Chapter One : Topological Spaces

Definition : **Topology & Topological Space**

Let X be a nonempty set and τ be a family of subsets of X (i.e., $\tau \subseteq IP(X)$). We say τ is a **topology** on X if satisfy the following conditions :

(1) $X, \phi \in \tau$

(2) If $U, V \in \tau$, then $U \bigcap V \in \tau$

The finite intersection of elements from τ is again an element of τ.

(3) If $U_\alpha \in \tau$; $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau \quad \forall\alpha \in \Lambda$

The arbitrary (finite or infinite) union of elements of τ is again an element of τ.

We called a pair (X, τ) **topological space.**

Remarks :

[1] The topological space (X, τ) is sometimes called the **space** X.

[2] The elements of X are called **points** of the space.

[3] When write τ we said **topology** and when write (X, τ) we said **topological space**.

Example : Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{a, c\}\}$, $\tau_3 = \{X, \phi, \{a, b\}, \{a, c\}\}$, $\tau_4 = \{X, \phi, \{a\}, \{b\},\{a, c\}\}$ and $\tau_5= \{X, \{a\}, \{b\}, \{a, b\}\}$. Is $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ topology on X.

Solution : Notes that τ_1 and τ_2 is topology on X since its satisfy the three conditions of topology.

τ_3 is not topology on X since $\{a, b\} \bigcap \{a, c\} = \{a\} \notin \tau_3$ (i.e., the condition two is not satisfy).

τ_4 is not topology on X since $\{a\} \bigcup \{b\} = \{a, b\} \notin \tau_4$ (i.e., the condition three is not satisfy).

τ_5 is not topology on X since $\phi \notin \tau_3$ (i.e., the condition one is not satisfy).

Example : Let $X = \{1, 2, 3, 4\}$. Let

[1] $\tau_1 = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$. Then, is τ_1 is a topology on X? (H.W)

[2] $\tau_2 = \{\emptyset, X, \{3, 4\}, \{2\}, \{3\}, \{1, 2, 3\}\}$. Then, is τ_2 is a topology on X? (H.W)

[3] $\tau_3 = \{\{1\}, \{2\}, \{1, 2\}\}$. Then, is τ_3 is a topology on X? (H.W)

[4] $\tau_4 = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$. Then, is τ_4 is a topology on X? (H.W)

[5] $\tau_5 = \{\emptyset, X, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Then, is τ_5 is a topology on X? (H.W)

Example : Let $X = \{1, 2, 3\}$. Let $\tau = \{X, \emptyset\} = I$ is a topology on X and is said to be the Indiscrete topology.
Also, we have, [1]. $X = \mathbb{R}, \tau = I = \{\emptyset, \mathbb{R}\}$, [2]. $X = \mathbb{Q}, \tau = I = \{\emptyset, \mathbb{Q}\}$,
[3]. $X = \mathbb{N}, \tau = I = \{\emptyset, \mathbb{N}\}$.

Example : Let $X = \{1, 2, 3\}$. Let $\tau = \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} = P(X) = D$ is a topology on X and is said to be the discrete topology.
Also, we have, [1]. $X = \mathbb{R}, \tau = D = IP(\mathbb{R})$, [2]. $X = \mathbb{Q}, \tau = D = IP(\mathbb{Q})$,
[3]. $X = \mathbb{N}, \tau = D = IP(\mathbb{N})$.

Remark : If $X \neq \phi$, then

[1] $\tau = \{X, \phi\}$ is a topology on X and its the smallest topology that we can defined on any set X and called **Indiscrete topology** and denoted by I. (i.e., $I = \{X, \phi\}$).

[2] $\tau = IP(X)$ is a topology on X and its the largest topology that we can defined on any set X and called **Discrete topology** and denoted by D. (i.e., $D = IP(X)$).

[3] If τ any topology on X then $I \subseteq \tau \subseteq D$.

[4] $\tau = D$ if and only if $\{x\} \in \tau \ \forall x \in X$.

Example : Let $X = \mathbb{N}, \tau = \{X, \emptyset, \{1\}, E\}$. Is τ a topology on X?
Solution : No, since $\{1\} \in \tau$, $E = \{2, 4, 6, 8, \dots\} \in \tau$ but $\{1\} \cup E = \{1, 2, 4, 6, \dots\} \notin \tau$

Example : Let $X = \mathbb{R}, \tau = \{\mathbb{R}, \emptyset, \mathbb{Q}, Irr\}$. Is τ a topology on X?
Solution : Yes, since

(1) $\mathbb{R}, \emptyset \in \tau$,

(2) $\emptyset \cap \mathbb{R} = \emptyset \in \tau, \emptyset \cap \mathbb{Q} = \emptyset \in \tau, \emptyset \cap Irr = \emptyset \in \tau, \mathbb{Q} \cap Irr = \emptyset \in \tau.$

(3) $\emptyset \cup \mathbb{R} = \mathbb{R} \in \tau$, ……. (H.W)

Example : Let $X = \mathbb{R}$, $\tau_1 = \{\mathbb{R}, \emptyset, (0,1], \{\frac{-1}{2}\}\}$ and let $\tau_2 = \{\mathbb{R}, \emptyset, (0,1], (-2,1)\}$. Is τ_1 and τ_2 topologies on $\mathbb{R}$?
Solution : $\tau_1$is not a topology on $\mathbb{R}$ since $(0,1] \in \tau_1, \{\frac{-1}{2}\} \in \tau_1$ but $(0,1] \cup \{\frac{-1}{2}\} \notin \tau_1$.
Also, τ_2 is not a topology on $\mathbb{R}$ since $(0,1] \in \tau_2, (-2,1) \in \tau_2$ but $(0,1] \cap (-2,1) = (0,1) \notin \tau_2$ and $(0,1] \cup (-2,1) = (-2,1] \notin \tau_2$.

Home works :

[1] Let $X = \{a, b, c, d\}$, then

- Is $\tau_1 = \{X, \emptyset, \{a, c\}, \{d\}\}$ a topology on X?
- Is $\tau_2 = \{\emptyset, \{c, d\}, \{d\}, \{c\}\}$ a topology on X?

- Is $\tau_3 = \{X, \emptyset, \{a, b\}, \{c, d\}\}$ a topology on X?
- Is $\tau_4 = \{X, \emptyset, \{h\}\}$ a topology on X?

[2] Let $X = \mathbb{N} = \{1,2,3, \dots\}$, then
- Define the indiscrete topology on $\mathbb{N}$.
- Is $\tau_1 = \{\mathbb{N}, \emptyset, E, O\}$ a topology on $\mathbb{N}$?
- Is $\tau_2 = \{\mathbb{N}, \emptyset, \{1,3\}, O\}$ a topology on $\mathbb{N}$?
- Is $\tau_3 = \{\mathbb{N}, \emptyset, \{2\}, \{4\}, E\}$ a topology on $\mathbb{N}$?

[3] Let $X = \mathbb{R}$, then
- Is $\tau_1 = \{\mathbb{R}, \emptyset, \{-1\}, \{2\}\}$ a topology on $\mathbb{R}$?
- Is $\tau_2 = \{\mathbb{R}, \emptyset, \{-1\}, (0, \infty)\}$ a topology on $\mathbb{R}$?
- Is $\tau_3 = \{\mathbb{R}, \emptyset, (-3,1], [1, \infty), (-3, \infty)\}$ a topology on $\mathbb{R}$?
- Is $\tau_4 = \{\mathbb{R}, \emptyset, (-\infty, 2), [-1,5), (-\infty, 5], [-1,2)\}$ a topology on $\mathbb{R}$?

<u>Remark :</u> there are 29 different topology on a set X contain only three elements.
If X = {1, 2, 3}, then all the following is a topology on X.
$\tau_1 = \{X, \phi\}$ Indiscrete Top., $\tau_2 = \{X, \phi, \{1\}\}$, $\tau_3 = \{X, \phi, \{2\}\}$, $\tau_4 = \{X, \phi, \{3\}\}$, $\tau_5 = \{X, \phi, \{1\}, \{1, 2\}\}$, $\tau_6 = \{X, \phi, \{1\}, \{1, 3\}\}$, $\tau_7 = \{X, \phi, \{1\}, \{1, 2\}, \{1, 3\}\}$, $\tau_8 = \{X, \phi, \{2\}, \{1, 2\}\}$, $\tau_9 = \{X, \phi, \{2\}, \{2, 3\}\}$, $\tau_{10} = \{X, \phi, \{2\}, \{1, 2\}, \{2, 3\}\}$, $\tau_{11} = \{X, \phi, \{3\}, \{1, 3\}\}$, $\tau_{12} = \{X, \phi, \{3\}, \{2, 3\}\}$, $\tau_{13} = \{X, \phi, \{3\}, \{1, 3\}, \{2, 3\}\}$, $\tau_{14} = \{X, \phi, \{1, 2\}\}$, $\tau_{15} = \{X, \phi, \{2, 3\}\}$, $\tau_{16} = \{X, \phi, \{1, 3\}\}$, $\tau_{17} = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$, $\tau_{18} = \{X, \phi, \{1\}, \{3\}, \{1, 3\}\}$, $\tau_{19} = \{X, \phi, \{2\}, \{3\}, \{2, 3\}\}$, $\tau_{20} = \{X, \phi, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$, $\tau_{21} = \{X, \phi, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}\}$, $\tau_{22} = \{X, \phi, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}\}$, $\tau_{23} = \{X, \phi, \{1\}, \{2, 3\}\}$, $\tau_{24} = \{X, \phi, \{2\}, \{1, 3\}\}$, $\tau_{25} = \{X, \phi, \{3\}, \{1, 3\}\}$, $\tau_{26} = \{X, \phi, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$, $\tau_{27} = \{X, \phi, \{1\}, \{3\}, \{1, 3\}, \{1,2\}\}$, $\tau_{28} = \{X, \phi, \{2\}, \{3\}, \{2, 3\}, \{1, 2\}\}$, $\tau_{29} = \{X, \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} = IP(X)$ Discrete Top..

<u>Remark :</u> If the number of elements of a set X four elements, then there are more than deference four hundred topology on X.

<u>Definition :</u> Open set & Closed set

Let (X, τ) be a topological space. The subsets of X belonging to τ are called **<u>open sets</u>** in the space X. i.e.,

$$\text{If } A \subseteq X \ \wedge \ A \in \tau \ \Rightarrow \ A \text{ open set}$$

The subset A of X is called a **<u>closed set</u>** in the space X if its complement X\A is open set. We will denoted the family of closed sets by $\mathcal{F}$. i.e.,

$$\text{If } A \subseteq X \ \wedge \ A \in \mathcal{F} \ \Rightarrow \ A \text{ closed set.}$$

Example : Let $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1,3\}\}$ be a topology on X. Then

[1] Is $\{2, 3\}$ is an open set in X? No, since $\{2,3\} \notin \tau$

[2] Is $\{2,3\}$ is a closed set in X? Yes, since $\{2,3\}^c = \{1\} \in \tau$.

[3] Find the family of all open sets.

Answer : $\{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}\}$ is the family of all open sets in X.

[4] Find the family of all closed sets.

Answer : $\mathcal{F} = \{X, \emptyset, \{2, 3\}, \{3\}, \{2\}\}$.

[5] Is $\{3\}$ open set? Closed set?

Answer : (H.W)

[6] Is $\emptyset, X$ are open sets? Closed sets?

Answer : $\emptyset, X$ are open sets since $\emptyset, X \in \tau$.

And, $\emptyset, X$ are closed sets since $\emptyset^c = X \in \tau$ or ($\emptyset \in \mathcal{F}$) and $X^c = \emptyset \in \tau$.

Example : Let $X = \mathbb{R}$, $\tau = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$. Then,

[1] Find the family of all closed sets.

Answer : $\mathcal{F} = \{\mathbb{R}, \emptyset, Irr\}$.

[2] Is an open interval $(0,1)$ open in $\mathbb{R}$?

Answer : No, since $(0,1) \notin \tau$.

[3] Is an open interval $(0,1)$ closed in $\mathbb{R}$?

Answer : No, since $(0,1)^c = (-\infty, 0] \cup [0, \infty) \notin \tau$.

Example : (H.W). Let $X = \mathbb{R}$, $\tau = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{N}^c\}$. Then,

[1] Is $\mathbb{N}$ open set? Closed set?

[2] Is $\mathbb{N}^c$ open set? Closed set?

[3] Is $(0,2)$ open set? Closed set?

[4] Is $(-\infty, 0)$ not open set?

[5] Find the family of all closed sets in $\mathbb{R}$.

Remark : The sets in (X, τ) may be

[1] open and not closed.
[2] closed and not open.
[3] closed and open (clopen).
[4] not open and not closed.

