Chapter Two : Continuity and Derived Topological Spaces

Definition : Continuous & discontinuous Functions

Let (X, τ) and (Y, τ') be two topological spaces and $f : (X, \tau) \to (Y, \tau')$. The function f is called **continuous** if the inverse image for any open set in Y is an open set in X. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is continuous } \Leftrightarrow f^{-1}(V) \in \tau \quad \forall V \in \tau'$$

and the function f is called **discontinuous** if there exist an open set in Y, but inverse image is not open in X. i.e.,

$$f \text{ is discontinuous } \Leftrightarrow \exists V \in \tau' \wedge f^{-1}(V) \notin \tau$$

Example : Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi, \{b\}\}$

(1) Define $f : (X, \tau) \to (Y, \tau')$; $f(1) = b, f(2) = f(3) = a$. Is f continuous??

The open sets in Y are $Y, \phi, \{b\}$. Now take the inverse image of this sets.

$Y \in \tau' \Rightarrow f^{-1}(Y) = X \in \tau$ the set of all element in X its image in Y

$\phi \in \tau' \Rightarrow f^{-1}(\phi) = \phi \in \tau$ the set of all element in ϕ its image in ϕ

$\{b\} \in \tau' \Rightarrow f^{-1}(\{b\}) = \{x \in X ; f(x) = b\} = \{1\} \in \tau$

the set of all element in X its image in $\{b\}$

Therefore, the inverse image of every element in τ' is element in τ, hence f is continuous.

(2) Define $g : (X, \tau) \to (Y, \tau')$; $g(1) = a, g(2) = g(3) = b$. Is g continuous??

$Y \in \tau' \Rightarrow g^{-1}(Y) = X \in \tau$

$\phi \in \tau' \Rightarrow g^{-1}(\phi) = \phi \in \tau$

$\{b\} \in \tau' \Rightarrow g^{-1}(\{b\}) = \{2, 3\} \notin \tau$

Therefore, f is discontinuous.

(3) Define $h : (X, \tau) \to (Y, \tau')$; $h(1) = h(2) = h(3) = a$. Is h continuous??

$Y \in \tau' \Rightarrow h^{-1}(Y) = X \in \tau$

$\phi \in \tau' \Rightarrow h^{-1}(\phi) = \phi \in \tau$

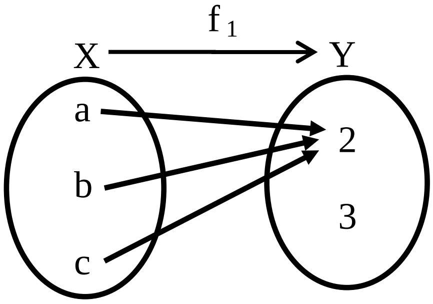
$\{b\} \in \tau' \Rightarrow h^{-1}(\{b\}) = \phi \in \tau$ since there is no element its image is b

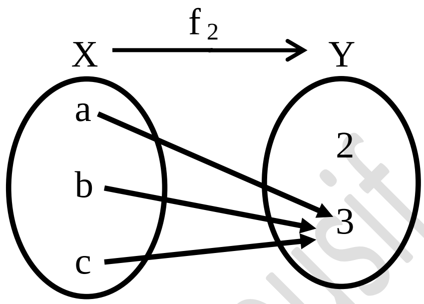
Therefore, f is continuous.

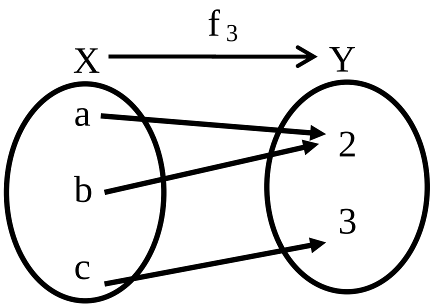
Remark : Always the inverse image of Y is X and the inverse image of ϕ is ϕ.

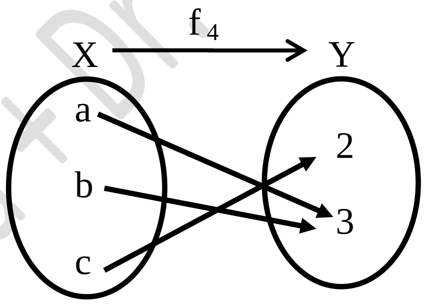
Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}\}$, $Y = \{2, 3\}$ and $\tau' = \{Y, \phi, \{2\}\}$. Find all continuous function define from (X, τ) to (Y, τ').

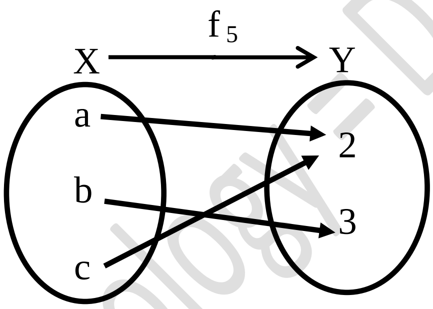
Solution : There are $2^3 = 8$ from difference functions from X to Y which are some of them continuous and some others of them discontinuous . Now we introduce the figure for all functions from X to Y and discuses there continuous.

