$$4 \cdot \overline{z_2} \cdot \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{z_2 z_1}{z_2}\right)}$$

$$= \overline{z_1}$$

$$\therefore \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

6. Let $Z = a + ib \neq 0$

$$\rightarrow \bar{z} = a - ib$$

$$z^{-1} = \frac{1}{z}$$

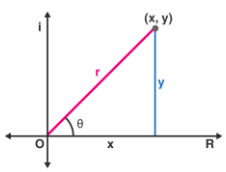
$$= \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib}$$

$$= \frac{a-ib}{a^2+b^2}$$

$$\rightarrow z^{-1} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

Polar representation of the complex numbers التمثيل القطبي للاعداد العقدية

اذا كان لدينا العدد العقدي $z=a+ib$ فعند تمثيل العدد z بنقطة في المستوي فأن مقياس القيمة المطلقة للعدد العقدي z هو المسافة r من نقطة الاصل 0 الى النقطة z.



$$|z| = r = \sqrt{a^2+b^2}.$$

زاوية العدد العقدي θ التي يرمز لها بالرمز $\arg(z)$ تعرف بأنها الزاوية المحصورة بين الاتجاه الموجب للمحور الحقيقي و بين المتجه الذي يمثل z.

So that we have the following relations

1- $\tan\theta = \frac{b}{a} \rightarrow \theta = \tan^{-1}\frac{b}{a}$

2- $a = r\cos\theta$

$b = r\sin\theta$

$\therefore Z = a + ib = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

Thus, the polar represent of z is

$Z = r(\cos\theta + i\sin\theta)$

Remark:

1- The argament of z $(\arg(z))$ has not unique define.

In fact if $k \in Z$ then

$a.$ $\sin(\theta + 2k\pi) = \sin\theta$
$b.$ $\cos\theta(\theta + 2\ddot{k}\pi) = \cos\theta$

Hence if $\theta = 30$, then we can use $\theta = 390$ or $\theta = 750$ or $\theta = -330$

2- Each complex number $z = a + ib$ can represent by polar as follows

$$r = \sqrt{a^2 + b^2}, \tan\theta = \frac{b}{a}$$

Example: Use polar representation of the complex numbers

1. $z_1 = 3 - 3i$

Solution:

$$a = 3, b = -3$$
$$\therefore r_1 = \sqrt{3^2 + (-3)^2} = \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$$
$$a = r_1\cos\theta \to \cos\theta = \frac{a}{r_1} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$b = r_1\sin\theta \longrightarrow \sin\theta = \frac{b}{r_1} = \frac{-3}{3\sqrt{2}} = \frac{-1}{\sqrt{2}}$$
$$\theta = \frac{\pi}{4}, 45^\circ$$

So that

$$360^\circ - 45^\circ = 315^\circ$$
$$\therefore z_1 = 3\sqrt{2}\big(\cos(315^\circ) + i\sin(315^\circ)\big)$$

2. $z_2 = -4$

Solution:

$a = -4, b = 0$
$\therefore r_2 = \sqrt{(-4)^2} = 4$
$a = r_2 \cos\theta \to \cos\theta = \dfrac{-4}{4} = -1$
$b = r_2 \sin\theta \to \sin\theta = \dfrac{0}{4} = 0$
$\theta \to= \pi, 180$
$\therefore Z_2 = 4(\cos\pi + i\sin\pi)$

Proposition:

Let

$Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then.

1. $z_1 \cdot z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
2. $|z_1 \cdot z_2| = r_1 \cdot r_2$
3. $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

4- If $z_2 \neq 0$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

5. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2}$
6. $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Remark: The complex number $\cos\theta + i\sin\theta$ can be represented as $e^{i\theta}$, i.e

$i\theta$
$e = \cos\theta + i\sin\theta$

This is called Euler formula.

De Mover's Theorem

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number, if n is positive integer then

$Z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

Example:

Find $(-\sqrt{3}+i)^7$

Solution:

we write $-\sqrt{3}+i$ in polar form

$$a=-\sqrt{3}, \quad b=1$$
$$r=\sqrt{a^2+b^2}=\sqrt{3+1}=2.$$
$$a=r\cos\theta \rightarrow \cos\theta=\frac{-\sqrt{3}}{2}$$
$$\sin\theta=\frac{b}{r}=\frac{1}{2}$$
$$\theta=\frac{\pi}{6}, 150^\circ\left(\pi-\frac{\pi}{6}\right)$$

Now By De Mover's formula:

$$\begin{aligned}(-\sqrt{3}+i)^7 &= [2(\cos 150^\circ + i\sin 150^\circ)]^7 = 2^7\left(\cos 7(150^\circ) + i\sin\big(7(150^\circ)\big)\right)\\ &= 128(\cos 1050 + i\sin 1050) = 128\big(\cos(330^\circ) + i\sin(330^\circ)\big)\\ &= 128(\cos(360-30) + i\sin(360-30))\\ &= 128(\cos 30^\circ - i\sin 30^\circ) = 128\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)\\ &= 64\sqrt{3} - i64\end{aligned}$$

The root of complex numbers جذور الاعداد العقدية

Let $z, w \in \mathbb{C}$ and let $w^n = z$ then w is the n-th root of z. z هو الجذر النوني للعدد w

Theorem (without proof)

Each complex number not equal zero and as in the form $z=r(\cos\theta+i\sin\theta)$ has n roots in the form

$$w_k=\sqrt[n]{r}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$$
$$k=0,1,\ldots,n-1$$

Example: Find the root complex for $\sqrt[3]{8i}$?

Solution: Let $z=\sqrt[3]{8i} \longrightarrow z^3=8i$

Suppose that $z=re^{i\theta}$

$\therefore z^3 = \left(re^{i\theta}\right)^3 = r^3 e^{i3\theta}$

$8i = 8\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 8e^{i\frac{\pi}{2}}$

$\therefore r^3 e^{i3\theta} = 8e^{i\frac{\pi}{2}}$

$\rightarrow r^3 = 8 \rightarrow r = \sqrt[3]{8} = 2,\ \ 3\theta = \frac{\pi}{2} + 2k\pi \rightarrow\ \ \theta = \frac{\frac{\pi}{2} + 2k\pi}{3}, k = 0,1,2$

$\because z = re^{\mathrm{i}\theta}$

$\rightarrow z_k = 2e^{i\left(\frac{\pi+4k\pi}{6}\right)} = 2\left[\cos\left(\frac{\pi + 4k\pi}{6}\right) + i\sin\left(\frac{\pi + 4k\pi}{6}\right)\right], k = 0,1,2$

if $k = 0$

$z_0 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = \sqrt{3} + i$

if $k = 1$

$z_1 = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = 2\left(\cos(150°) + i\sin(150°)\right)$

$= -\sqrt{3} + i$

if $k = 2$

$z_2 = 2\left(\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right) = 2\left(\cos(270°) + i\sin(270°)\right)$

$= -2i$

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