Theorem : Let (X, τ) be a topological space and $\mathcal{F}$ be the family of closed sets on X, then :

(1) $X, \phi \in \mathcal{F}$

(2) If $A, B \in \mathcal{F}$, then $A \bigcup B \in \mathcal{F} \quad \forall A, B \in \mathcal{F}$

(3) If $A_\alpha \in \mathcal{F}$; $\alpha \in \Lambda$, then $\bigcap_{\alpha\in\Lambda} A_\alpha \in \mathcal{F} \quad \forall \; A_\alpha \in \mathcal{F}$

Proof :

(1) $\because\ \phi \in \tau \Rightarrow \phi^c \in \mathcal{F} \Rightarrow X \in \mathcal{F}$

$\because\ X \in \tau \Rightarrow X^c \in \mathcal{F} \Rightarrow \phi \in \mathcal{F}$

(2) Let $A, B \in \mathcal{F} \Rightarrow A^c, B^c \in \tau$ (def. of closed sets)

$\Rightarrow A^c \bigcap B^c \in \tau$ (second condition of def. of top.)

$\Rightarrow (A \bigcup B)^c \in \tau$ (De Morgan's laws)

$\Rightarrow A \bigcup B \in \mathcal{F}$ (def. of closed sets)

(3) Let $A_\alpha \in \mathcal{F} \quad \forall\ \alpha \in \Lambda$

$\Rightarrow A^c{}_\alpha \in \tau \quad \forall\ \alpha \in \Lambda$

$\Rightarrow \bigcup_{\alpha\in\Lambda} A^c{}_\alpha \in \tau$ (third condition of def. of top.)

$\Rightarrow (\bigcap_{\alpha\in\Lambda} A_\alpha)^c \in \tau$ (De Morgan's laws)

$\Rightarrow \bigcap_{\alpha\in\Lambda} A_\alpha \in \mathcal{F}$ (def. of closed sets)

Proposition : Let X be a nonempty set and $\mathcal{F}$ be the family of subsets of X which has properties

(1) $X, \phi \in \mathcal{F}$

(2) If $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}$, then $F_1 \bigcup F_2 \in \mathcal{F}$

(3) If $\{F_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{F}$;, then $\bigcap_{\alpha\in\Lambda} F_\alpha \in \mathcal{F}$

Then the family $\tau = \{X - F : F \in \mathcal{F}\}$ is a topology on X, and $\mathcal{F}$ is the family of all closed sets in the topological space (X, τ).

Remark : The topology τ is called the topology generated by the family of closed sets $\mathcal{F}$.

Now we introduce some important examples of topological spaces and show that the open sets and closed sets in this examples :

Example : Usual Topology on $\mathbb{R}$

Let $\tau_u = \{\mathbb{R}, \phi, U ; \forall\ x \in U\ \exists$ open interval (a, b) ; $x \in (a, b) \subseteq U\}$

or $\tau_u = \{U \subseteq \mathbb{R}$; U = union of family of open interval$\}$

show that $(\mathbb{R}, \tau_u)$ is a topological space.

Solution :

(1) $\mathbb{R} = (-\infty, \infty) \in \tau_u$ (i.e., $\mathbb{R}$ is open interval and every open interval is open set)

$\phi = (a, a) \in \tau_u$

(2) Let $U, V \in \tau_u$

if U or $V = \phi \Rightarrow U \bigcap V = \phi \in \tau_u$

if U or $V = \mathbb{R} \;\Rightarrow\; U \bigcap V = V \in \tau_u$ (if $U = \mathbb{R}$)

$\Rightarrow\; U \bigcap V = U \in \tau_u$ (if $V = \mathbb{R}$)

Otherwise,

Let $x \in U \bigcap V \;\Rightarrow\; x \in U \wedge x \in V$

$\because\; x \in U \;\Rightarrow\; \exists$ open interval (a, b) ; $x \in (a, b) \subseteq U$

$\because\; x \in V \;\Rightarrow\; \exists$ open interval (c, d) ; $x \in (c, d) \subseteq V$

$\Rightarrow\; x \in (a, b) \bigcap (c, d) \subseteq U \bigcap V$

$\Rightarrow x \in (\max\{a, c\}, \min\{b, d\}) \subseteq U \bigcap V$

$\Rightarrow \exists$ open interval $(\max\{a, c\}, \min\{b, d\})$;

$x \in (\max\{a, c\}, \min\{b, d\}) \subseteq U \bigcap V$

$\Rightarrow U \bigcap V \in \tau_u$

(3) Let $U_\alpha \in \tau_u$; $\alpha \in \Lambda$

if $U_\alpha = \mathbb{R}$ for some $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \mathbb{R} \in \tau_u \;\forall\, \alpha \in \Lambda$

if $U_\alpha = \phi$ for all $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \phi \in \tau_u \;\forall\, \alpha \in \Lambda$

if $U_\alpha = \phi$ for some $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcup_{\alpha\in\Lambda} U_\alpha$

Now,

Let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow x \in U_\alpha$ for some α

$\because\; x \in U_\alpha \Rightarrow \exists$ open interval (a, b) ; $x \in (a, b) \subseteq U_\alpha$

$\Rightarrow x \in (a, b) \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_u$

$\therefore$ $(\mathbb{R}, \tau_u)$ is a topological space.

Remarks :

[1] The sets $(0, 1) \cup (2, 4)$, $(-2, 1)$ … etc are open sets in τ_u.

[2] The natural numbers $\mathbb{N}$ is not open set since its cannot represented as a union of open intervals, but its closed set since $\mathbb{N}^c = (-\infty, 1) \cup (1, 2) \cup \ldots$ is open set in τ_u.

[3] Every set contains discrete points is closed set in τ_u.

[4] Every closed interval is closed set in τ_u.

[5] The rational numbers set $\mathbb{Q}$ and the irrational numbers set Irr are not open sets and not closed sets in τ_u.

Example : Let $X = \mathbb{R}$ and $\tau = \{\mathbb{R}, \phi, \mathbb{Q}, \text{Irr}\}$.

τ is a topology on $\mathbb{R}$ and τ is a topology different from τ_u in the previous example. In this example the open intervals is not open sets since it's not contain in τ, while $\mathbb{Q}$, Irr are open and closed in the same time.

How being a topology on any set :

Let X be any nonempty set and A be a proper nonempty subset of X, then

[1] $\tau = \{X, \phi, A\}$ is a topology on X for any X and for any A.

[2] $\tau = \{X, \phi, A, A^c\}$ is a topology on X and this topology has the property that every open set is close set in same time (i.e., $\tau = \mathscr{F}$).

Example : Cofinite Topology

Let X be infinite set and $\tau_{cof} = \{U \subseteq X, U^c = \text{finite set}\} \bigcup \{\phi\}$

Show that (X, τ_{cof}) is a topological space.

Solution :

(1) $\phi \in \tau_{cof}$ (def. of τ_{cof})

$\because$ $X^c = \phi$ and ϕ is finite set, then $X \in \tau_{cof}$

(2) Let $U, V \in \tau_{cof}$

if U or $V = \phi \Rightarrow U \bigcap V = \phi \in \tau_{cof}$

if $U = X \Rightarrow U \bigcap V = V \in \tau_{cof}$

if $V = X \Rightarrow U \bigcap V = U \in \tau_{cof}$

if U and $V \neq \phi$, X

$\Rightarrow U^c$ and V^c finite set

So, $(U \bigcap V)^c = U^c \bigcup V^c$ = finite set $\bigcup$ finite set = finite set

$\Rightarrow U \bigcap V \in \tau_{cof}$

(3) Let $U_\alpha \in \tau_{cof}$; $\alpha \in \Lambda$

if $U_\alpha = X$ for some $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = X \in \tau_{cof} \quad \forall\, \alpha \in \Lambda$

if $U_\alpha = \phi$ for all $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \phi \in \tau_{cof} \quad \forall\, \alpha \in \Lambda$

if $U_\alpha \neq \phi$ or X for all $\alpha \Rightarrow (\bigcup_{\alpha\in\Lambda} U_\alpha)^c = \bigcap_{\alpha\in\Lambda} U^c_\alpha = \bigcap$ finite sets = finite set

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_{cof}$

$\therefore$ (X, τ_{cof}) is a topological space.

Remarks :

[1] Notes that X is any set, so there are infinite number of the topological spaces that satisfy this definition according to the set which put replace from X which has a condition infinite set, so we can replies X by $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{Q}$ or Irr or [0, 1] or $(-\infty, 2]$ or $\mathbb{C}$ …. etc.

Now : Take a special case when $X = \mathbb{N}$ and study the open and closed sets in the space $(\mathbb{N}, \tau_{cof})$.

Notes that, $\mathbb{N} \setminus \{1\}$ is open set since its complement is $\{1\}$ which is finite and the set of even numbers E^+ and odd numbers O^+ are not open sets since the complement of E^+ is O^+ and the complement of O^+ is E^+ and all E^+ and O^+ are not finite.

[2] In general : every open set in the space (X, τ_{cof}) is infinite set, but if the set is infinite this not mean its open i.e.

$$U \in \tau_{cof} \quad \underset{\not\Leftarrow}{\Rightarrow} \quad U \text{ infinite set}$$

[3] In general : every finite set is closed set and every closed set (except X) is finite set. i.e.,

$$A \in \mathscr{F}_{cof} \quad \Leftrightarrow \quad A \text{ finite set } (A \neq X)$$

Example : Let X be any set contain more than one element and let x_0 any element in X and $\tau = \{U \subseteq X \; ; \; x_0 \in U\} \bigcup \{\phi\}$. Show that (X, τ) is a topological space.

Solution :

(1) $\phi \in \tau$ (def. of τ)

$X \in \tau$ (since X contains all its elements, therefore its contain x_0)

(2) Let $U, V \in \tau$

if U or $V = \phi \Rightarrow U \bigcap V = \phi \in \tau$

if $U = X \Rightarrow U \bigcap V = V \in \tau$

if $V = X \Rightarrow U \bigcap V = U \in \tau$

if U and $V \neq \phi, X$

$\Rightarrow x_0 \in U \wedge x_0 \in V$ (def. of τ)

$\Rightarrow x_0 \in U \bigcap V$ (def. of intersection)

$\Rightarrow U \bigcap V \in \tau$

(3) Let $U_\alpha \in \tau \; ; \alpha \in \Lambda$

if $U_\alpha = \phi \quad \forall \alpha \in \Lambda \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \phi$

if $U_\alpha \neq \phi$ for some $\alpha \in \Lambda \Rightarrow x_0 \in U_\alpha$ for some $\alpha \in \Lambda$

$\Rightarrow x_0 \in \bigcup_{\alpha\in\Lambda} U_\alpha$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

$\therefore$ (X, τ) is a topological space.

Remarks :

[1] Notes that any set not contained x_0 is a closed set and any set contained x_0 is open set.

Special case : Suppose that $X = \mathbb{R}$ and $x_0 = 0$, then

the sets $\{0\}$, $(-\infty, 2)$, $\mathbb{Q}$, $[0, 1]$ … etc are open. And
the sets $\{-4\}$, Irr, $\mathbb{N}$, $(6, \infty)$, $[3, 5]$ … etc are closed since it is not contained 0.

[2] In general : we can replace 0 by 2 or $\sqrt{5}$ or any other real number.
And we can replace $\mathbb{R}$ by any other set.

Example : Let X be a nonempty set contain more than one element and let x_0 any element in X and $\tau = \{U \subseteq X \,;\, x_0 \notin U\} \bigcup \{X\}$. Show that (X, τ) is a topological space.

Solution :

(1) $X \in \tau$ (def. of τ)
$\phi \in \tau$ (since $x_0 \notin \phi$ by def. of τ)

(2) Let $U, V \in \tau$
if $U = X \quad \wedge \quad V = X \quad \Rightarrow U \bigcap V = X \in \tau$
if $x_0 \notin U$ or $x_0 \notin V \Rightarrow x_0 \notin U \bigcap V$
$\Rightarrow U \bigcap V \in \tau$ (def. of τ)

(3) Let $U_\alpha \in \tau \,;\, \alpha \in \Lambda$
if $U_\alpha = X$ for some $\alpha \in \Lambda \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = X \in \tau$
if $\exists\, U_\alpha \neq X \;\forall\, \alpha \in \Lambda \Rightarrow x_0 \notin U_\alpha$ (def. of τ)
$\Rightarrow x_0 \notin \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$ (def. of τ)

$\therefore$ (X, τ) is a topological space.

Remarks :

[1] Special case of this example we can take $X = \mathbb{N}$ and $x_0 = 2$, then the open sets are $\mathbb{N}$ and every subset from $\mathbb{N}$ not contain the element 2, while every set contain the element 2 is closed set.

[2] There are infinite number of spaces from this types when replace X by any set and x_0 by any element.

Example : Let $X = \mathbb{N}$ and $\tau = \{A_n \subseteq \mathbb{N} : A_n = \{1, 2, \dots, n\} \,;\, n \in \mathbb{N}\} \bigcup \{\mathbb{N}, \phi\}$ show that τ is a topology on $\mathbb{N}$.