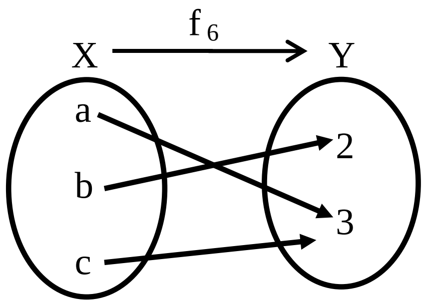


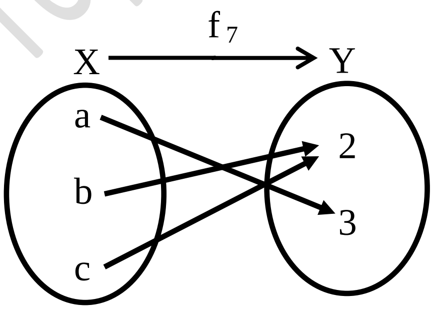


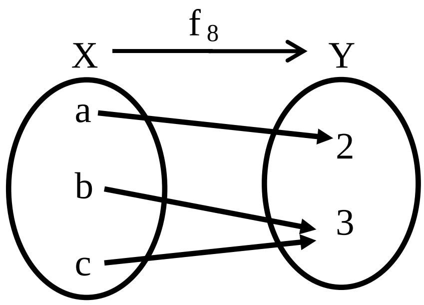












From remark above $f_i^{-1}(Y) = X$ and $f_i^{-1}(\phi) = \phi$, i = 1, 2, 3, 4, 5, 6, 7, 8.
f_1 is continuous, since $f_1^{-1}(\{2\}) = \{a, b, c\} = X \in \tau$.
f_2 is continuous, since $f_2^{-1}(\{2\}) = \phi \in \tau$.
f_3 is discontinuous, since $f_3^{-1}(\{2\}) = \{a, b\} \notin \tau$.
f_4 is discontinuous, since $f_4^{-1}(\{2\}) = \{c\} \notin \tau$.
f_5 is discontinuous, since $f_5^{-1}(\{2\}) = \{a, c\} \notin \tau$.
f_6 is discontinuous, since $f_6^{-1}(\{2\}) = \{b\} \notin \tau$.
f_7 is continuous, since $f_7^{-1}(\{2\}) = \{b, c\} \in \tau$.
f_8 is discontinuous, since $f_8^{-1}(\{2\}) = \{a\} \notin \tau$.
Therefore, the continuous functions in this example are f_1, f_2, f_7 only.

Remark : There are special cases of continuous functions.

[1] Every constant function from a space (X, τ) to a space (Y, τ') is continuous. i.e.,

$$f : (X, \tau) \to (Y, \tau') ; f(x) = c \quad \forall x \in X \text{ and } c = \text{constant in } Y.$$

To show that f is continuous.

Let $V \in \tau' \Rightarrow V$ is open in Y, then

$$f^{-1}(V) = \begin{cases} X & \text{if} \quad c \in V \\ \emptyset & \text{if} \quad c \notin V \end{cases}$$

$\Rightarrow X, \phi \in \tau \Rightarrow$ f is continuous.

[2] If $\tau' = I$, then the function $f : (X, \tau) \to (Y, I)$ is continuous for any set Y and any topological space (X, τ). i.e., $I = \{Y, \phi\}$ and $f^{-1}(Y) = X \in \tau$, $f^{-1}(\phi) = \phi \in \tau$.

Special case : $f : (X, I) \to (Y, I)$ is continues

And the function $f : (X, I) \to (Y, \tau')$; $\tau' \neq I$ is discontinuous in general for example :

$$f : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u) \quad ; \quad f(x) = x$$

f is discontinuous since $(0, 1) \in \tau_u$ and $f^{-1}((0, 1)) = (0, 1) \notin I = \{\mathbb{R}, \phi\}$.

[3] If $\tau = D$, then the function $f : (X, D) \to (Y, \tau')$ is continuous for any set X and any topological space (Y, τ') and for any function f since, if $V \in \tau'$, then $f^{-1}(V) \subseteq X$ this means $f^{-1}(V) \in IP(X)$, but $D = IP(X)$ and this implies $f^{-1}(V) \in D$. Therefore f is continuous.

Special case : $f : (X, D) \to (Y, D)$ is continuous

And the function $f : (X, \tau) \to (Y, D)$; $\tau \neq D$ is discontinuous in general for example :

$$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, D) \quad ; \quad f(x) = x$$

f is discontinuous since $\{1\} \in D$ and $f^{-1}(\{1\}) = \{1\} \notin \tau_u$.

Notes that the function $f : (X, D) \rightarrow (Y, I)$ is continuous always for any set X and any set Y since its add the remark [2] and [3] such that $\tau = D$ and $\tau' = I$.

[4] Every identity function from a spaces to the same space is continuous. i.e.,

$$f : (X, \tau) \rightarrow (X, \tau) \quad ; \quad f(x) = x \quad \forall\, x \in X$$

is continuous function since $f^{-1}(V) = V$ for any open set V in (X, τ) and this implies $f^{-1}(V)$ is open in (X, τ)

notes that the identity function from a space to another space may be continuous and its clear in example in remark [2] and [3] above.

Theorem : If $f : (X, \tau) \rightarrow (Y, \tau')$ and $g : (Y, \tau') \rightarrow (Z, \tau'')$ are both continuous functions, then the composition $gof : (X, \tau) \rightarrow (Z, \tau'')$ is continuous.

Proof :

Let $W \in \tau'' \Rightarrow g^{-1}(W) \in \tau'$ (since g is continuous)

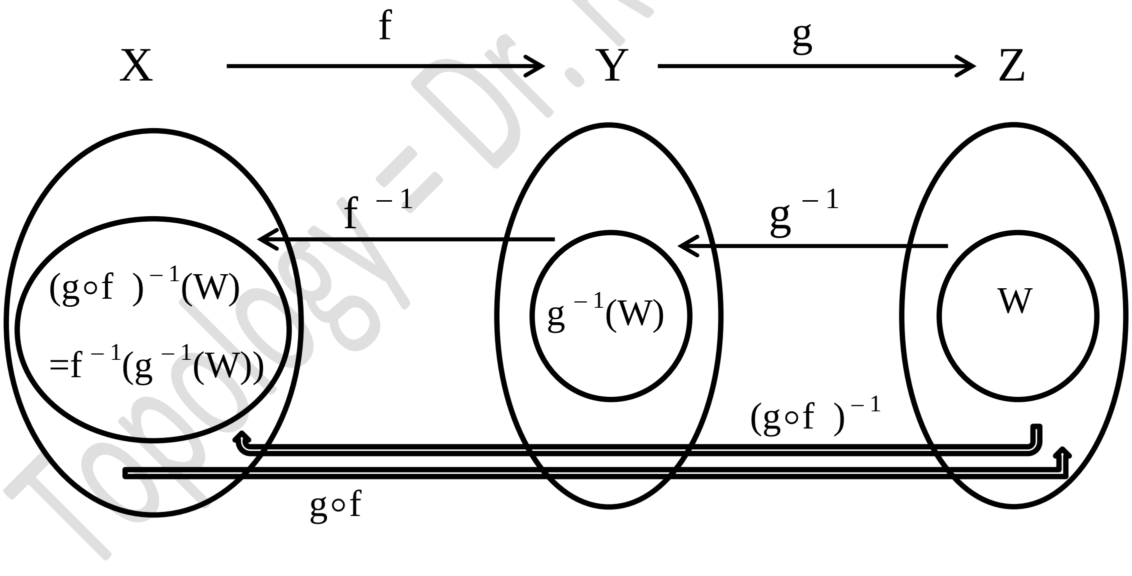
notes that $g^{-1}(w) \subseteq Y$

$\Rightarrow f^{-1}(g^{-1}(W)) \in \tau$ (since f is continuous)

$\Rightarrow (f^{-1}og^{-1})(W) \in \tau$ (by composition of function)

$\Rightarrow (gof)^{-1}(W) \in \tau$ (since $(gof)^{-1} = f^{-1}og^{-1}$)

$\therefore$ gof is continuous. The figure below clear this theorem.



