Universal Quantifier

السور الكلي

Let $p(x)$ be an open sentence on a set A. Then

$$(\forall x \in A)\, p(x) \quad \text{or} \quad \forall x,\, p(x)$$

is a statement which reads "For every element x in A, $p(x)$ is a true statement", or, simply, "For all x, $p(x)$". The symbol $\forall$ which reads "for all" or "for every" is called the universal quantifier.

Remark: If $\{x \mid x \in A, p(x)\} = A$, then $\forall x, p(x)$ is true.

if $\{x \mid x \in A, p(x)\} \neq A$, then $\forall x, p(x)$ is false.

Example ① The proposition $(\forall n \in \mathbb{N})(n+4 > 3)$, where $\mathbb{N}$ is the set of natural numbers, is true since

$$\{n \mid n+4 > 3\} = \{1, 2, 3, \ldots\} = \mathbb{N}$$

② The proposition $\forall n\, (n+2 > 8)$ is false since

$$\{n \mid n+2 > 8\} = \{7, 8, 9, \ldots\} \neq \mathbb{N}$$

③ The symbol $\forall$ can be used in defining the intersection of a family of sets $\{A_i\}_{i \in I}$ as follows

$$\bigcap_{i \in I} A_i = \{x \mid \forall i \in I,\ x \in A_i\}$$

Existential Quantifier

السور الجزئي

Let $p(x)$ be an open sentence on a set A. Then

$$(\exists x \in A)\, p(x) \quad \text{or} \quad \exists x\, p(x)$$

is a proposition which reads, "There exists an $x \in A$ such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$". The symbol

$$\exists$$

which reads "there exists" or "for some" is called the existential quantifier.

Remark:- If $\{x \mid p(x)\} \neq \phi$, then $\exists x\, p(x)$ is true,

if $\{x \mid p(x)\} = \phi$, then $\exists x\, p(x)$ is false.

Example ① The statement $(\exists n \in \mathbb{N})(n+4 < 7)$ is true since

$$\{n \mid n+4 < 7\} = \{1, 2\} \neq \phi$$

② The proposition $\exists n (n+6 < 4)$ is false since

$$\{n \mid n+6 < 4\} = \phi$$

③ The symbol $\exists$ can be used in defining the union of a family of sets $\{A_i\}_{i \in I}$ as follows

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I, x \in A_i\}$$

Negation of propositions which contain Quantifiers

The negation of the proposition "ALL men are clever" reads "It is not true that all men are clever"; in other words, there exists at least one man who is not clever. Symbolically, then, if M denotes the set of men, then the above can be written as

$$\sim(\forall x \in M)(x \text{ is clever}) \equiv (\exists x \in M)(x \text{ is not clever})$$

Furthermore, if $p(x)$ denotes "x is clever", then the above can be written

$$\sim(\forall x \in M)\, p(x) \equiv (\exists x \in M) \sim p(x)$$

The above is true in general. Specifically

Theorem $\sim(\forall x \in A)\, p(x) \equiv (\exists x \in A) \sim p(x)$

In other words, the statement

"It is not true that, for every $a \in A$, $p(a)$ is true"

is equivalent to the statement

"There exists an $a \in A$ such that $p(a)$ is false".

There is an analogous theorem for the negation of a proposition which contains the existential quantifier

Theorem $\sim(\exists x \in A)\, p(x) \equiv (\forall x \in A) \sim p(x)$

i.e., the statement

"It is not true that there exists an $a \in A$ such that $p(a)$ is true".

is equivalent to the statement

"For all $a \in A$, $p(a)$ is false".

Example The negation of the proposition "For all natural numbers n, $n+2 > 8$" is equivalent to the proposition "There exists an n such that $n+2 \not> 8$".

In other words,

$$\sim(\forall n \in \mathbb{N})(n+2 > 8) \equiv (\exists n \in \mathbb{N})(n+2 \not> 8)$$

Remark The statement $p(x) \wedge q(x)$ reads "$p(x)$ and $q(x)$", and $p(x) \vee q(x)$ reads "$p(x)$ or $q(x)$". It can be shown that the same Laws for propositions hold for propositional functions (open sentence),

i.e

$$\sim(p(x) \wedge q(x)) \equiv \sim p(x) \vee \sim q(x)$$

$$\sim(p(x) \vee q(x)) \equiv \sim p(x) \wedge \sim q(x)$$

Example Let $M = \{$ALi, Raed, Saad, Ahmed$\}$, Let $W = \{$Noor, Hiba$\}$, and Let $p(x,y)$ be "x is the brother of y". Then

$$\forall x \in M [\exists y \in W, p(x,y)] \equiv \forall x \in M\ \exists y \in W\ p(x,y)$$

reads "For every x in M there exists a y in W such that x is the brother of y

In other words, each member of M is the brother of Noor or Hiba.

The negation of a proposition which contains quantifiers can be found as follows:-

$$\sim \forall x [\exists y \, p(x,y)] \equiv \exists x \sim [\exists y \, p(x,y)] \equiv \exists x \, \forall y \sim p(x,y)$$

Example:- Let M, W and $p(x,y)$ be as in above example. Then

$$\sim \forall x \in M \; \exists y \in W \; p(x,y) \equiv \exists x \in M \; \forall y \in W \sim p(x,y)$$

In other words, the statement "It is false that each man is the brother of at least one woman" is equivalent to "At least one of the men is not the brother of any of the women".

Example:- give the negation of the following statement.

"$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \; x+y=0$"

The negation of this statement is

"$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}; \; x+y \neq 0$"

Example:- Let A is a positive integer number

$P(x)$:- x is prime number

$O(x)$:- x is odd number

$E(x)$:- x is even number

$G(x)$:- $x > 2$

write the following statements by quantifier statement.

① Exists not negative integer number not prime.

$\exists x \in A, \sim p(x)$

② Exists not negative integer number greater than 2.

$\exists x \in A, G(x)$

③ Each non negative integer number if it is even then it is odd.

$\forall x \in A, E(x) \longrightarrow O(x)$

④ Each non negative integer number then it is prime and not odd.

$\forall x \in A, p(x) \wedge \sim O(x)$