Solution : Notes that the elements of τ as follow

$$A_1 = \{1\},\ A_2 = \{1, 2\},\ A_3 = \{1, 2, 3\}, \dots$$
$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots\dots \subseteq A_n \subseteq \dots$$

(1) $X, \phi \in \tau$ (def. of τ)

(2) Let $A_i, A_j \in \tau$, then $A_i \bigcap A_j = \begin{Bmatrix} A_i \in \tau & if \;\; i \le j \\ A_j \in \tau & if \;\; i > j \end{Bmatrix}$

(3) Let $A_\alpha \in \tau$, $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} A_\alpha = \begin{Bmatrix} A_\delta \in \tau & if\ \delta \geq \alpha\ and\ \alpha\ finite \\ \mathbb{N} \in \tau & if \qquad \alpha\ infinite \end{Bmatrix}$

$\therefore$ $(\mathbb{N}, \tau)$ is a topological space.

The following sets are open in this example :

$$A_{100} = \{1, 2, 3, \ldots, 100\},\ A_{30} = \{1, 2, 3, \ldots, 30\}$$

The following sets are closed in this example :

$$\{3, 4, 5, \ldots\} = \mathbb{N} \setminus \{1, 2\},\ \ \mathbb{N} \setminus \{1, 2, 3, \ldots, 10\},\ \ \mathbb{N} \setminus \{1, 2, 3, 4, 5\}$$

<u>Example :</u> Let $X = \mathbb{N}$ and $\tau = \{B_n \subseteq \mathbb{N} : B_n = \{n, n+1, n+2, \ldots\}\ ;\ n \in \mathbb{N}\} \bigcup \{\phi\}$ show that τ is a topology on $\mathbb{N}$.

<u>Solution :</u> Notes that the elements of τ as follow

$$B_1 = \{1, 2, 3, \ldots\},\ B_2 = \{2, 3, 4, 5, \ldots\},\ B_3 = \{3, 4, 5, \ldots\}$$

$$B_1 = \mathbb{N},\ B_2 = \mathbb{N} \setminus \{1\},\ B_3 = \mathbb{N} \setminus \{1, 2\},\ \ldots \text{ etc}$$

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots..$$

(1) $\mathbb{N}, \phi \in \tau$ (def. of τ)

(2) Let $B_i, B_j \in \tau$, then $B_i \bigcap B_j = \begin{Bmatrix} B_i \in \tau & if\ \ i \geq j \\ B_j \in \tau & if\ \ i < j \end{Bmatrix}$

(3) Let $B_\alpha \in \tau$, $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} B_\alpha = \begin{Bmatrix} B_\delta \in \tau & if & \delta \leq \alpha\ and\ \alpha\ finite \\ \mathbb{N} \in \tau & if & 1 \in \Lambda \qquad \alpha\ infinite \end{Bmatrix}$

$\therefore$ $(\mathbb{N}, \tau)$ is a topological space.

<u>Remark :</u> The open sets in this example are a closed sets in the previous example and vise verse.

<u>Definition :</u> Equal Topological Spaces

Let (X, τ), (Y, τ') be two topological spaces, we say that **(X, τ) <u>equal to</u> (Y, τ')** if the sets and topologies are equal, i.e.,

$$(X, \tau) = (Y, \tau') \iff X = Y \wedge \tau = \tau'$$

<u>Definition :</u> Finer Than & Coarser Than

Let τ_1, τ_2 be two topologies on X, we say the topology **τ_2 is <u>Finer than</u> τ_1** if the family τ_1 is subset of the family τ_2 and we say **τ_1 <u>Coarser than</u> τ_2** and denoted by $\tau_1 \leq \tau_2$, i.e.,

$$\tau_2 \text{ Finer than } \tau_1 \text{ or } \tau_1 \text{ Coarser than } \tau_2 \text{ iff } \tau_1 \subseteq \tau_2.$$

Remarks : Let τ_1, τ_2 be topologies on X, then

[1] $\tau_1 \bigcap \tau_2$ is a topology on X since

(1) $\because$ X, $\phi \in \tau_1$ and X, $\phi \in \tau_2 \Rightarrow$ X, $\phi \in \tau_1 \bigcap \tau_2$

(2) Let U, V $\in \tau_1 \bigcap \tau_2 \Rightarrow$ U , V $\in \tau_1$ and U, V $\in \tau_2$

$\Rightarrow U \bigcap V \in \tau_1$ and $U \bigcap V \in \tau_2$ (since τ_1, τ_2 are topologies on X)

$\Rightarrow U \bigcap V \in \tau_1 \bigcap \tau_2$

(3) Let $U_\alpha \in \tau_1 \bigcap \tau_2$; $\alpha \in \Lambda$

$\Rightarrow U_\alpha \in \tau_1$ and $U_\alpha \in \tau_2 \ \forall \alpha \in \Lambda$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_1$ and $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_2$ (since τ_1, τ_2 are topologies on X)

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_1 \bigcap \tau_2$

$\therefore$ $\tau_1 \bigcap \tau_2$ is a topology on X.

[2] $\tau_1 \bigcup \tau_2$ is not topology on X in general, for example :

Let X = {1, 2, 3}, τ_1 = {X, ϕ, {1}} and τ_2 = {X, ϕ, {2}}. Notes that τ_1, τ_2 are topologies on X, but $\tau_1 \bigcup \tau_2$ = {X, ϕ, {1}, {2}} is not topology on X.

Remarks :

[1] Intersection of infinite number of open sets need not open set.

Example : Let $(X, \tau) = (\mathbb{R}, \tau_u)$ and $U_n = (-\frac{1}{n}, \frac{1}{n})$ such that $n \in \mathbb{N}$. We know the open intervals is open sets in the space $(\mathbb{R}, \tau_u)$, so $\{U_n\}_{n\in\mathbb{N}}$ is a family of open sets, but the intersection of this family is not open set since

$$\bigcap_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) = \{0\} \text{ not open.}$$

[2] Union of any family of closed sets need not closed set.

Example : Let $(X, \tau) = (\mathbb{N}, \tau_{cof})$ and $A_n = \{2n\}_{n\in IN}$ i.e.,

$$A_1 = \{2\}, A_2 = \{4\}, A_3 = \{6\}, \ldots\ldots .$$

Notes that every set A_n is closed for all n in this space, but the union of this family is the positive even number {2, 4, 6, …….} and this set is not closed in this space (see example $(\mathbb{N}, \tau_{cof})$, page 5).

Definition : Neighborhood

Let (X, τ) be a topological space, $x \in X$ and $A \subseteq X$. We called A is a **neighborhood** for a point x if there exist an open set U contains x and contain in A and denoted by nbhd. i.e.,

$$A \text{ is a nbhd for } x \Leftrightarrow \exists\, U \in \tau \,;\, x \in U \subseteq A$$

If A is open set and contains x we called A is open neighborhood for a point x.

Example : In the space ($\mathbb{R}$, τ_u), every an open interval is an open nbhd for any point in this interval, while the closed interval or half open interval is nbhd for every point in this intervals except the end point in the closed interval.

Example : In the space ($\mathbb{N}$, τ_{cof}), find three open nbhds for the point 2 and two open nbhds for the point 3.

Solution :

A = {2, 3, 4, 5, …. }, B = {2, 10, 11, 12, …..} and C = {2, 20, 21, ….} are open nbhds for the element 2.

U = {3, 4, 5, ….} and V = {3, 6, 7,8, ….} are open nbhds for the element 3.

Theorem : Let (X, τ) be a topological space and $A \subseteq X$, then A is open iff A contains an open nbhd for every point in A.

Definition : Basis or Base

Let (X, τ) be a topological space and β be a subfamily from τ. We called β is a **basis or base** for τ, if every element in τ is a union numbers of elements of β. i.e.,

β is a basis or base for τ $\Leftrightarrow$ (1) $\beta \subseteq \tau$

(2) $\forall\, U \in \tau$; $U = \bigcup_i B_i$; $B_i \in \beta$ $\forall\, i$

Remark : From the definition of the base we notes that the number of bases are not determined, so the number of bases is open, may be finite number and may be infinite number.

Example : Let X = {a, b, c} and τ = {X, ϕ, {a}, {b}, {a, b}}, define a base for τ?

Solution : Let β = {X, ϕ, {a}, {b}}

Clearly $\beta \subseteq \tau$ and X, ϕ, {a}, {b} $\in \tau$ and also X, ϕ, {a}, {b} $\in \beta$ and

$\{a, b\} \in \tau \Rightarrow \{a, b\} = \underbrace{\{a\}}_{\in\beta} \bigcup \underbrace{\{b\}}_{\in\beta}$

$\therefore$ β is a base for τ.

Remark : τ is a base for τ (i.e., we can chose $\beta = \tau$) and this is a special case and conclude from this case there is not exists topology has no base and this base called **trivial base**.

Example : Let X = {1, 2, 3} and τ = IP(X) = {X, ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2,3}}. Define two different bases for τ?

Solution : Let β_1 = {ϕ, {1}, {2}, {3}}, β_2 ={ϕ, {1}, {2}, {3}, {1, 2}}

We can show by a simple way that β_1 and β_2 are bases for τ since every one of them generated the elements of τ.

Example : Define base for the usual topology $(\mathbb{R}, \tau_u)$.

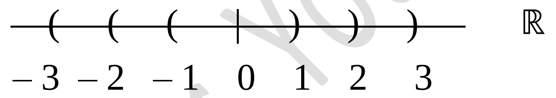
Solution : Let $\beta = \{(a, b) : a \in \mathbb{R} \wedge b \in \mathbb{R}\}$

Notes that β contain every open intervals which end points are real numbers (i.e., $(-3, 2) \in \beta$ while $(-\infty, 5) \notin \beta$)

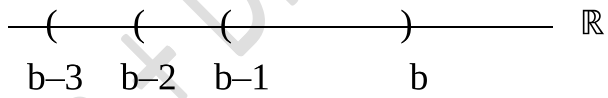
Notes that $\phi \in \beta$ since $\phi = (a, a)$ such that a is real number.

To prove β is a base for τ_u, it is enough to prove $\mathbb{R} = (-\infty, \infty)$, $(-\infty, b)$ and (a, ∞) equal union of family of elements of β. So we introduce this prove :

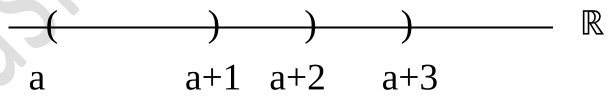
$\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$; $(-n, n) \in \beta \ \ \forall\, n \in \mathbb{N}$



$(-\infty, b) = \bigcup_{n=1}^{\infty}(b-n, b)$; $(b-n, b) \in \beta \ \ \forall\, n \in \mathbb{N}$



$(a, \infty) = \bigcup_{n=1}^{\infty}(a, a+n)$; $(a, a+n) \in \beta \ \ \forall\, n \in \mathbb{N}$



This is a prove that β is a base for τ_u.

Remarks :

[1] For one topology we can fined for more than one base (i.e., the base is not unique).

[2] Every base for any topology must contains the empty set ϕ (i.e., $\phi \in \beta$) since $\phi \in \tau$ must be equal union element from β (by def. of β).

[3] X may be not belong to the base β and the previous example clear that.

[4] If the singleton set $\{x\} \in \tau$, then $\{x\} \in \beta$.

Theorem : Let (X, τ) be a topological space and β be a base for τ, then

(1) X is a union elements of β.

(2) If $B_1, B_2 \in \beta$, then $B_1 \bigcap B_2$ is a union elements of β.

Proof :

(1) $\because\ X \in \tau \ \Rightarrow\ X =$ union of elements of β (def. of base)

(2) $\because\ B_1 , B_2 \in \beta$ and $\beta \subseteq \tau$

$\Rightarrow\ B_1 , B_2 \in \tau$

$\Rightarrow\ B_1 \bigcap B_2 \in \tau$ (second condition from def. of top.)

$\Rightarrow B_1 \bigcap B_2 =$ union of elements of β (def. of base)

This theorem clear the properties of base and the next theorem is a new method to get a topology by using a family of sets from β which satisfy the condition of previous theorem.

Theorem : Let X be a nonempty set and β be a family of subsets of X satisfying the following properties:

(1) X = union of family of elements of β

(2) The intersection any two elements of β is a union elements of β.

Then τ which is define as follows:

$\tau = \{U \subseteq X\ ;\ U = \text{union of elements of } \beta\}$

Is a topology on X and this is the unique topology on X such that β is a base for τ.

Proof : To prove τ is a topology on X must prove the three condition for topology.