Remake : The composition of finite number of continuous functions is continuous. i.e., the composition of three or five or hundred continuous functions is continuous. For example if f , g, h, k are continuous, then kohogof is continuous …. etc.

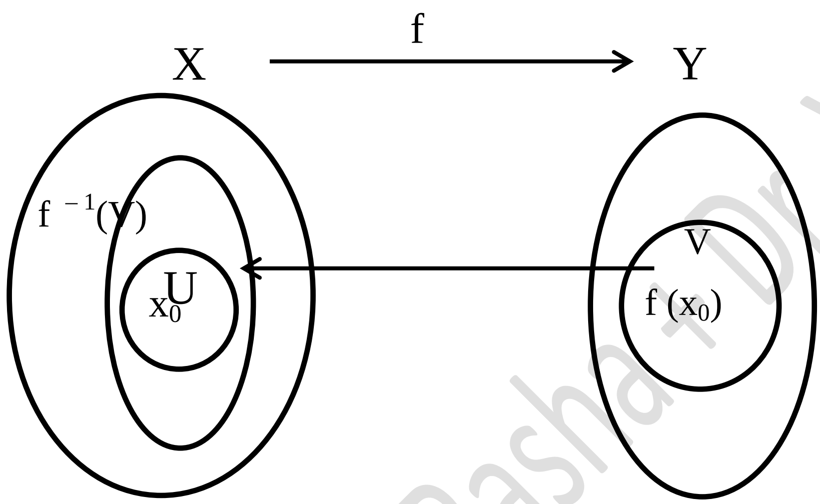
Now we introduce the definition of continuous function at a point :

<u>Definition :</u> Continuous at a Point

Let (X, τ) and (Y, τ') be topological spaces and $f : (X, \tau) \to (Y, \tau')$. the function f is called **<u>continuous at a point</u>** $x_0 \in X$ if the inverse image for any open nbd for $f(x_0)$ in Y contains an open nbd for x_0 in X. i.e.,

$$f \text{ is continuous at } x_0 \in X \Leftrightarrow \forall\, V \in \tau' \,;\, f(x_0) \in V\ \exists\, U \in \tau \,;\, x_0 \in U \wedge U \subseteq f^{-1}(V)$$

The following figure clear this definition :



Such that V is an open nbd for $f(x_0)$ in Y and $f^{-1}(V)$ is inverse image for V and U is an open nbd for x_0 in X contains in $f^{-1}(V)$.

<u>Remark :</u> If f is continuous function. Then its continuous at every point in the domain. Also, if f is continuous at every point in the domain, then its continuous.

<u>Notes that</u>, if f is continuous at a point in the domain, then its discontinuous in general and the following example show that :

<u>Example :</u> Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, $Y = \{1, 2\}$ and $\tau' = \{Y, \phi, \{1\}\}$. Define f as follow :

$$f : (X, \tau) \to (Y, \tau') \;;\quad f(a) = f(b) = 2,\ f(c) = 1$$

notes that f is discontinuous since $\{1\} \in \tau'$, but $f^{-1}(\{1\}) = \{c\} \notin \tau$.

On the other hand, thought that f is discontinuous in general, but its continuous at a point a as follow :

$f(a) = 2$ and the open nbd of 2 is Y only and $f^{-1}(Y) = X$ and X is an open nbd

There are several characterizations of continuous functions and, hence, that any one of them may be used to show continuity of a function. These are given in the next theorem :

Theorem : Let $f : (X, \tau) \to (Y, \tau')$ be a function. Then f is continuous iff satisfy one of the following properties :

(1) $f^{-1}(F) \in \mathscr{F} \quad \forall \quad F \in \mathscr{F}'$; $\mathscr{F}$ family of closed sets in X and $\mathscr{F}'$ family of closed sets in Y i.e., The inverse image of every closed set in Y is closed in X.

(2) $f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$; β' is a basis for τ'.

i.e., The inverse image of every element in any basis for τ' is open set in X.

(3) $f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathscr{S}'$; $\mathscr{S}'$ is a subbasis for τ'.

i.e., The inverse image of every element in any subbasis for τ' is open set in X.

(4) $f^{-1}(N_y) \in \tau \quad \forall \quad y \in Y \quad \forall \quad N_y \in \eta_y$; η_y is a family of open nbd for a point y in Y.

i.e., The inverse image of every open nbd for any element Y is open set in X.

(5) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

(6) $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$.

Proof :

(1) To prove, f is continuous $\Leftrightarrow X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$

($\Rightarrow$) Suppose that f is cont. , to prove $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$

Let F closed set in Y $\Rightarrow$ Y – F open set in Y (def. of closed set)

$\Rightarrow f^{-1}(Y - F) \in \tau$ (since f is continuous)

But, $f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F)$

$= X - f^{-1}(F)$ (since $f^{-1}(Y) = X$)

$\because \quad f^{-1}(Y - F) \in \tau \Rightarrow X - f^{-1}(F) \in \tau$

($\Leftarrow$) Suppose that $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\therefore$ Y – V closed set in Y since V open

$\Rightarrow f^{-1}(Y - V)$ closed set in X (by hypothesis)

$\Rightarrow X - f^{-1}(Y - V) \in \tau$

But, $X - f^{-1}(Y - V) = X - [f^{-1}(Y) - f^{-1}(V)] = X - [X - f^{-1}(V)] = f^{-1}(V)$

$\Rightarrow f^{-1}(V) \in \tau$

$\therefore$ f is continuous.

