 = n_2$ (Cancellation Law)

2 - For each $n, m \in \mathbb{N}$, there exist unique element $m \cdot n \in \mathbb{N}$ is called multiplication elements n, m such that:

a) $m \cdot 1 = 1.m = m$
b) $n \cdot m = m \cdot n$ (Commutative Law)
c) $(m \cdot n) \cdot k = m \cdot (n \cdot k)$ (associative Law)

d) $k \cdot (m + n) = k \cdot m + k \cdot n$

e) If $m \cdot n_1 = m \cdot n_2$ then $n_1 = n_2$

Theorem: For each $n \in \mathbb{N}$ then $n^+ \neq n$ (i.e n $+1 \neq n$)

Proof:

By using principle of mathematical intuition

Let S $= \{n \in \mathbb{N}; n + 1 \neq n \}$

1. $1 \in$ S.

If $1 \notin S \longrightarrow 1 + 1 = 1$ c !

معناه (1) هو تابع و هذا تناقض مع فرضية بيانو

2. Suppose that $k \in S$

We have to show that $k^+ = k + 1 \in S$

Suppose that $k^+ \notin S$

$\to k^+ + 1 = k^+$ $(Def.$ of $S)$
$\to (k^+)^+ = k^+$
$\to k^+ =$ k (if $n^+ = m^+ \to n = m)$
$\to k \notin S$ c! (with 2)

So that $k^+ = k + 1 \in S$

By mathematical induction we get $S = \mathbb{N}$

Definition: Let $n, m \in \mathbb{N}$ then $m < n$ iff there exists a unique $\dot{k} \in \mathbb{N}$ such that $m + k = n$.

Remark: For each $n \in \mathbb{N}, n < n + 1$.

Proposition: For each $n \in \mathbb{N}, n \neq 1$ then $n > 1$

(i.e 1 is smallest element in $\mathbb{N}$)

Proof: Since $n \neq 1$ then n is successor for another element say m hence $n = m + 1$

$\rightarrow m + 1 > 1$
$\rightarrow n > 1$

Proposition: Let $n, m \in \mathbb{N}$, if $m < n$ then $m^+ \leqslant n$

Proof: Since $m < n \longrightarrow \exists k \in \mathbb{N}$ s.t $m + k = n, \quad k \geqslant 1$

1.If $k = 1 \longrightarrow m + 1 = n \longrightarrow m^+ = n$

2.If $k > 1 \rightarrow \exists t \in \mathbb{N}$ s.t $k = t + 1.$

$$\therefore n = m + k = m + (t + 1) = (m + 1) + t$$
$$= m^+ + t$$

$\rightarrow m^+ < n$

From (1) and (2) we get $m^+ \leqslant n$

Proposition: For each n $\in \mathbb{N}$, then, $\nexists k \in \mathbb{N}$ s.t $n < k < n + 1$:

Proof:

Suppose that there exists $k \in \mathbb{N}$ s.t $n < k < n + 1$

Now,
$n < k \rightarrow \exists k_1 \in \mathbb{N}$ s.t $n + k_1 = k$
$k < n + 1 \rightarrow \exists k_2 \in \mathbb{N}$ s.t $k + k_2 = n + 1$
$(n + k_1) + k_2 = n + 1 \rightarrow n + (k_1 + k_2) = n + 1$
$\rightarrow k_1 + k_2 = 1$
$\rightarrow k_1 < 1$ C! since $k_1 \in \mathbb{N}$

Proposition: Let $m, n, k \in \mathbb{N}$. Then

1. Either $m = n$ or $m < n$ or $n < m$

2. If $m < n$ and $k < m$ then $k < n$

3. If $m < n$ then $m + t < n + t\ , \quad t \in \mathbb{N}$

4. If $m < n$, then $mt < nt$

5. If $m + t < n + t$ then $m < n$

6. If $mt < nt$ then $m < n$

Proof:

(1) If $n = 1$ or $m = 1$ then by preceding proposition the proof is clear [Suppose that $n \neq 1$ and $m = 1 \overset{\text{pro.}}{\rightarrow} n > 1 = m$]

Thus suppose that $n \neq 1$ and $m \neq 1$, we use mathematical induction

Let $S_1 = \{n\}$, $S_2 = \{x \in \mathbb{N}; x > n\}$, $S_3 = \{x \in \mathbb{N}; x < n\}$

Let $S = S_1 \cup S_2 \cup S_3$

1.$1 \in S$. Since $\forall n \in \mathbb{N}$ &$n \neq 1 \rightarrow n > 1$
$\rightarrow 1 \in S_3 \rightarrow 1 \in S$

2. Let $k \in S \rightarrow k \in S_1$ or $k \in S_2$ or $k \in S_3$
Case (1): if $k \in S_1 \longrightarrow k = n \longrightarrow k^+ > n \longrightarrow k^+ \in S_2 \longrightarrow k^+ \in S$

Case (2): $k \in S_2 \longrightarrow k > n \rightarrow \exists t \in \mathbb{N}$ s.t

$k = n + t \rightarrow k^+ = (t + n)^+$
$= (t + n) + 1 = n + (t + 1)$
$\rightarrow k^+ > n \rightarrow k^+ \in S_2 \longrightarrow k^+ \in S$

Case (3): if $k \in S_3 \longrightarrow k < n \rightarrow k^+ \leqslant n$

if $k^+ = n \longrightarrow k^+ \in S_1 \longrightarrow k^+ \in S$

if $k^+ < n \rightarrow k^+ \in S_3 \longrightarrow k^+ \in S$

By (1),(2) &(3) $\forall k \in S \rightarrow k^+ \in S$, thus

by mathematical induction $S = \mathbb{N}$

2. $m < n \longrightarrow \exists t_1 \in \mathbb{N}$ s.t $n = m + t_1$
$k < m \longrightarrow \exists t_1 \in \mathbb{N}$ s.t $m = k + t_2$
$n = m + t_1 = (k + t_2) + t_1 = k + (t_1 + t_2)$
$n = \text{k} + \text{t} \in \mathbb{N}$; where $t = t_1 + t_2 \in \mathbb{N}$
$\rightarrow k < n$

3. $m < n \longrightarrow \exists k \in \mathbb{N}$ s.t $n = m + k$

$n + t = m + k + t$ (بأضافة t)
$= (m + t) + k$

$\rightarrow m + t < n + t$
4. $m < n \longrightarrow \exists k \in \mathbb{N}$ s.t $m + k = n$