(1) $\phi \in \tau$ (def. of τ)

$X \in \tau$ (from (1))

(2) Let $U, V \in \tau$ to prove $U \bigcap V \in \tau$

$\because\ U, V \in \tau \ \Rightarrow\ U = \bigcup_i B_i \ \wedge\ V = \bigcup_j B_j \ \ni\ B_i, B_j \in \beta\ \forall\ i, j$ (def. of τ)

$\Rightarrow\ U \bigcap V = (\bigcup_i B_i) \bigcap (\bigcup_j B_j) = \bigcup_{i,j} (B_i \bigcap B_j)$

$= \bigcup (\bigcup_k B_k)\ ;\ B_k \in \beta$ (from (2))

(3) Let $U_\alpha \in \tau\ \ \forall\ \alpha \in \Lambda$ to prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

$\because\ U_\alpha \in \tau \ \Rightarrow\ U_\alpha = \bigcup_i B_i \ \ni\ B_i \in \beta\ \forall\ i$ (def. of τ)

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcup_{\alpha\in\Lambda}(\bigcup_i B_i) = \bigcup_k B_k$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$ (def. of τ)

This prove that τ is a topology on X by define of τ.

To prove that τ is the unique topology generated from β. Suppose there exists another topology say τ' generated from β, this means that τ' = all possible union for elements of β, but τ = all possible union for elements of $\beta \Rightarrow \tau' = \tau$.

Definition : Subbasis

Let (X, τ) be a topological space and β be a base for τ and $\mathcal{S}$ be a subfamily from τ. We called $\mathcal{S}$ is a **subbasis** for τ if every element of basis β equal finite intersection numbers of elements of $\mathcal{S}$. i.e.,

$\mathcal{S}$ is a subbasis of $\tau \Leftrightarrow \forall\ B \in \beta\ \Rightarrow\ B = \bigcap_{j=1}^{n} S_j \ \ni\ S_j \in \mathcal{S}\ \ \forall\ j\ ;\ j = 1, 2, \ldots, n$

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\beta = \{\phi, \{a\}, \{b\}, \{a, c\}\}$. Define a subbasis for τ.

Solution : Let $\mathcal{S} = \{X, \{a, c\}, \{a, b\}, \{b\}\}$.

Clear, $\mathcal{S} \subseteq \tau$, to prove $\mathcal{S}$ is a subbasis for τ we compute all different intersection for elements of $\mathcal{S}$, if we get β, then $\mathcal{S}$ is subbasis for τ.

$\phi = \{a, c\} \bigcap \{b\}$, $\{a\} = \{a, c\} \bigcap \{a, b\}$, $\{b\} = \{b\} \bigcap \{b\}$,
$\{a, c\} = \{a, c\} \bigcap \{a, c\}$, $\{a, b\} = \{a, b\} \bigcap \{a, b\}$, $X = X \bigcap X$,
So, we get all elements of β, this means that $\mathcal{S}$ is a subbasis for τ.

Example : Define a subbasis for a usual topological space $(\mathbb{R}, \tau_u)$.

Solution : From the previous example we prove that $\beta = \{(a, b) \,;\, a, b \in \mathbb{R}\}$ is a basis for τ_u. We must define a subbasis $\mathcal{S}$ for τ_u such that $\mathcal{S}$ generated β.

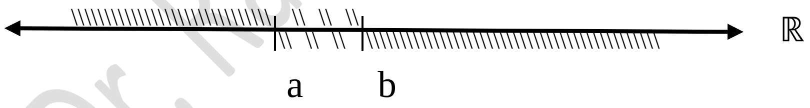
Define $\mathcal{S} = \{(a, b) \,;\, a = -\infty \vee b = \infty\}$

Notes that $\mathcal{S} \subseteq \tau_u$ and

$(-1, 4) \in \beta \wedge (-1, 4) \notin \mathcal{S}$, $(-\infty, 3) \in \mathcal{S} \wedge (-\infty, 3) \notin \beta \Rightarrow \mathcal{S} \not\subset \beta \wedge \beta \not\subset \mathcal{S}$

Now, to show that $\mathcal{S}$ is subbasis for τ, we take an element of β and prove that its equal finite intersection numbers of elements of $\mathcal{S}$ as follow :

Let $(a, b) \in \beta$; $a, b \in \mathbb{R}$



$$(a, b) = \underbrace{(-\infty, b)}_{\in \mathcal{S}} \bigcap \underbrace{(a, \infty)}_{\in \mathcal{S}}$$

$\therefore$ $\mathcal{S}$ is a subbasis for τ_u.

Remarks :

[1] We can find more than one subbasis for one topology.

[2] ϕ may be not contain in a subbasis.

[3] $X \in \mathcal{S}$.

[4] τ is a subbasis for τ.

Theorem : Let X be a nonempty set and $\mathcal{S}$ be a subfamily of subsets of X, then the set $\beta = \{B \subseteq X \,;\, B = \text{finite intersection numbers of elements of } \mathcal{S}\}$ is a basis for the unique topology on X define as follow :

$$\tau = \{U \subseteq X \,;\, U = \text{union of elements of } \beta\}.$$

Proof : without prove.

This theorem show that there exists a method to generated a topology on X if we have a set $\mathcal{S}$ such that $\mathcal{S}$ generated β and β generated τ and this topology is unique such that β is a basis for τ and $\mathcal{S}$ is a subbasis for τ.

<u>Definition :</u> Open Neighborhood System

Let (X, τ) be a topological space and $x \in X$ and η_x be a family of open sets (i.e., $\eta_x \subseteq \tau$) and satisfying the following conditions:

(1) $\eta_x \neq \phi$ for all $x \in X$.

(2) $x \in N \ \ \forall \ \ N \in \eta_x$.

(3) $\forall\ N_1, N_2 \in \eta_x \ \Rightarrow\ \exists\ N_3 \in \eta_x$ such that $N_3 \subseteq N_1 \bigcap N_2$.

(4) $\forall\ N \in \eta_x \ \ \forall\ y \in N \ \ \exists\ N' \in \eta_y$ such that $N' \subseteq N$.

(5) $U \in \tau \ \Leftrightarrow\ \forall\ x \in U\ \exists\ N \in \eta_x$ such that $N \subseteq U$.

We called the family $\eta = \{\eta_x ; x \in X\}$ **<u>open neighborhood system</u>** for τ and denoted by (o.n.s)

<u>Example :</u> Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define open neighborhood system for τ.

<u>Solution :</u> We must prove for all element in X a family of this element satisfy the five conditions in the definition as follow :

$\eta_a = \{\{a\}, \{a, b\}\}$, $\eta_b = \{\{b\}, \{a, b\}\}$, $\eta_c = \{X\}$, notes that

(1) η_a, η_b and η_c are nonempty and contain of a family of open sets and

(2) $\forall\ N \in \eta_a \Rightarrow a \in N$, $\forall\ N \in \eta_b \Rightarrow b \in N$ and $\forall\ N \in \eta_c \Rightarrow c \in N$,

(3) Intersection of any two element in η_a or η_b or η_c is an element in η_a or η_b or η_c

(4) $\{a, b\} \in \eta_a$; $b \in \{a, b\}$ and $b \in \{b\}$; $\{b\} \subseteq \{a, b\}$

(5) Every open set satisfy the five condition

$\therefore\ \eta = \{\eta_a, \eta_b, \eta_c\}$ is open neighborhood system for τ

<u>Example :</u> Define open neighborhood system for a usual topological space $(\mathbb{R}, \tau_u)$.

<u>Solution :</u> Let $x \in \mathbb{R}$, define η_x as follow : $\eta_x = \{(a, b) ; x \in (a, b)\}$

i.e., η_x is a family of every open sets that contain x.

clear $\eta_x \subseteq \tau_u$ and η_x satisfy the five conditions of open neighborhood system for τ as follow :

(1) $\eta_x \neq \phi$ since $(x - \varepsilon, x + \varepsilon) \in \eta_x$; $\varepsilon > 0$

(2) if $N \in \eta_x$, then $N = (a, b)$ and $x \in (a, b)$ (def. of η_x)

(3) if $N_1, N_2 \in \eta_x$

$\Rightarrow N_1$ open interval s.t. $x \in N_1$ and N_2 open interval s.t. $x \in N_2$

$\Rightarrow N_1 \bigcap N_2 \neq \phi$

$\Rightarrow N_1 \bigcap N_2$ open interval s.t. $x \in N_1 \bigcap N_2$

$\Rightarrow N_1 \bigcap N_2 \in \eta_x$.

(4) let $N \in \eta_x$ and $y \in N \Rightarrow N$ open interval s.t. $y \in N \Rightarrow N \in \eta_y$.

(5) This condition is satisfy from definition of usual topology on $\mathbb{R}$.

<u>Theorem :</u> Let X be a nonempty and η_x be a family of subsets from X. for all $x \in X$; η_x satisfy the condition (1), (2), (3), (4) in the definition of open neighborhood system , then τ which define as follow :

$$\tau = \{U \subseteq X \,;\ \forall\, x \in U \ \ \exists\, N \in \eta_x \text{ such that } N \subseteq U \}$$

Is a topology on X such that η_x is open neighborhood system for τ.

<u>Proof :</u> We must prove τ satisfy the three conditions for topology.

(1) $X \in \tau$ (since X contains all subset of X)

$\phi \in \tau$ (since, $x \in \phi \Rightarrow \exists\, N \in \eta_x \,;\, N \subseteq \phi$)

$(F \Rightarrow F) = T$

(2) Let $U, V \in \tau$ To proof $U \bigcap V \in \tau$

Let $x \in U \bigcap V \Rightarrow x \in U$ and $x \in V$ (def. of $\bigcap$)

$\Rightarrow \exists\, N_1 \in \eta_x$ such that $N_1 \subseteq U$ and $\exists\, N_2 \in \eta_x$ such that $N_2 \subseteq V$ (def. of τ)

$\Rightarrow \exists\, N_3 \in \eta_x$ such that $N_3 \subseteq N_1 \bigcap N_2$

(condition (3) from open neighborhood system)

$\Rightarrow N_3 \subseteq U$ and $N_3 \subseteq V$

$\Rightarrow N_3 \subseteq U \bigcap V$ (def. of $\bigcap$)

$\Rightarrow U \bigcap V \in \tau$

(3) Let $U_\alpha \in \tau$; $\alpha \in \Lambda$ To proof $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

Let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow \exists\, \alpha \in \Lambda \,;\, x \in U_\alpha$ (def. of $\bigcup$)

$\Rightarrow \exists\, N \in \eta_x$ such that $N \subseteq U_\alpha$ (since $U_\alpha \in \tau$ and def. of τ)

$\Rightarrow N \subseteq U_\alpha \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

$\Rightarrow N \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

$\therefore \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

$\therefore$ τ is a topology on X and from prove above we have η_x $\forall\, x \in X$ is open neighborhood system for τ.

Remark : From information above we have five deference method to define topology on a nonempty set as follow :

[1] Direct definition for τ by write $\tau = \{\dots\ \}$.

[2] Using the family $\mathscr{F}$ such that the complement of this family is topology.

[3] Using the family β such that the union of all possible of elements of β is topology.

[4] Using the family $\mathscr{S}$ such that the finite intersection of elements of $\mathscr{S}$ is a basis for topology.

[5] Using the family η_x ; $x \in X$ and τ is the family of every sets that contain open neighborhood for every element.

Derived Sets

Definition : Interior points and Interior set

Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in A$ is called an **interior point** of A iff there exists an open set $U \in \tau$ containing x such that $x \in U \subseteq A$. The set of all interior points of A is called the **interior** of A and is denoted by A^o or Int(A). i.e.,

$$A^o = \{x \in A : \exists\ U \in \tau\ ; x \in U \subseteq A\ \}$$

$$x \in A^o \Leftrightarrow \exists\ U \in \tau\ ; x \in U \subseteq A$$

if $x \notin A^o$, we define

$$x \notin A^o \Leftrightarrow \forall\ U \in \tau \text{ such that } x \in U \text{ and } U \not\subset A$$

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $A = \{b\}$, $B = \{a, c\}$ and $C = \{c\}$. Find A^o, B^o and C^o.

Solution :

$A^o = \{b\} = A$ since $b \in U = \{b\} \subseteq A = \{b\}$

$B^o = \{a\}$ since $a \in U = \{a\} \subseteq B = \{a\}$

$C^o = \phi$ since the only open set contain in C is ϕ.

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A^o \subseteq A$

(2) $A \subseteq B \Rightarrow A^o \subseteq B^o$

(3) $A \in \tau$ (i.e., A is open) $\Leftrightarrow A^o = A$

(4) $A^o = \bigcup\{U \in \tau\ ; U \subseteq A\}$ (this means A^o is the large open set contain in A)

(5) $A^o \bigcap B^o = (A \bigcap B)^o$

(6) $A^o \bigcup B^o \subseteq (A \bigcup B)^o$

Proof :

(1) From definition of A^o

(2) Suppose that $A \subseteq B$ to prove $A^o \subseteq B^o$

Let $x \in A^o \Rightarrow \exists\, U \in \tau\,;\, x \in U \subseteq A$ (def of A^o)

$\Rightarrow \exists\, U \in \tau\,;\, x \in U \subseteq B$ (since $A \subseteq B$)

$\Rightarrow x \in B^o$ (def of B^o)

(3) ($\Rightarrow$) Suppose that A is open, to prove $A^o = A$

From (1) $A^o \subseteq A$ ----------------(1)

Let $x \in A \Rightarrow x \in A \subseteq A$ (since $A \in \tau$)

$\Rightarrow x \in A^o$ (def of A^o)

$\Rightarrow A^o \subseteq A$ ----------------(2)

From (1) and (2), we have $A^o = A$

($\Leftarrow$) Suppose that $A^o = A$, to prove A is open

$\forall\; x \in A \Rightarrow \exists\, U_x \in \tau\,;\, x \in U_x \subseteq A$ (since $A^o = A$)

$\Rightarrow \bigcup_{x\in A} U_x \subseteq A \quad \wedge \quad A \subseteq \bigcup_{x\in A} U_x$

$\Rightarrow A = \bigcup_{x\in A} U_x$

But, U_x open set $\forall\, x \Rightarrow \bigcup_{x\in A} U_x$ is open

$\Rightarrow A$ is open (by three condition of def. of top.)