(2) To prove, f is continuous $\Leftrightarrow f^{-1}(B') \in \tau \ \forall B' \in \beta'$; β' is a basis for τ'.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(B') \in \tau \ \ \forall \ \ B' \in \beta'$

Let β' be a base of τ' and $B' \in \beta'$

$\Rightarrow B' \in \tau'$ (since $\beta' \subseteq \tau'$)

$\Rightarrow f^{-1}(B') \in \tau$ (since f is continuous)

$\Rightarrow f^{-1}(B') \in \tau \ \ \forall \ \ B' \in \beta'$

($\Leftarrow$) Suppose that $f^{-1}(B') \in \tau \ \ \forall \ \ B' \in \beta'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow \ V = \bigcup_i B_i' \quad ; \quad B_i' \in \beta'$ (def. of basis)

$\Rightarrow f^{-1}(V) = f^{-1}(\bigcup_i B_i') = \bigcup_i f^{-1}(B_i')$

$\Rightarrow \bigcup_i f^{-1}(B_i') \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $f^{-1}(V) = f^{-1}(\bigcup_i B_i') = \bigcup_i f^{-1}(B_i')$)

$\therefore$ f is continuous

(3) To prove, f is continuous $\Leftrightarrow f^{-1}(S') \in \tau \ \forall \ S' \in \mathcal{S}'$; $\mathcal{S}'$ is a subbasis for τ'.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(S') \in \tau \ \ \forall \ \ S' \in \mathcal{S}'$

Let $\mathcal{S}'$ be a subbase of τ' and $S' \in \mathcal{S}'$

$\Rightarrow S' \in \tau'$ (since $\mathcal{S}' \subseteq \tau'$)

$\Rightarrow f^{-1}(S') \in \tau$ (since f is continuous)

$\Rightarrow f^{-1}(S') \in \tau \ \ \forall \ \ S' \in \mathcal{S}'$

($\Leftarrow$) Suppose that $f^{-1}(S') \in \tau \ \ \forall \ \ S' \in \mathcal{S}'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow V = \bigcup_i (\bigcap_{j=1}^{n} S_j')$ (def. of basis and subbasis)

$\Rightarrow f^{-1}(V) = f^{-1}(\bigcup_i (\bigcap_{j=1}^{n} S_j'))$

$= \bigcup_i f^{-1}(\bigcap_{j=1}^{n} S_j')$ (inverse image distribution on union)

$= \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j'))$ (inverse image distribution on intersection)

$\because f^{-1}(S_j') \in \tau \quad \Rightarrow \quad f^{-1}(S_j')$ open in X

$\Rightarrow \bigcap_{j=1}^{n} f^{-1}(S_j') \in \tau$ (by second condition of def. of top.)

$\Rightarrow \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j')) \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $f^{-1}(V) = \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j'))$)

$\therefore$ f is continuous,

(4) To prove, f is continuous $\Leftrightarrow f^{-1}(N_y) \in \tau \ \ \forall \ \ y \in Y \ \ \forall \ \ N_y \in \eta_y$

($\Rightarrow$) Suppose that f is continuous, to prove $f^{-1}(N_y) \in \tau \ \ \forall \ \ y \in Y \ \ \forall \ \ N_y \in \eta_y$

Let $N_y \in \eta_y$

$\because$ η_y is open nbd system for y, then η_y is a family of open set, therefore

$\Rightarrow N_y \in \tau'$

$\Rightarrow f^{-1}(N_y) \in \tau$ (since f is continuous)

($\Leftarrow$) Suppose that $f^{-1}(N_y) \in \tau \ \forall \ y \in Y \ \ \forall \ \ N_y \in \eta_y$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow V = \bigcup_{y \in V} N_y$ i.e., V = union of a family of open sets for every point in V by using the fifth condition of def. of o.n.s

$(U \in \tau \Leftrightarrow \exists \ N_y \in \eta_y \,;\, N_y \subseteq U \ \ \forall \ \ y \in U)$

$\therefore f^{-1}(V) = f^{-1}(\bigcup_{y \in V} N_y) = \bigcup_{y \in V} f^{-1}(N_y)$ (inverse image distribution on union)

$\because f^{-1}(N_y) \in \tau$ (by hypothesis)

$\Rightarrow \bigcup_{y \in V} f^{-1}(N_y) \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $(f^{-1}(V) = \bigcup_{y \in V} f^{-1}(N_y))$)

$\therefore$ f is continuous.

(5) To prove, f is continuous $\Leftrightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \ ; \ B \subseteq Y$.

($\Rightarrow$) Suppose that f is continuous , to prove $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \ ; \ B \subseteq Y$.

Let $B \subseteq Y \Rightarrow B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$

$\because \overline{B}$ closed set in Y by (1) $f^{-1}(\overline{B})$ closed set in X

$\Rightarrow \bigcap\{F \subseteq X : F^c \in \tau \ \wedge \ f^{-1}(B) \subseteq F\} \subseteq f^{-1}(\overline{B})$

since $\overline{f^{-1}(B)}$ is intersection of all closed sets that contain $f^{-1}(B)$ and $f^{-1}(\overline{B})$ is one of the closed set that contain $f^{-1}(B)$, then

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ (since $\bigcap\{F : F^c \in \tau \ \wedge \ f^{-1}(B) \subseteq F\} = \overline{f^{-1}(B)}$)

($\Leftarrow$) Suppose that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \ ; \ B \subseteq Y$. , to prove f is continuous.

We will use part (1) in this theorem

Let F closed set in Y, to prove $f^{-1}(F)$ closed in X i.e., $\overline{f^{-1}(F)} = f^{-1}(F)$.

$f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ (since $A \subseteq \overline{A}$) ---------------(1)

$\because$ F closed $\Rightarrow F = \overline{F} \Rightarrow f^{-1}(F) = f^{-1}(\overline{F})$

By hypnoses, $\overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) = f^{-1}(F)$

$\Rightarrow \overline{f^{-1}(F)} \subseteq f^{-1}(F)$ ---------------(2)

From (1) and (2) we have $\overline{f^{-1}(F)} = f^{-1}(F)$

$\therefore f^{-1}(F)$ closed in X.

$\therefore$ f is continuous.

(6) To prove, f is continuous $\Leftrightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o \ ; \ B \subseteq Y$.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(B^o) \subseteq (f^{-1}(B))^o \ ; \ B \subseteq Y$

Let $B \subseteq Y \Rightarrow B^o \subseteq B \Rightarrow f^{-1}(B^o) \subseteq f^{-1}(B)$

$\because$ B^o is open in Y $\Rightarrow f^{-1}(B^o)$ is open in X (since f is continuous)
$f^{-1}(B^o) \subseteq \bigcup\{O \subseteq X \,;\, O \in \tau \,,\, O \subseteq f^{-1}(B)\}$
since $(f^{-1}(B))^o$ is union of all open sets that contain in $f^{-1}(B)$ and $f^{-1}(B^o)$ is one of the open set that contain in $f^{-1}(B)$, then

$\Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ (since $\bigcup\{O \subseteq X \,;\, O \in \tau \,,\, O \subseteq f^{-1}(B)\} = (f^{-1}(B))^o$)
($\Leftarrow$) Suppose that $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$, to prove f is continuous.
Let V open set in Y i.e., $V \in \tau'$ to prove $f^{-1}(V)$ open in X i.e., $f^{-1}(V) = (f^{-1}(V))^o$.
$\because$ $V^o \subseteq V \Rightarrow ((f^{-1}(V))^o \subseteq f^{-1}(V)$ ---------------(1)
$\because$ V open $\Rightarrow V = V^o \Rightarrow f^{-1}(V) = f^{-1}(V^o)$
By hypnoses, $f^{-1}(V) = f^{-1}(V^o) \subseteq (f^{-1}(V))^o$
$\Rightarrow f^{-1}(V) \subseteq (f^{-1}(V))^o$ ---------------(2)
From (1) and (2) we have $f^{-1}(V) = (f^{-1}(V))^o$
$\therefore$ $f^{-1}(V)$ open in X
$\therefore$ f is continuous.