(4) To prove $A^o = \bigcup\{U \in \tau\,;\, U \subseteq A\}$

$x \in A^o \Leftrightarrow \exists\, U \in \tau\,;\, x \in U \subseteq A$ (def of A^o)

$\Leftrightarrow x \in \bigcup\{U \in \tau\,;\, U \subseteq A\}$

Since the element x belong to one of this sets in the union then its belong to union $\therefore A^o = \bigcup\{U \in \tau\,;\, U \subseteq A\}$

(5) To prove $A^o \bigcap B^o = (A \bigcap B)^o$, we must prove

$(A \bigcap B)^o \subseteq A^o \bigcap B^o \;\wedge\; A^o \bigcap B^o \subseteq (A \bigcap B)^o$

$(A \bigcap B) \subseteq A \quad \wedge \quad (A \bigcap B) \subseteq B$ (def. of $\bigcap$)

$\Rightarrow (A \bigcap B)^o \subseteq A^o \quad \wedge \quad (A \bigcap B)^o \subseteq B^o$ (from (2) above)

$\Rightarrow (A \bigcap B)^o \subseteq A^o \bigcap B^o$ ----------------(1)

From (1) $A^o \subseteq A \;\wedge\; B^o \subseteq B$

$\Rightarrow A^o \bigcap B^o \subseteq A \bigcap B$

$\because A^o \bigcap B^o$ open set containing in $A \bigcap B$

and $(A \bigcap B)^o$ large open set containing in $A \bigcap B$

$\Rightarrow A^o \bigcap B^o \subseteq (A \bigcap B)^o$ ----------------(2)

From (1) and (2), we have $(A \bigcap B)^o = A^o \bigcap B^o$

(6) $A \subseteq A \bigcup B \;\wedge\; B \subseteq A \bigcup B$ (def. of $\bigcup$)

$\Rightarrow\ A^o \subseteq (A \bigcup B)^o \quad \wedge \quad B^o \subseteq (A \bigcup B)^o$
$\Rightarrow\ A^o \bigcup B^o \subseteq (A \bigcup B)^o$

Remarks :

[1] The converse of property (2) is not true, i.e.,

$$A^o \subseteq B^o \not\Rightarrow A \subseteq B$$

The following example show that :

Example : Let X = {a, b, c}, τ = {X, ϕ, {b}, {c}, {b, c}}, A = {a}, B = {b, c}.

$A^o = \phi$ and B^o = {b, c} = B

Notes that, $A^o \subseteq B^o$ but $A \not\subset B$

[2] The converse contains of property (5) is not true in general. i.e.,

$$(A \bigcup B)^o \not\subset A^o \bigcup B^o$$

In the previous example show that :

$A \bigcup B = X \ \Rightarrow\ (A \bigcup B)^o = X$

But, $A^o = \phi$ and $B^o = \{b, c\} \Rightarrow A^o \bigcup B^o = \{b, c\}$ and $X \not\subset \{b, c\}$.

[3] There exists a special cases of property (3) as follow :

$X \in \tau \Rightarrow X^o = X$ and $\phi \in \tau \Rightarrow \phi^o = \phi$ and $(A^o)^o = A^o$

In a space (X, I) the only open sets are X and ϕ, so if $A \subsetneq X$, then $A^o = \phi$.

In a space (X, D) every subset of X is open, so $\forall\ A \subseteq X$, then $A^o = A$.

[4] If {x} open set in any topological space, then x is interior point of any set contain x, i.e., $\{x\} \in \tau \ \Rightarrow\ x \in A^o \ \forall\ A$ such that $x \in A$

Example : In usual topological space $(\mathbb{R}, \tau_u)$, find the interior of the following sets :

$$A = [a, b],\ B = \mathbb{N},\ C = \mathbb{Q},\ D = [0, \infty)$$

Solution :

Interior of any set in this example is the largest open set containing in this set.

$[a, b]^o = [a, b)^o = (a, b]^o = (a, b)^o = (a, b)$

$\mathbb{N}^o = \mathbb{Z}^o = \mathbb{P}^o = E^o = O^o = \phi$

$\mathbb{Q}^o = Irr^o = \phi$

$[a, \infty)^o = (a, \infty)$ and $(-\infty, b]^o = (-\infty, b)$.

Example : In cofinite topological space $(\mathbb{N}, \tau_{cof})$, let $A \subseteq \mathbb{N}$. Find A^o.

Solution : If A is open set, then $A^o = A$. For example A = {4, 5, 6, … }

If A is not open set, so there exists two cases either A closed set or A is not closed set, then $A^o = \phi$ [[since A^o is open set, this means by definition τ_{cof} that the complement of A^o is finite set and since $A^o \subseteq A$ (in general), then the complement of A must be finite if $A^o \neq \phi$. This means the interior of a set in this space either ϕ or A.

Definition : **Exterior points and Exterior set**

Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in A^c$ is called an **exterior point** of A iff there exists an open set $U \in \tau$ containing x such that $x \in U \subseteq A^c$. The set of all exterior points of A is called the **exterior** of A and is denoted by A^x or Ext(A). i.e.,

$$A^x = \{x \in A^c : \exists\, U \in \tau \,;\, x \in U \subseteq A^c\}$$

$$x \in A^x \Leftrightarrow \exists\, U \in \tau \,;\, x \in U \subseteq A^c$$

if $x \notin A^x$, we define

$$x \notin A^x \Leftrightarrow \forall\, U \in \tau \text{ such that } x \in U \not\subset A^c$$

Remark : From definition we have $A^x \subseteq A^c$ or $A^x \cap A = \phi$ and $A^x = (A^c)^o$.

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $A = \{b\}$, $B = \{a, c\}$ and $C = \{c\}$. Find A^x, B^x and C^x.

Solution :

$A^x = (A^c)^o = \{a\}$ (largest open set contain in A^c)

$B^x = (B^c)^o = \phi$ and $C^x = \{a, b\}$.

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A^o \bigcap A^x = \phi$

(2) $A \subseteq B \Rightarrow B^x \subseteq A^x$

(3) $(A \bigcup B)^x = A^x \bigcap B^x$

(4) $A^c \in \tau$ (i.e., A closed) $\Leftrightarrow A^x = A^c$

(5) $A^x \bigcup B^x \subseteq (A \bigcap B)^x$

Proof :

(7) From definition of A^o $\Rightarrow A^o \subseteq A$ and $A^x \subseteq A^c$

$\Rightarrow A^o \bigcap A^x \subseteq A \bigcap A^c$

$\Rightarrow A^o \bigcap A^x \subseteq \phi$

$\Rightarrow A^o \bigcap A^x = \phi$

(8) Suppose that $A \subseteq B$ to prove $B^x \subseteq A^x$

Let $x \in B^x \Rightarrow \exists\, U \in \tau \,;\, x \in U \subseteq B^c$ (def. of B^x)

$\Rightarrow \exists\, U \in \tau \,;\, x \in U \subseteq A^c$ (since $A \subseteq B \Rightarrow B^c \subseteq A^c$)

$\Rightarrow x \in A^x$ (def. of B^x)

$\therefore B^x \subseteq A^x$

(9) To prove $(A \bigcup B)^x = A^x \bigcap B^x$

$(A \bigcup B)^x = ((A \bigcup B)^c)^o = (A^c \bigcap B^c)^o = (A^c)^o \bigcap (B^c)^o = A^x \bigcap B^x$

(10) ($\Rightarrow$) Suppose that A is closed or A^c is open, to prove $A^x = A^c$

$A^c \in \tau \Rightarrow (A^c)^o = A^c$ (by theorem, $A \in \tau \Leftrightarrow A^o = A$)

$\Rightarrow A^x = A^c$ (since $(A^c)^o = A^x$)

($\Leftarrow$) Suppose that $A^x = A^c$, to prove A is closed or A^c is open

$A^x = A^c \Rightarrow (A^c)^o = A^c$ (since $(A^c)^o = A^x$)

$\Rightarrow A^c \in \tau$ (by theorem, $A \in \tau \Leftrightarrow A^o = A$)

$\Rightarrow$ A is closed

(11) $A \bigcap B \subseteq A$ and $A \bigcap B \subseteq B \Rightarrow A^x \subseteq (A \bigcap B)^x$ and $B^x \subseteq (A \bigcap B)^x$

$\Rightarrow A^x \bigcup B^x \subseteq (A \bigcap B)^x$.

Example : In usual topological space $(\mathbb{R}, \tau_u)$, find the exterior of the following sets :

$$\mathbb{N}, \mathbb{Q}, (6, 7), \{-\sqrt{2}, \sqrt{2}\}, (-\infty, 5], [-1, \infty), [2, 4]$$

Solution :

exterior of any set in this example is the largest open set exterior this set.

$\mathbb{N}^x = (\mathbb{N}^c)^o$

$\mathbb{R}$ —— 1| —— 2| —— 3| —— 4| ——

$\mathbb{N}^c = \mathbb{R} - \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup \ldots\ldots$

Clear that $\mathbb{N}^c$ is a union of open interval, so it's open set

$\mathbb{N}^x = (\mathbb{N}^c)^o = \mathbb{R} - \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup \ldots\ldots$

$\mathbb{Q}^x = \phi$ and $(Irr)^x = \phi$

$(6, 7)^x = (-\infty, 6) \cup (7, \infty) \subseteq \mathbb{R} \setminus [6, 7]$

—— 6| —— 7|

$\{-\sqrt{2}, \sqrt{2}\}^x = (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty) = \mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\}$

$(-\infty, 5]^x = (5, \infty)$ and $[-1, \infty)^x = (-\infty, -1)$

$[2, 4]^x = \mathbb{R} - [2, 4] = (-\infty, 2) \cup (4, \infty)$

Remarks :

[1] In a space (X, τ), every one X, ϕ are closed sets, so property (4) apply of them, i.e., $X^x = \phi$, $\phi^x = X$.

[2] In a space (X, I), if $\phi \neq A \subseteq X$, then $A^x = \phi$ because the only sets in I are X, ϕ and since $A \neq \phi$, then $A^x \neq X$, so the unique open set contain in A^c is ϕ.

[3] In a space (X, D), if $A \subseteq X$, then $A^x = A^c$ because every sets in D are open and closed.

Example : Let $X = \mathbb{R}$ and $\tau = \{X, \phi, \mathbb{N}, \mathbb{P}\}$; $\mathbb{P}$ is prime numbers set and $\mathbb{P} \subseteq \mathbb{N}$. Clear that τ is a topology on $\mathbb{R}$ and the open sets in this space are $\mathbb{R}, \phi, \mathbb{N}, \mathbb{P}$ only. Find exterior set of the following sets :

$$\mathbb{Q}, \quad \text{Irr}, \quad [2, 6], \quad \mathbb{N}, \quad \mathbb{Z}, \quad (-\infty, 1]$$

Solution :

$\mathbb{Q}^x = \phi$ since $\mathbb{Q}^c = \text{Irr}$ and there is no open set contain in Irr except ϕ.

$\text{Irr}^x = \mathbb{N}$ since $(\text{Irr})^c = \mathbb{Q}$ and $\mathbb{N}$ is large open set contain in $\mathbb{Q}$.

$[2, 6]^x = \phi$ since $[2, 6]^c = (-\infty, 2) \cup (6, \infty)$ and there is no open set contain in $(-\infty, 2) \cup (6, \infty)$ except ϕ.

$\mathbb{N}^x = \phi$ since $\mathbb{N}^c = \mathbb{R} \setminus \mathbb{N}$ and $\mathbb{R} \setminus \mathbb{N}$ not contain $\mathbb{R}, \mathbb{N}, \mathbb{P}$.

$\mathbb{Z}^x = \phi$ since $\mathbb{Z}^c = \mathbb{R} \setminus \mathbb{Z}$ and $\mathbb{R} \setminus \mathbb{Z}$ not contain $\mathbb{R}, \mathbb{N}, \mathbb{P}$.

$(-\infty, 1]^x = \mathbb{P}$ since $(-\infty, 1]^c = (1, \infty)$ and $\mathbb{P} \subseteq (1, \infty)$ while $\mathbb{N} \not\subseteq (1, \infty)$.