Remark : The six characterizations in the previous theorem for definition of continuity is not unique, but there are another characterizations for example :

$$f \text{ is continuous } \Leftrightarrow f(\overline{A}) \subseteq \overline{f(A)} \;;\; A \subseteq X.$$
$$f \text{ is continuous } \Leftrightarrow (f(A))^o \subseteq f(A^o) \;;\; A \subseteq X.$$

Remarks :

[1] Notes that, if $f : (X, \tau) \to (Y, \tau')$ is continuous function, then it's not necessary that the direct image of open set in X is open set in Y. i.e.,

$$U \in \tau \not\Rightarrow f(U) \in \tau' \text{ (in general not true)}$$
$$V \in \tau' \Rightarrow f^{-1}(V) \in \tau \text{ (is true)}$$

This two statements is difference and the following example show this :
Example : Let $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$ be a function, then f is continuous (see, page 37). Now we will show that the direct image of open set is not open in general :
$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall\; x \in \mathbb{R}$
Let $U = (0, 1)$ open set in $(\mathbb{R}, \tau_u)$ and $f(U) = U$
$\Rightarrow f(U) = U$ is not open in $(\mathbb{R}, I)$ since $U = (0, 1) \notin I = \{\mathbb{R}, \phi\}$
So, we show that $U \in \tau \wedge f(U) \notin \tau'$.

[2] Notes that, if $f : (X, \tau) \to (Y, \tau')$ is continuous function, then it's not necessary that the direct image of closed set in X is closed set in Y. i.e.,

$$F^c \in \tau \not\Rightarrow (f(F))^c \in \tau' \qquad \text{(in general not true)}$$
$$F^c \in \tau' \Rightarrow (f^{-1}(F))^c \in \tau \qquad \text{(is true)}$$

This two statements is difference and the following example show this :

Example : In the previous example :

$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall\ x \in \mathbb{R}$, f is continuous

Let $F = [0, 1]$ closed set in $(\mathbb{R}, \tau_u)$

$\Rightarrow f(F) = F$ is not closed in $(\mathbb{R}, I)$ since $F^c = [0, 1]^c \notin I = \{\mathbb{R}, \phi\}$

So, we show that $[0, 1]^c \in \tau_u \wedge (f([0, 1])^c = [0, 1]^c \notin I$.

Now we will introduce a new definitions for functions satisfy the condition [1] and [2] in the previous remark as follows :

Definition : Open & Closed Functions

Let $f : (X, \tau) \to (Y, \tau')$ be a function.

(1) The function f is called **open** if the direct image for any open set in X is open set in Y. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is open function } \Leftrightarrow \forall\ U \in \tau \Rightarrow f(U) \in \tau'$$
$$f : (X, \tau) \to (Y, \tau') \text{ is not open function } \Leftrightarrow \exists\ U \in \tau \wedge f(U) \notin \tau'$$

(2) The function f is called **closed** if the direct image for any closed set in X is closed set in Y. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is closed function } \Leftrightarrow \forall\ F^c \in \tau \Rightarrow (f(F))^c \in \tau'$$
$$f : (X, \tau) \to (Y, \tau') \text{ is not closed function } \Leftrightarrow \exists\ F^c \in \tau \wedge (f(F))^c \notin \tau'$$

Remark : There are no relation between the concepts continuous, open, closed functions and the following table show that :

f continuous function	f open function	f closed function
T	T	T
T	T	F
T	F	T
F	T	T
T	F	F
F	T	F
F	F	T
F	F	F

Such that T = True (i.e., the function is satisfy) and F = False (i.e., the function is not satisfy). Also, there are eight probability may be taken the function for example (T F T) means that the function f is continuous, not open , closed. Therefore we will introduce an eight examples satisfy this probability :

Example (1) : (T T T) means the function f is continuous, open , closed.
Define the identity function $f : (X, \tau) \to (X, \tau)$; $f(x) = x \quad \forall x \in X$.
f is continuous since $\forall\ V \in \tau$ in rang X $\Rightarrow$ $f^{-1}(V) = V \in \tau$ in domain X.
f is open since $\forall\ U \in \tau$ open in domain X $\Rightarrow$ $f(U) = U$ is open in rang X.
f is closed since $\forall$ F closed in domain X $\Rightarrow$ $f(F) = F$ is closed in rang X.

Example (2) : (T T F) means the function f is continuous, open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}\}$
Define the constant function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = f(3) = a$
f is continuous since it is constant.
f is open since : $X \in \tau \Rightarrow f(X) = \{a\} \in \tau'$, $\phi \in \tau \Rightarrow f(\phi) = \phi \in \tau'$ and $\{1\} \in \tau \Rightarrow f(\{1\}) = \{a\} \in \tau'$ (i.e., $\forall\ U \in \tau \Rightarrow f(U) \in \tau'$).
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathscr{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$)
But, $f(\{2, 3\}) = \{a\} \notin \mathscr{F}'$ since $\{a\}^c = \{b, c\} \notin \tau'$

Example (3) : (T F T) means the function f is continuous, not open , closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a, b\}\}$
Define the constant function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = f(3) = c$
f is continuous since it is constant.
f is not open since $\exists$ open set $\{1\} \in \tau$,but $f(\{1\}) = \{c\} \notin \tau'$.
f is closed since : The family of closed sets in X is $\mathscr{F} = \{X, \phi, \{2, 3\}\}$ and the family of closed set in Y is $\mathscr{F}' = \{Y, \phi, \{c\}\}$, then
$f(X) = \{c\}$, $f(\phi) = \phi$, $f(\{2, 3\}) = \{c\}$ (i.e., $\forall\ F \in \mathscr{F} \Rightarrow f(F) \in \mathscr{F}'$).

Example (4) : (F T T) means the function f is not continuous, open , closed.
Define the function $f : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$; $f(x) = x \quad \forall x \in \mathbb{R}$
f is not continuous since $\exists$ (0, 1) open in $(\mathbb{R}, \tau_u)$, but $f^{-1}((0, 1)) = (0, 1)$ not open in $(\mathbb{R}, I)$.
f is open and closed since the only open and closed sets in $(\mathbb{R}, I)$ are $\mathbb{R}$, ϕ and $f(\mathbb{R}) = \mathbb{R}$ and $f(\phi) = \phi$ (i.e., the direct image of open (rep., closed) set is open (rep., closed) set)

Example (5) : (T F F) means the function f is continuous, not open , not closed.
Define the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall\, x \in \mathbb{R}$
f is continuous since the rang is $(\mathbb{R}, I)$ (see, page 37).
f is not open since $\exists$ open set $(0, 1)$ in $(\mathbb{R}, \tau_u)$, but $f((0, 1)) = (0, 1)$ is not open in $(\mathbb{R}, I)$.
f is not closed since $\exists$ closed set $\{0\}$ in $(\mathbb{R}, \tau_u)$, but $f(\{0\}) = \{0\}$ is not closed in $(\mathbb{R}, I)$.

Example (6) : (F T F) means the function f is not continuous, open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}, \{a, b\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{a\} \in \tau'$,but $f^{-1}(\{a\}) = \{1, 2\} \notin \tau$.
f is open since : $X \in \tau \Rightarrow f(X) = \{a, b\} \in \tau'$, $\phi \in \tau \Rightarrow f(\phi) = \phi \in \tau'$ and $\{1\} \in \tau \Rightarrow f(\{1\}) = \{a\} \in \tau'$ (i.e., $\forall\; U \in \tau \Rightarrow f(U) \in \tau'$).
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$), but, $f(\{2, 3\}) = \{a, b\} \notin \mathcal{F}'$ since $\{a, b\}^c = \{c\} \notin \tau'$.

Example (7) : (F F T) means the function f is not continuous, not open , closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{c\}, \{b, c\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{b, c\} \in \tau'$,but $f^{-1}(\{b, c\}) = \{3\} \notin \tau$.
f is not open since $\exists$ open set $\{1\} \in \tau$,but $f(\{1\}) = \{a\} \notin \tau'$.
f is closed since : The family of closed sets in X is $\mathcal{F} = \{X, \phi, \{2, 3\}\}$ and the family of closed set in Y is $\mathcal{F}' = \{Y, \phi, \{a, b\}, \{a\}\}$, then
$f(X) = \{a, b\}$, $f(\phi) = \phi$, $f(\{2, 3\}) = \{a, b\}$ (i.e., $\forall\; F \in \mathcal{F} \Rightarrow f(F) \in \mathcal{F}'$).

Example (8) : (F F F) means the function f is not continuous, not open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{a\} \in \tau'$,but $f^{-1}(\{a\}) = \{1, 2\} \notin \tau$.
f is not open since $\exists$ open set $X \in \tau$,but $f(X) = \{a, b\} \notin \tau'$.
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$), but, $f(\{2, 3\}) = \{a, b\} \notin \mathcal{F}'$ since $\{a, b\}^c = \{c\} \notin \tau'$.

Remark : The open function is closed and the closed function is open if the function is bijective (injective and surjective). i.e.,

$$f \text{ is bijective function } \Rightarrow (f \text{ open} \Leftrightarrow f \text{ closed})$$

$$f \text{ is bijective function } \Rightarrow (f \text{ not open} \Leftrightarrow f \text{ not closed})$$

This means if we wanted to get a function is open not closed or closed not open must be define a function not bijection (not injective or not surjective) since if we define a bijective function then it's either open and closed or not open and not closed.

Remark : We talking about the function $f : (X, \tau) \to (Y, \tau')$ is continuous or discontinuous. Now we will question if the function f is bijective and continuous this implies that f^{-1} is continuous (i.e., if f^{-1} exists function and f is continuous, then that implies to f^{-1} is continuous ?? or conversaly). The answer of this question is no since we can find a continuous function but your inverse is not continuous and the following example show that :

Example : Define the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall x \in \mathbb{R}$

Notes that f is continuous since the rang is $(\mathbb{R}, I)$ (see, page 37).

Notes that f is bijective , then f^{-1} is exists function and

$f^{-1} : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$, but this function not continuous since

$\exists$ open set $(0, 1)$ in $(\mathbb{R}, \tau_u)$, but $(f^{-1})^{-1}((0, 1)) = (0, 1)$ is not open in $(\mathbb{R}, I)$.

Now the question : Is there are functions is continuous and there inverse is continuous two ??. The answer is yes and the following definition introduce this functions :

Definition : Homeomorphism Functions

Let $f : (X, \tau) \to (Y, \tau')$ be a function. The function f is called **homeomorphism** if its injective, surjective, continuous and f^{-1} continuous. i.e.,

$f : (X, \tau) \to (Y, \tau')$ is homeomorphism $\Leftrightarrow$ f 1-1, onto, continuous and f^{-1} continuous

$f : (X, \tau) \to (Y, \tau')$ is not homeomorphism $\Leftrightarrow$

$$f \text{ not 1-1} \vee f \text{ not onto} \vee f \text{ not continuous} \vee f^{-1} \text{ not continuous}$$

Remark : Clear that every homeomorphism function is continuous, but the converse is not true for example :

$$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I) \ ; \ f(x) = x \quad \forall \ x \in X$$

The function f is1-1, onto, continuous, but f^{-1} is not continuous. Therefore f is not homeomorphism.

Remark : If $f : (X, \tau) \to (Y, \tau')$ is homeomorphism function, this means :
$(f^{-1})^{-1}(U) \in \tau' \quad \forall\ U \in \tau$ (def of continuity), but $(f^{-1})^{-1}(U) = f(U)$ (since f bijective), so we can said

$$f^{-1} \text{ is continuous } \Leftrightarrow f(U) \in \tau' \quad \forall\ U \in \tau$$

but this is the definition of open function, so if f^{-1} is continuous this means f is open and vise versa with property that f is bijective. i.e.,

$$f^{-1} \text{ is continuous } \Leftrightarrow f \text{ is open}$$

if f is bijective (by previous remark, p. 47, f is open $\Leftrightarrow$ f is closed), so that

$$f^{-1} \text{ is continuous } \Leftrightarrow f \text{ is open } \Leftrightarrow f \text{ is closed}$$

i.e., the three concepts are equivalent and we can replace the definition of homeomorphism as follow :

f is homeomorphism $\Leftrightarrow$ f is 1-1, onto, continuous and open.

f is homeomorphism $\Leftrightarrow$ f is 1-1, onto, continuous and closed.

such that we replace the statement f^{-1} is continuous in definition of homeomorphism by either f open function or f closed function.