Definition : Boundary points and boundary set

Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called a **boundary point** of A iff every open set in X containing x contains at least one point of A, and at least one point of A^c. The set of all boundary points of A is called the **boundary** of A and is denoted by A^b or Bd(A) or b(A) or $\partial(A)$. i.e.,

$$A^b = \{x \in X : \forall\ U \in \tau\ ; x \in U\ , U \cap A \neq \phi\ \wedge\ U \cap A^c \neq \phi\}$$

$$x \in A^b \Leftrightarrow \forall\ U \in \tau\ ; x \in U\ , U \cap A \neq \phi\ \wedge\ U \cap A^c \neq \phi$$

if $x \notin A^b$, we define

$$x \notin A^b \Leftrightarrow \exists\ U \in \tau\ ; x \in U\ , U \cap A = \phi\ \vee\ U \cap A^c = \phi.$$

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a, c\}$, $B = \{c\}$ and $C = \{a, b\}$. Find A^b, B^b and C^b.

Solution :

$A^b = ?$

To find the boundary of any set we must choose every open sets for every point in X and notes satisfy the definition or not.

$a \in X$ and the open sets contain a are X, {a}, {a, b}

notes that : $\{a\} \cap A = \{a\} \neq \phi$ while $\{a\} \cap A^c = \{a\} \cap \{b\} = \phi \Rightarrow a \notin A^b$.

$b \in X$ and the open sets contain a are X, {b}, {a, b}

notes that : $\{b\} \cap A = \phi \Rightarrow b \notin A^b$.

$c \in X$ and the only open set contain c is X.

notes that : $X \cap A \neq \phi$ and $X \cap A^c \neq \phi \Rightarrow c \in A^b$.

Therefore $A^b = \{c\}$

$B^b = ?$

Since $\{a\} \cap B = \{a\} \cap \{c\} = \phi \Rightarrow a \notin B^b$

Since $\{b\} \cap B = \{b\} \cap \{c\} = \phi \Rightarrow b \notin B^b$

Since $X \cap B \neq \phi$ and $X \cap B^c \neq \phi \Rightarrow c \in B^b$.

Therefore $B^b = \{c\}$

$C^b = ?$. by similar way we have $a \notin C^b$, $b \notin C^b$, $c \in C^b$

Therefore $C^b = \{c\}$.

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, c\}\}$, $A = \{b\}$, $B = \{a, b\}$ and $C = \{b, c\}$. Find A^b, B^b and C^b.

Solution :

$A^b = ?$

$a \in X$ and the open sets contain a are $X, \{a, c\}$

notes that : $\{a, c\} \cap A = \phi \Rightarrow a \notin A^b$.

$b \in X$ and the only open set contain b is X

notes that : $X \cap A \neq \phi$ and $X \cap A^c \neq \phi \Rightarrow b \in A^b$.

$c \in X$ and the open sets contain a are $X, \{a, c\}$

notes that : $\{a, c\} \cap A = \phi \Rightarrow c \notin A^b$.

Therefore $A^b = \{b\}$

$B^b = ?$

Since $X \cap B \neq \phi$ and $X \cap B^c \neq \phi$ also

$\{a, c\} \cap B \neq \phi$ and $\{a, c\} \cap B^c \neq \phi \Rightarrow a \in B^b$

Since $X \cap B \neq \phi$ and $X \cap B^c \neq \phi \Rightarrow b \in B^b$.

Since $X \cap B \neq \phi$ and $X \cap B^c \neq \phi$ also

$\{a, c\} \cap B \neq \phi$ and $\{a, c\} \cap B^c \neq \phi \Rightarrow c \in B^b$

Therefore $B^b = \{a, b, c\} = X$.

$C^b = ?$

$a \in C^b$, $b \in C^b$, $c \in C^b$

Therefore $C^b = \{a, b, c\} = X$.

Remarks :

[1] Notes that : $A^b \subseteq A$ or $A^b \subseteq A^c$ or $A^b \cap A \neq \phi$ or $A^b \cap A^c \neq \phi$. i.e., anything possible.

[2] If $\{a\} \in \tau$ in any topological space (X, τ) ; $a \in X$, then a is not boundary point for any set A in X since if $a \in A$, then $\{a\} \cap A^c = \phi$ and if $a \notin A$, then $\{a\} \cap A = \phi$, so in this two case $a \notin A^b$. Therefore, we can use this idea to have a set contain number of boundary points we determent, for example :

Example : Give an example for a subset A of topological space (X, τ) contains six boundary points.

Solution : Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, $\tau = \{X, \phi, \{1\}\}$ and let $A \subseteq X$; $\phi \neq A \neq \{1\}$, then $A^b = \{2, 3, 4, 5, 6, 7\}$.

We can generalizations this example for any numbers of boundary points.

[3] In a space (X, I), if $\phi \neq A \subseteq X$, then $A^b = X$ because the only open set in I is X for every element in X and $X \cap A \neq \phi$ and $X \cap A^c \neq \phi$.

[4] In a space (X, D), if $A \subseteq X$, then $A^b = \phi$ because $\{x\} \in D$ for all $x \in X$ and by Remake (2) every point is not boundary.

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A^b \cap A^o = \phi$ and $A^b \cap A^x = \phi$

(2) $A^b = (A^c)^b$

(3) $(A \cup B)^b \subseteq A^b \cup B^b$

(4) $A \in \tau \Leftrightarrow A^b \subseteq A^c$ and $A^b \cap A = \phi$

(5) $A^c \in \tau \Leftrightarrow A^b \subseteq A$ and $A^b \cap A^c = \phi$

(6) $A, A^c \in \tau \Leftrightarrow A^b = \phi$

Proof :

(1) To prove $A^b \cap A^o = \phi$, suppose that $A^b \cap A^o \neq \phi$

$\Rightarrow \exists\, x \in A^b \cap A^o \Rightarrow x \in A^b \wedge x \in A^o$

$\Rightarrow \exists\, U \in \tau ; x \in U \subseteq A$ (def. of A^o)

$\Rightarrow U \cap A^c = \phi$ (since $U \subseteq A$)

$\Rightarrow x \notin A^b$ contradiction !!!

$\therefore A^b \cap A^o = \phi$

By similar way, to proof $A^b \cap A^x = \phi$, suppose that $A^b \cap A^x \neq \phi$

$\Rightarrow \exists\, x \in A^b \cap A^x \Rightarrow x \in A^b \wedge x \in A^x$

$\Rightarrow \exists\, U \in \tau ; x \in U \subseteq A^c$ (def. of A^x)

$\Rightarrow U \cap A = \phi$ (since $U \subseteq A^c$)

$\Rightarrow x \notin A^b$ contradiction !!!

$\therefore A^b \cap A^x = \phi$

(2) By definition of A^b, we have

$x \in A^b \Leftrightarrow \forall\, U \in \tau ; x \in U , U \cap A \neq \phi \wedge U \cap A^c \neq \phi$

$\Leftrightarrow \forall\, U \in \tau ; x \in U , U \cap (A^c)^c \neq \phi \wedge U \cap A^c \neq \phi$ (since $A = (A^c)^c$)

$\Leftrightarrow x \in (A^c)^b$

$\therefore A^b = (A^c)^b$

(3) To prove $(A \bigcup B)^b \subseteq A^b \bigcup B^b$

$x \in (A \bigcup B)^b \Rightarrow \forall\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap (A \bigcup B) \neq \phi\ \wedge\ U \bigcap (A \bigcup B)^c \neq \phi$

$\Rightarrow\ (U \bigcap A) \bigcup (U \bigcap B) \neq \phi\ \wedge\ U \bigcap (A^c \bigcap B^c) \neq \phi$

$\Rightarrow [(U \bigcap A) \bigcup (U \bigcap B) \neq \phi]\ \wedge\ [(U \bigcap A^c) \bigcap (U \bigcap B^c) \neq \phi]$

$\Rightarrow [U \bigcap A \neq \phi \vee U \bigcap B \neq \phi]\ \wedge\ [U \bigcap A^c \neq \phi \wedge U \bigcap B^c \neq \phi]$

$\Rightarrow [U \bigcap A \neq \phi \wedge U \bigcap A^c \neq \phi]\ \vee\ [U \bigcap B \neq \phi \wedge U \bigcap B^c \neq \phi]$

$\Rightarrow x \in A^b \vee\ x \in B^b$

$\Rightarrow x \in A^b \bigcup B^b$

$\therefore\ (A \bigcup B)^b \subseteq A^b \bigcup B^b$

(4) ($\Rightarrow$) Suppose that $A \in \tau$, to prove $A^b \subseteq A^c$ and $A^b \cap A = \phi$

Let $x \in A^b \Rightarrow \forall\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap A \neq \phi\ \wedge\ U \bigcap A^c \neq \phi$

$\Rightarrow$ i.e., every open set contain x intersect A and A^c

But, A is open (since $A \in \tau$) and $A \bigcap A^c = \phi$

$\Rightarrow x \notin A\ \ \Rightarrow x \in A^c\ \Rightarrow A^b \subseteq A^c$ and $A^b \cap A = \phi$.

($\Leftarrow$) Suppose that $A^b \subseteq A^c$, to prove A is open ($A \in \tau$)

To prove A is open, we must prove that A contains open nbhd for every point in A

Let $x \in A\ \Rightarrow x \notin A^c\ \Rightarrow x \notin A^b$ (since $A^b \subseteq A^c$)

$\Rightarrow \exists\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap A = \phi\ \vee\ U \bigcap A^c = \phi$

$\Rightarrow U \bigcap A \neq \phi$ (since $x \in U \wedge\ x \in A$)

$\Rightarrow U \bigcap A^c = \phi$

$\Rightarrow U \subseteq A$

$\Rightarrow\ A \in \tau$ (since A contains open nbhd for every point in A)

$\therefore$ A is open

(5) ($\Rightarrow$) Suppose that $A^c \in \tau$, to prove $A^b \subseteq A$

Let $x \in A^b \Rightarrow \forall\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap A \neq \phi\ \wedge\ U \bigcap A^c \neq \phi$ (def. of A^b)

Since A^c open set $\Rightarrow x \notin A^c$

$\because$ every open set contains x intersect A and A^c , then x cannot in A^c since A^c contains open nbhd for every point in A^c.

$\Rightarrow x \in A$ (since $X = A \bigcup A^c$)

$\therefore A^b \subseteq A$

($\Leftarrow$) Suppose that $A^b \subseteq A$, to prove A is closed ($A^c \in \tau$)

we will prove A^c open set i.e., A^c contains open nbhd for every point in A^c.

Let $x \in A^c\ \Rightarrow x \notin A\ \Rightarrow x \notin A^b$ (since $A^b \subseteq A$)

$\Rightarrow \exists\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap A = \phi$ (def. of boundary point and since $x \in A^c$)

$\Rightarrow U \subseteq A^c \neq \phi$ (since $X = A \bigcup A^c$)

So, A^c contains open nbhd for every point in A^c.

$\Rightarrow A^c \in \tau$ (i.e., A^c open set)

$\Rightarrow A$ closed set.

(6) ($\Rightarrow$) Suppose that $A, A^c \in \tau$, to prove $A^b = \phi$

$\because$ A open set $\Rightarrow A^b \subseteq A^c$ (By (4))

$\because$ A closed set $\Rightarrow A^b \subseteq A$ (By (5))

$\therefore A^b \subseteq A \bigcap A^c = \phi \Rightarrow A^b \subseteq \phi \Rightarrow A^b = \phi$

($\Leftarrow$) Suppose that $A^b = \phi$, to prove $A, A^c \in \tau$

$\because A^b = \phi$ and $\phi \subseteq A$ and $\phi \subseteq A^c$

$\Rightarrow A^b \subseteq A \Rightarrow A^c \in \tau$ (By (5))

$A^b \subseteq A^c \Rightarrow A \in \tau$ (By (4))

$\Rightarrow A, A^c \in \tau$ (i.e., A is closed and open))

Remarks :

[1] Notes that : $X = A^o \bigcup A^x \bigcup A^b$ and $\phi = A^o \bigcap A^x \bigcap A^b$, this means the sets A^o, A^x, A^b, being a partition for X, also if $x \in X$, then $x \in A^o$ or $x \in A^x$ or $x \in A^b$.

[2] The set A^b is closed set since $A^b = X \setminus (A^o \bigcup A^x)$ and we know that the sets A^o and A^x are open sets, therefore $A^o \bigcup A^x$ is open set, so $X \setminus (A^o \bigcup A^x)$ is closed set, hence A^b closed set.

<u>Example :</u> Let X = {1, 2, 3, 4, 5} and τ be a topology on X and $A \subseteq X$ such that A^o = {1} and A^x = {2, 3}. Find A^b.

<u>Solution :</u> using previous remark $A^b = X \setminus (A^o \bigcup A^x)$

$A^b = \{1, 2, 3, 4, 5\} \setminus (\{1\} \bigcup \{2, 3\}) = \{4, 5\}$

Notes that we find A^b thought we unknown the topology τ.