Remarks :

[1] If f is homeomorphism function, then f^{-1} is also homeomorphism function.
since f is 1-1 and onto, then f^{-1} is 1-1 and onto
since f is homeomorphism, then f^{-1} is continuous, also, $f = (f^{-1})^{-1}$ is continuous
Therefore, f^{-1} is homeomorphism function.

[2] If $f : (X, \tau) \to (Y, \tau')$ and $g : (Y, \tau') \to (Z, \tau'')$ are both homeomorphism functions, then the composition $gof : (X, \tau) \to (Z, \tau'')$ is homeomorphism.
since f and g are 1-1 and onto, then gof is 1-1 and onto
since f and g are continuous, then gof is continuous (by previous theorem)
since f and g are homeomorphism, then f^{-1} and g^{-1} are continuous also $f^{-1}og^{-1}$ is continuous (by previous theorem)
but, $f^{-1}og^{-1} = (gof)^{-1}$ is continuous.
Therefore, gof is homeomorphism function.

Definition : Homeomorphic Topologies

We called two topological spaces (X, τ) and (Y, τ') are **homeomorphic** if there exists a homeomorphism function from (X, τ) to (Y, τ') and denoted by $(X, \tau) \cong (Y, \tau')$ or $(Y, \tau') \cong (X, \tau)$. i.e.,

$$(X, \tau) \cong (Y, \tau') \Leftrightarrow \exists \text{ homeomorphism function } f : (X, \tau) \to (Y, \tau')$$

Theorem : The relation $\cong$ is an equivalent relation on the family of topological spaces.

Proof : We must prove the relation $\cong$ is reflexive, symmetric and transitive.

(1) To prove $\cong$ is reflexive. i.e., $(X, \tau) \cong (X, \tau)$??

Define the identity function $f : (X, \tau) \to (X, \tau)$; $f(x) = x \quad \forall x \in X$

Clear that f is 1-1, onto, continuous and $f = f^{-1}$ so that f^{-1} is continuous

Therefore, $\cong$ is reflexive.

(2) To prove $\cong$ is symmetric. i.e., if $(X, \tau) \cong (Y, \tau') \Rightarrow (Y, \tau') \cong (X, \tau)$??

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists$ homo. funct. $f : (X, \tau) \to (Y, \tau')$

by remark [1] above, we have f^{-1} is homo. funct. and

$f^{-1} : (Y, \tau') \to (X, \tau) \Rightarrow (Y, \tau') \cong (X, \tau)$

Therefore, $\cong$ is symmetric.

(3) To prove $\cong$ is transitive. i.e., if $(X, \tau) \cong (Y, \tau') \cong (Z, \tau'') \Rightarrow (X, \tau) \cong (Z, \tau'')$??

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists$ homo. funct. $f : (X, \tau) \to (Y, \tau')$ and

$\because (Y, \tau') \cong (Z, \tau'') \Rightarrow \exists$ homo. funct. $g : (Y, \tau') \to (Z, \tau'')$ and

by remark [2] above, we have gof is homo. funct. and

$gof : (X, \tau) \to (Z, \tau'') \Rightarrow (X, \tau) \cong (Z, \tau'')$

Therefore, $\cong$ is transitive.

Theorem :

(1) The bijective function $f : (X, \tau) \to (Y, \tau')$ is homeomorphism iff $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$; $B \subseteq Y$.

(2) The bijective function $f : (X, \tau) \to (Y, \tau')$ is homeomorphism iff $f^{-1}(B^{o}) = (f^{-1}(B))^{o}$; $B \subseteq Y$.

Proof :

(1) ($\Leftarrow$) Suppose that $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and f^{-1} is cont.

$\because \overline{f^{-1}(B)} = f^{-1}(\overline{B}) \Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$)

and $\because\ \overline{f^{-1}(B)} = f^{-1}(\overline{B}) \Rightarrow f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)} \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow f(\overline{B}) \subseteq \overline{f(B)}$; $B \subseteq X$ and f replace by f^{-1})

$\therefore$ f is Home.

($\Rightarrow$) Suppose that f is Home., to prove $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$

$\because$ f is home. $\Rightarrow$ f is cont. and f^{-1} is cont.

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ and $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$

$\Rightarrow \overline{f^{-1}(B)} = f^{-1}(\overline{B})$

(2) ($\Leftarrow$) Suppose that $f^{-1}(B^o) = (f^{-1}(B))^o$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and f^{-1} is cont.

$\because f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$)

and $\because\ f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow (f^{-1}(B))^o \subseteq f^{-1}(B^o) \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow (f(B))^o \subseteq f(B^o)$; $B \subseteq X$ and f replace by f^{-1})

$\therefore$ f is Home.

($\Leftarrow$) Suppose that f is Home., to prove $f^{-1}(B^o) = (f^{-1}(B))^o$

$\because$ f is home. $\Rightarrow$ f is cont. and f^{-1} is cont.

$\Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ and $(f^{-1}(B))^o \subseteq f^{-1}(B^o)$

$\Rightarrow f^{-1}(B^o) = (f^{-1}(B))^o$

<u>Definition :</u> Topological Property

A property "P" of a topological space (X, τ) is called a **<u>topological property</u>** iff every topological space (Y, τ') homeomorphic to (X, τ) also has the same property. i.e., if $(X, \tau) \cong (Y, \tau')$ and (X, τ) has a property "P", then (Y, τ') has the same property and vise versa.

<u>Subspace or Induced space</u>

<u>Definition :</u> Let (X, τ) be a topological space and $W \subseteq X$. Define the family τ_W as a family of subset of W as follow :

$$\tau_W = \{W \cap U \ :\ U \in \tau\}$$

Notes that the elements of τ_W are intersection W with every open set in X.

<u>Theorem :</u> Let (X, τ) be a topological space and $W \subseteq X$. Then $\tau_W = \{W \cap U \ :\ U \in \tau\}$ is a topology on W.

<u>Proof :</u> We will satisfy the three conditions in the definition of topological space.