<u>Definition :</u> Derived set

Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called a **<u>cluster point</u>** (or **<u>accumulation point</u> or <u>Limit point</u>)** of A iff every open set containing x contains at least one point of A different from x. The set of all cluster points of A is called the **<u>derived set</u>** of A and is denoted by $A^/$. i.e.,

$$A^/ = \{x \in X : \forall\, U \in \tau \,;\, x \in U \ \wedge\ U \setminus \{x\} \bigcap A \neq \phi\}$$

$$\text{or} \quad x \in A^/ \Leftrightarrow \forall\, U \in \tau \,;\, x \in U \ \wedge\ U \setminus \{x\} \bigcap A \neq \phi$$

if $x \notin A^/$, we define

$$x \notin A^/ \Leftrightarrow \exists\, U \in \tau \,;\, x \in U \ \wedge\ U \setminus \{x\} \bigcap A = \phi$$

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$, $A = \{b, c\}$, $B = \{c\}$, $C = \{a, b\}$ and $D = \{a\}$. Find $A^/$, $B^/$, $C^/$ and $D^/$.

Solution : $A^/ = ?$

To find the cluster set of any set must choose every open sets for every point in X and notes satisfy the definition or not.

$a \in X$ and the open sets contain a are $X, \{a\}, \{a, b\}, \{a, c\}$

$b \in X$ and the open sets contain b are $X, \{a, b\}$

$c \in X$ and the open sets contain c are $X, \{a, c\}$

notes that : $\{a\} \setminus \{a\} \bigcap A = \phi \bigcap A = \phi \Rightarrow a \notin A^/$.

notes that : $\{a, b\} \setminus \{b\} \bigcap A = \{a\} \bigcap A = \phi \Rightarrow b \notin A^/$.

notes that : $\{a, c\} \setminus \{c\} \bigcap A = \{a\} \bigcap A = \phi \Rightarrow c \notin A^/$.

Therefore $A^/ = \phi$

By the similar way compute the other sets such that

$B^/ = \phi$, $C^/ = \{b, c\}$, $D^/ = \{b, c\}$.

Remarks :

[1] If $\{x\} \in \tau$ in any topological space (X, τ), then $x \notin A^/$ for any subset $A \subseteq X$. Since $\{x\} \in \tau$ this means $\{x\}$ is open set of X and $\{x\} \setminus \{x\} \bigcap A = \phi \bigcap A = \phi$ so the definition not satisfy (in the previous example take the element a).

[2] If $A = \{a\}$ singleton set, then $a \notin A^/$ since $U \setminus \{a\} \bigcap A = \phi$ (in the previous example take the set B).

[3] Notes that $A^/ \not\subset A$ and $A \not\subset A^/$ and sometime $A^/ \bigcap A = \phi$ or $A^/ \bigcap A \neq \phi$ (in the previous example notes that $C^/ \not\subset C$ and $C \not\subset C^/$ and $D^/ \bigcap D = \phi$).

[4] In a space (X, I), if $A \neq \phi$ and A contains more than one element, then $A^/ = X$ because the only open set in I is X for every element in X and $X \setminus \{x\} \bigcap A \neq \phi$.

[5] In a space (X, D), if $A \subseteq X$, then $A^/ = \phi$ because $\{x\} \in D$ for every $x \in X$ and by Remake (1) every point is not cluster point for any set.

[6] In any topological space (X, τ), we have $\phi^/ = \phi$ since for every open set U any for every element x is $U \setminus \{x\} \bigcap \phi = \phi$.

[7] The derived set of X is change by change the topology may be $X^/ = \phi$ (remake (5)) or may be $X^/ \neq \phi$ or $X^/ \neq X$. In the previous example $X^/ = \{b, c\}$).

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A \subseteq B \Rightarrow A^/ \subseteq B^/$ (In general the converse is not true)

(2) $(A \bigcup B)^/ = A^/ \bigcup B^/$

(3) $(A \bigcap B)^/ \subseteq A^/ \bigcap B^/$ (In general the equality is not true)

(4) $A^c \in \tau \;\Leftrightarrow\; A' \subseteq A$

or A is closed $\Leftrightarrow A' \subseteq A$.

Proof :

(1) $x \in A' \Rightarrow \forall\, U \in \tau\,;\, x \in U \;\wedge\; U \setminus \{x\} \bigcap A \neq \phi$

$\Rightarrow \forall\, U \in \tau\,;\, x \in U \;\wedge\; U \setminus \{x\} \bigcap B \neq \phi$ (since $A \subseteq B$)

$\Rightarrow x \in B'$ (def. of cluster point)

$\therefore\; A' \subseteq B'$

(2) To prove $(A \bigcup B)' = A' \bigcup B'$

$A \subseteq A \bigcup B$ (def. of $\bigcup$) $\Rightarrow A' \subseteq (A \bigcup B)'$ (By (1))

$B \subseteq A \bigcup B$ (def. of $\bigcup$) $\Rightarrow B' \subseteq (A \bigcup B)'$ (By (1))

$\Rightarrow\; A' \bigcup B' \subseteq (A \bigcup B)'$ ---------------(1)

Let $x \notin A' \bigcup B' \Rightarrow x \notin A' \wedge x \notin B'$

$\Rightarrow \exists\, U \in \tau\,;\, x \in U \wedge U \setminus \{x\} \bigcap A = \phi \;\wedge\; \exists\, V \in \tau\,;\, x \in V \wedge V \setminus \{x\} \bigcap B = \phi$

(def. of cluster point)

$\Rightarrow U \bigcap V \in \tau\,;\, x \in U \bigcap V \wedge (U \bigcap V) \setminus \{x\} \bigcap (A \bigcup B) = \phi$

$\Rightarrow x \notin (A \bigcup B)'$

$(A \bigcup B)' \subseteq A' \bigcup B'$ ---------------(2)

From (1) and (2), we have $(A \bigcup B)' = A' \bigcup B'$

(3) To prove $(A \bigcap B)' \subseteq A' \bigcap B'$

Let $x \in (A \bigcap B)' \Rightarrow \forall\, U \in \tau\,;\, x \in U \wedge U \setminus \{x\} \bigcap (A \bigcap B) \neq \phi$

(def. of cluster point)

$\Rightarrow \forall\, U \in \tau\,;\, x \in U \wedge [(U \setminus \{x\} \bigcap A) \bigcap (U \setminus \{x\} \bigcap B)] \neq \phi$

($\bigcap$ distribution on $\bigcap$)

$\Rightarrow \forall\, U \in \tau\,;\, x \in U \wedge (U \setminus \{x\} \bigcap A) \neq \phi \wedge (U \setminus \{x\} \bigcap B) \neq \phi$

$\Rightarrow x \in A' \wedge x \in B'$ (def. of cluster point)

$\Rightarrow x \in A' \bigcap B'$

$\therefore\; (A \bigcap B)' \subseteq A' \bigcap B'$

(4) ($\Rightarrow$) Suppose that $A^c \in \tau$, to prove $A' \subseteq A$

To prove $A' \subseteq A$, we must prove that $A^c \subseteq (A')^c$

Let $x \notin A \Rightarrow x \in A^c$

$\Rightarrow\; \exists\, U \in \tau\,;\, x \in U \;\wedge\; U \subseteq A^c$ (def. of open set and $A^c \in \tau$)

$\Rightarrow U \subseteq A^c \;\Rightarrow U \bigcap A = \phi$

$\Rightarrow U \setminus \{x\} \bigcap A = \phi$ (since $x \notin A$)

$\Rightarrow x \notin A'$ (def. of cluster point)

($\Leftarrow$) Suppose that $A' \subseteq A$, to prove A is closed, i.e., A^c is open

Let $x \in A^c \Rightarrow x \notin A$ (def. of complement)

$\Rightarrow x \notin A^/$ (since $A^/ \subseteq A$)

$\Rightarrow \exists\, U \in \tau \,;\, x \in U \;\wedge\; U \setminus \{x\} \bigcap A = \phi$

$\Rightarrow U \setminus \{x\} \subseteq A^c \;\wedge x \in A^c$

$\Rightarrow U \subseteq A^c$

$\Rightarrow A^c \in \tau$

$\therefore$ A is closed

Remarks :

[1] The converse of property (1) is not true in general for example :

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, $A = \{a\}$, $B = \{b\}$.

Notes that, $A^/ = \{b, c\}$ and $B^/ = \{c\}$, so $B^/ \subseteq A^/$, but $B \not\subset A$.

[2] The equality of property (3) is not true in general i.e., $A^/ \bigcap B^/ \not\subset (A \bigcap B)^/$ for example :

Example : In the previous example notes that

$A \bigcap B = \{a\} \bigcap \{b\} = \phi \quad \Rightarrow \quad (A \bigcap B)^/ = \phi^/ = \phi$

But, $A^/ \bigcap B^/ = \{b, c\} \bigcap \{c\} = \{c\}$

$\therefore A^/ \bigcap B^/ \not\subset (A \bigcap B)^/$

Definition : Closure of a set

Let (X, τ) be a topological space and $A \subseteq X$. The **closure** of a set A is $A \bigcup A^/$ and is denoted by $\overline{A}$ or Cl(A). i.e.,

$$\overline{A} = A \bigcup A^/$$

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$, $A = \{a, c\}$. Find $\overline{A}$.

Solution :

$\overline{A} = ?$

To find the closure set of A we must find $A^/$.

$a \in X$ and the open sets contain a are X, {a, b},

$b \in X$ and the open sets contain b are X, {a, b}

$c \in X$ and the open set contain c is X

notes that : $\{a, b\} \setminus \{a\} \bigcap A = \{b\} \bigcap A = \phi \;\Rightarrow\; a \notin A^/$.

notes that : $\{a, b\} \setminus \{b\} \bigcap A = \{a\} \bigcap A \neq \phi$ and

$X \setminus \{b\} \bigcap A = \{a, c\} \bigcap A \neq \phi \;\Rightarrow\; b \in A^/$.

notes that : $X \setminus \{c\} \bigcap A = \{a, b\} \bigcap A \neq \phi \;\Rightarrow\; c \in A^/$.

Therefore $A^/ = \{b , c\}$

$\therefore \overline{A} = A \bigcup A^/ = \{a, c\} \bigcup \{b, c\} = \{a, b, c\} = X$

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A \subseteq \overline{A}$

(2) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$ (In general the converse is not true)

(3) $\overline{A} = \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$ (i.e., $\overline{A}$ is smallest closed set contains A)

(4) $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$

(5) $\overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$ (In general the equality is not true)

(6) $A^c \in \tau$ (i.e., A is closed) $\Leftrightarrow \overline{A} = A$

(7) $\overline{\overline{A}} = \overline{A}$

Proof :

(1) $\because \overline{A} = A \bigcup A'$ (def. of $\overline{A}$) $\Rightarrow A \subseteq \overline{A}$

(2) Suppose that $A \subseteq B$, to prove $\overline{A} \subseteq \overline{B}$

$\because A \subseteq B \Rightarrow A' \subseteq B'$ (property of cluster set)

$\Rightarrow A \subseteq B$ and $A' \subseteq B'$

$\Rightarrow A \bigcup A' \subseteq B' \bigcup B$

$\Rightarrow \overline{A} \subseteq \overline{B}$ (def. of $\overline{A}$)

(3) To prove, $\overline{A} = \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

First we prove, $\overline{A} \subseteq \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

Let $x \in \overline{A} \Rightarrow x \in A \bigcup A'$ (def. of $\overline{A}$)

$\Rightarrow x \in A \vee x \in A'$

if $x \in A \Rightarrow x \in A \subseteq F \Rightarrow x \in F \;\forall F \subseteq X ; F^c \in \tau$

$\Rightarrow x \in \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

if $x \in A'$

suppose $x \notin \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

$\exists F \in \mathscr{F} ; x \notin F$

$\Rightarrow x \in F^c = U$ open set containing x.

$\because x \in A' \Rightarrow A \bigcap F^c \setminus \{x\} \neq \phi$

$\Rightarrow A \bigcap F^c \neq \phi$

but $A \subseteq F \Rightarrow A \bigcap F^c = \phi$ C!! contradiction.

$\therefore x \in \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

$\therefore \overline{A} \subseteq \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$ ---------------(1)

Second we prove, $\bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\} \subseteq \overline{A}$

Let $x \in \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$ and suppose $x \notin \overline{A}$

$\Rightarrow x \notin A \bigcup A'$ (since $\overline{A} = A \bigcup A'$)

$\Rightarrow x \notin A \wedge x \notin A'$

$\Rightarrow \exists\, U \in \tau\, ;\, x \in U \;\wedge\; U \setminus \{x\} \bigcap A = \phi$

$\Rightarrow U \bigcap A = \phi$ (since $x \notin A$)

$\Rightarrow A \subseteq U^c$ (since $U \bigcap A = \phi$)

But U^c is closed (since U open)

$\Rightarrow x \in U^c$ (since $x \in \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$) (say $U^c = F$)

$\Rightarrow x \in U$ and $x \in U^c$ contradiction !!!

$\Rightarrow x \in \overline{A}$

$\therefore \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\} \subseteq \overline{A}$ ---------------(2)

From (1) and (2), we have $\overline{A} = \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

(4) To prove $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$

First we prove, $\overline{A \bigcup B} \subseteq \overline{A} \bigcup \overline{B}$

From (1), $A \subseteq \overline{A}$ and $B \subseteq \overline{B} \Rightarrow A \bigcup B \subseteq \overline{A} \bigcup \overline{B}$

From (5), $\overline{A}$, $\overline{B}$ are closed sets

$\therefore \overline{A} \bigcup \overline{B}$ is closed set contain $A \bigcup B$ (i.e., $A \bigcup B \subseteq \overline{A} \bigcup \overline{B}$)

but $\overline{A \bigcup B}$ is smallest closed set contain $A \bigcup B$ (i.e., $A \bigcup B \subseteq \overline{A \bigcup B}$).

$\Rightarrow \overline{A \bigcup B} \subseteq \overline{A} \bigcup \overline{B}$ -------------(1)

Now, we prove, $\overline{A} \bigcup \overline{B} \subseteq \overline{A \bigcup B}$

$\because A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$ (def. of $\bigcup$)

From (2), $\overline{A} \subseteq \overline{A \bigcup B}$ and $\overline{B} \subseteq \overline{A \bigcup B}$

$\Rightarrow \overline{A} \bigcup \overline{B} \subseteq \overline{A \bigcup B}$ --------------(2)

From (1) and (2), we have $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$

(5) To prove, $\overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$

$\because A \bigcap B \subseteq A$ and $A \bigcap B \subseteq B$ (def. of $\bigcap$)

From (2), $\overline{A \bigcap B} \subseteq \overline{A}$ and $\overline{A \bigcap B} \subseteq \overline{B}$

$\Rightarrow \overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$

(6) To prove, $A^c \in \tau \Leftrightarrow \overline{A} = A$

Suppose that $A^c \in \tau$, to prove $\overline{A} = A$

$A \subseteq \overline{A}$ (from (1)) ---------------(1)

$\because A^c \in \tau \Rightarrow A$ is closed and also $A \subseteq A$ and $A' \subseteq A$ (by theorem)

$\Rightarrow A \bigcup A' \subseteq A$

$\Rightarrow \overline{A} \subseteq A$ ---------------(2)

Hence, from (1) and (2) we have $\overline{A} = A$

Suppose that $\overline{A} = A$, to prove $A^c \in \tau$ (i.e., to prove A is closed set)

$\because \overline{A} = A$ and $\overline{A}$ is closed $\Rightarrow$ A is closed $\Rightarrow A^c \in \tau$.

(7) To prove $\overline{\overline{A}} = \overline{A}$

A is closed $\Leftrightarrow$ $\overline{A} = A$ (by (6))

$\overline{A}$ is closed $\Leftrightarrow$ $\overline{A} = \overline{\overline{A}}$ (by (6))

Remarks :

[1] We can using property (5) to find closure set for any set in topological space instead of definition of closure set such that $\overline{A}$ is smallest closed set contains A.

[2] From property (6) and since X, ϕ are closed sets then $\overline{X} = X$ and $\overline{\phi} = \phi$.

[3] The converse of property (2) is not true in general for example :

<u>**Example :**</u> Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $A = \{b, c\}$ and $B = \{a, b\}$.

Notes that $\mathscr{F} = \{X, \phi, \{b, c\}, \{c\}\}$, then

$\overline{A} = \{b, c\} = A$ and $\overline{B} = X$

$\Rightarrow$ $\overline{A} \subseteq \overline{B}$ but $A \not\subset B$.

[4] The equality of property (4) is not true in general i.e., $\overline{A} \cap \overline{B} \not\subset \overline{A \cap B}$ for example :

<u>**Example :**</u> In the usual topology $(\mathbb{R}, \tau_u)$, let $A = [1, 2]$ and $B = (2, 3)$

Clear, $A \cap B = \phi$ $\Rightarrow$ $\overline{A \cap B} = \overline{\phi} = \phi$ $\Rightarrow$ $\overline{A \cap B} = \phi$

But, $\overline{A} \cap \overline{B} = [1, 2] \cap [2, 3] = \{2\}$

$\overline{A} \cap \overline{B} \not\subset \overline{A \cap B}$. Hence, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

[5] In a space (X, I), if $\phi \neq A \subseteq X$, then $\overline{A} = X$ since $\overline{A}$ is closed set contain A and the only closed set in I contain A is X.

[6] In a space (X, D), every subsets of X is open and closed in the sometime, then $\overline{A} = A$ for all $A \subseteq X$ (by property (6)).

[7] In the usual topological space $(\mathbb{R}, \tau_u)$, if A is closed interval or open interval or half closed (open) as follow : $A = [a, b]$ or $A = (a, b)$ or $A = [a, b)$ or $A = (a, b]$, then $\overline{A} = [a, b]$ since $[a, b]$ is smallest closed set contains A.

If A is a discreet set in real number (finite or infinit), then $\overline{A} = A$ since A is closed set for example :

$A = \mathbb{N}$, $A = O$, $A = \mathbb{P}$, $A = E$, $A = \{1, 2, 3\}$, $A = \{-\sqrt{2}, 0, \sqrt{2}\}$

Either the rational numbers $\mathbb{Q}$ and irrational numbers, then $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{Irr} = \mathbb{R}$ since the only closed set in τ_u contain $\mathbb{Q}$ and Irr is $\mathbb{R}$.

[8] In the topological space $(\mathbb{N}, \tau_{cof})$, if A is finite set, then A is closed (def. of τ_{cof}), so $\overline{A} = A$.

If A is infinite, then $\overline{A} = \mathbb{N}$ since $\overline{A}$ closed set contain A and the only closed set in τ_{cof} contain A is $\mathbb{N}$.

Definition : **Dense set**

Let (X, τ) be a topological space. Then $A \subseteq X$ is called **dense** set in X iff $\overline{A} = X$.

Examples :

[1] In the usual topological space $(\mathbb{R}, \tau_u)$, the rational numbers $\mathbb{Q}$ and irrational numbers are dense in $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\text{Irr}} = \mathbb{R}$.

[2] In the cofinite topological space $(\mathbb{N}, \tau_{cof})$, every infinite set is dense in $\mathbb{N}$, for example if $A = \{5, 10, 15, \dots\}$, then $\overline{A} = \mathbb{N}$.

[3] In a space (X, I), every nonempty subset of X is dense.

[4] In a space (X, D), the only dense set is X.

[5] In every topological space (X, τ) is X dense set always. So, every topological space contain at least one dense set.

Topological Space Generated by Metric Space

Definition : **Metric & Metric Space**

Let $X \neq \phi$ a function $d : X \times X \to \mathbb{R}$ is called a **metric** on X if :

(1) $d(x, y) \geqslant 0 \ \forall x, y \in X$

(2) $d(x, y) = d(y, x) \quad \forall x, y \in X$

(3) $d(x, y) = 0 \Leftrightarrow x = y \ \forall x, y \in X$

(4) $d(x, y) + d(y, z) \geqslant d(x, z) \ \forall x, y, z \in X$

The pair (X, d) is called a **metric space**.

Definition : **Open Ball**

Let (X, d) be a metric space and let $x \in X$, $\varepsilon > 0$, the set

$$B_\varepsilon(x) = \{y \in X; d(y, x) < \varepsilon\}$$

is called an **open ball** in X with center x and radius ε.

Definition : **Open Set in Metric Space**

Let (X, d) be a metric space and let $U \subseteq X$, U is said to be **open** in (X, d) if $\forall x \in U \ \exists \ \varepsilon > 0 \ ; B_\varepsilon(x) \subseteq U$.

Proposition : Let (X, d) be a metric space and $U \subseteq X$, U is open in X iff U is the union of open balls.

Proposition : Let (X, d) be a metric space and let τ_d be the family of all open sets in (X, d). i.e., $\tau_d = \{U \subseteq X;\ U \text{ is open in } (X, d)\}$. Then τ_d is a topological space on X.

Proof :

(1) $\forall\, x \in X\ \exists\, \varepsilon > 0$, $B_\varepsilon(x) \subseteq X \Rightarrow X \in \tau_d$.

$\phi \in \tau_d$ since $\nexists\, x \in \phi$.

(2) Let $U, V \in \tau_d$, to prove $U \bigcap V \in \tau_d$.

Let $x \in U \bigcap V \Rightarrow x \in U \wedge x \in V$

$\Rightarrow \exists\, \varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $B_{\varepsilon 1}(x) \subseteq U \wedge B_{\varepsilon 2}(x) \subseteq V$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

$\Rightarrow B_\varepsilon(x) \subseteq B_{\varepsilon 1}(x) \bigcap B_{\varepsilon 2}(x) \subseteq U \bigcap V$.

$\Rightarrow U \bigcap V \in \tau_d$.

(3) Let $U_\alpha \in \tau_d\ \ \forall\, \alpha \in \Lambda$, to prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_d$

Let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow \exists\, \alpha_0 \in \Lambda$; $x \in U_{\alpha 0}$

$\Rightarrow \exists\, \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U_{\alpha 0}$.

but $U_{\alpha 0} \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

$\Rightarrow B_\varepsilon(x) \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$.

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_d$.

So τ_d is a topology on X induced by d.

Example : Let $X = \mathbb{R}$ and $d = |\ \ |$, then $(X, d) = (\mathbb{R}, |\ \ |)$ is a metric space.

Now, let $x \in X$, $\varepsilon > 0$ then

$$\begin{aligned} B_\varepsilon(x) &= \{y \in \mathbb{R}\ ;\ |\, y - x\,| < \varepsilon\} \\ &= \{y \in \mathbb{R}\ ;\ -\varepsilon < y - x < \varepsilon\} \\ &= \{y \in \mathbb{R}\ ; x - \varepsilon < y < x + \varepsilon\} \\ &= (x - \varepsilon, x + \varepsilon) \text{ open interval} \end{aligned}$$

So the open balls here is an open interval, and hence the open sets is the union of open intervals.

i.e., $\tau_d = \{U \subseteq \mathbb{R}\ ;\ U = \text{union of open intervals}\} = \tau_u$

We shall denote this topology by τ_u = the usual topology on $\mathbb{R}$ = the set of real.

Note that $\mathbb{R} \in \tau_u$ and $\mathbb{R} = (-\infty, \infty)$ which is an open interval and $\phi = (a, a)$; $a \in \mathbb{R}$.

Example : Which of the following subsets of $\mathbb{R}$ is open (closed) in $(\mathbb{R},\tau_u)$??
$(-1, 1)$, $(0, 1) \bigcup (10, 20)$, $\mathbb{N}$, $[2, 3]$, $[-1/2, 3)$, $\mathbb{Q}$, Irr, $\{3, 4, 5\}$.

Solution :
$(-1, 1)$ and $(0, 1) \bigcup (10, 20)$ are open but not closed.
$\mathbb{N}$ is not open , but closed
$[2, 3]$ and $\{3, 4, 5\}$ are closed but not open.
$[-1/2, 3)$, $\mathbb{Q}$ and Irr are not open and not closed.

Remark : We can get a topological space from any metric space, but we cannot get a metric space from any topological space.

Definition : (Metrizable Space)
The topological space (X, τ) is called **Metrizable** iff there exists a metric d for X such that the topology τ_d induced by τ (i.e, $\tau = \tau_d$). Otherwise, X is said to be nonmeterizable.

Remark : (X, D) is a metrizable topological space.
i.e., There is a metric d on X such that $\tau_d = D$.
where $d : X\times X \to \mathbb{R}$; $d(x, y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$
$B_r(x) = \{x\}$ if $r < 1$
$\therefore \ \{x\} \in \tau_d \ \ \forall x \in X \Rightarrow \tau_d = D$.

Example : If $X = \{1, 2, 3\}$, τ topology on X, (X, τ) is metrizable $\Rightarrow \tau = D$.
Suppose that $\exists\ d : X\times X \to \mathbb{R}$; $\tau_d = \tau$
$\Rightarrow d(1, 1) = d(2, 2) = d(3, 3) = 0$
$d(1, 2) = d(2, 1) = C_1$
$d(1, 3) = d(3, 1) = C_2$
$d(2, 3) = d(3, 2) = C_3$
$B_\varepsilon(1) = \{1\}$ if $\varepsilon < \min\{C_1, C_2\}$
$B_\varepsilon(2) = \{2\}$ if $\varepsilon < \min\{C_1, C_3\}$
$B_\varepsilon(3) = \{3\}$ if $\varepsilon < \min\{C_2, C_3\}$
$\therefore\ \tau = D$
Therefore, every topology (τ) on X not discrete (D) is space not generated by metric.