(1) To prove, $W \in \tau_W$ and $\phi \in \tau_W$.

$\because X \in \tau \wedge W \subseteq X \Rightarrow W = W \bigcap X \Rightarrow W \in \tau_W$ (def. of τ_W)

$\because \phi \in \tau \wedge \phi \subseteq X \Rightarrow \phi = W \bigcap \phi \Rightarrow \phi \in \tau_W$ (def. of τ_W)

(2) Let $V_1, V_2 \in \tau_W$, to prove $V_1 \bigcap V_2 \in \tau_W$

$\because V_1 \in \tau_W \Rightarrow \exists\ U_1 \in \tau\ ;\ V_1 = W \bigcap U_1$ (def. of τ_W)

$\because V_2 \in \tau_W \Rightarrow \exists\ U_2 \in \tau\ ;\ V_2 = W \bigcap U_2$ (def. of τ_W)

$\Rightarrow V_1 \bigcap V_2 = (W \bigcap U_1) \bigcap (W \bigcap U_2)$

$\Rightarrow V_1 \bigcap V_2 = W \bigcap \underbrace{(U_1 \bigcap U_2)}_{\in\ \tau}$ (since $\bigcap$ distribution on $\bigcap$)

$\Rightarrow V_1 \bigcap V_2 \in \tau_W$ (def. of τ_W)

(3) Let $V_\alpha \in \tau_W$; $\alpha \in \Lambda$, to prove $\bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_W$

$\because V_\alpha \in \tau_W \Rightarrow \exists\ U_\alpha \in \tau\ ;\ V_\alpha = W \bigcap U_\alpha\ ;\ \alpha \in \Lambda$ (def. of τ_W)

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha = \bigcup_{\alpha\in\Lambda}(W \bigcap U_\alpha)$

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha = W \bigcap \underbrace{(\bigcup_{\alpha\in\Lambda} U_\alpha)}_{\in\ \tau}$

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_W$ (def. of τ_W)

Therefore, τ_W is a topology on W.

<u>Definition :</u> Subspace (or Induced) Topology

Let (X, τ) be a topological space and $W \subseteq X$. Then the topology τ_W is called the **<u>subspace</u> (or <u>induced) topology</u>** for W and the pair (W, τ_W) is called **<u>subspace</u>** of (X, τ).

<u>Example :</u> Let X = {a, b, c}, τ = {X, ϕ, {a}, {a, c}}, W = {a, b}, Z = {b} and K = {a, c}. Find τ_W , τ_Z , τ_K .

<u>Solution :</u>

$\tau_W = \{W \bigcap U\ :\ U \in \tau\}$

$\tau_W = \{W \bigcap X, W \bigcap \phi, W \bigcap \{a\}, W \bigcap \{a, c\}\}$

$= \{W, \phi, \{a\}\}$

By similar way we compute τ_Z , τ_K .

$\tau_Z = \{Z \bigcap U\ :\ U \in \tau\}$

$\tau_Z = \{Z \bigcap X, Z \bigcap \phi, Z \bigcap \{a\}, Z \bigcap \{a, c\}\}$

$= \{Z, \phi\} = I_Z$ = **indiscrete topology on Z**

$\tau_K = \{K \bigcap U\ :\ U \in \tau\}$

$\tau_K = \{K \bigcap X, K \bigcap \phi, K \bigcap \{a\}, K \bigcap \{a, c\}\} = \{K, \phi, \{a\}\}$.

Remarks :

[1] Notes that there is an open set in the subspace but it's not open in the space. In the previous example the set W = {a, b} is open in the subspace (W, τ_W) but it is not open in the space (X, τ) i.e., $W \notin \tau$, so we have :

$$V \in \tau_w \not\Rightarrow V \in \tau$$

In other word $\tau_W \not\subset \tau$ (in general).

[2] If $W \in \tau$, then $\tau_W \subseteq \tau$.

[3] Notes that, in the previous example $\tau_Z = I_Z = \{Z, \phi\}$, but $\tau \neq I_X = \{X, \phi\}$.

[4] There are some example $\tau_W = D_W$ = **discrete topology on W**, but $\tau \neq D_X$ i.e.,

$$(\tau_w = D_W \not\Rightarrow \tau = D_X)$$

For example : Let X = {a, b, c, d}, $\tau = \{X, \phi, \{a, b\}, \{c, d\}\}$ and W = {a, c}, then $\tau_W = \{W, \phi, \{a\}, \{c\}\} = D_W$

[5] If $\tau_X = D_X$, then $\tau_W = D_W$ for all $W \subseteq X$. (i.e., $\tau_X = D_X \Rightarrow \tau_W = D_W$)

To prove this property it's enough to prove that every singleton subset of W is open set in W.

Let $y \in W \Rightarrow \{y\} \subseteq W$, to prove $\{y\} \in \tau_W$??

$\because \{y\} \subseteq W \subseteq X \Rightarrow \{y\} \subseteq X \Rightarrow \{y\} \in D_X$

since $\{y\} = W \cap \{y\} \Rightarrow \{y\} \in \tau_W$

$$\Rightarrow \tau_W = D_W.$$

[6] If $\tau = I_X = \{X, \phi\}$, then $\tau_W = I_W = \{W, \phi\}$. (i.e., $\tau = I_X \Rightarrow \tau_W = I_W$)

To prove this property

$\tau_W = \{W \cap U : U \in \tau\} = \{W \cap X, W \cap \phi\} = \{W, \phi\}$.

Example : In the usual topological space ($\mathbb{R}$, τ_u). Find the induced topology for the following sets : W = [0, 1] , H = $\mathbb{N}$, M = $\mathbb{Q}$, K = [2, 3).

Solution : The open sets in ($\mathbb{R}$, τ_u) is the union of family of open interval and the family of open interval is a basis for topology τ_u . So we will use the open interval to compute the basis for induce topology for given set as follow :

W = [0, 1] ??

if, $a, b \le 0 \vee a, b \ge 1$ — ( ) [//////////////] ( ) $\mathbb{R}$ (a b 0 1 a b)

then, $[0, 1] \cap (a, b) = \phi$

if, $b > 1 \wedge a < 0$ — ([//////////////]) $\mathbb{R}$ (a 0 1 b)

then, $[0, 1] \cap (a, b) = [0, 1]$

if, $b \le 1 \wedge a \ge 0$ — [(///////)] $\mathbb{R}$ (0 a b 1)

then, $[0, 1] \cap (a, b) = (a, b)$

if, $a < 0 \;\wedge\; 0 < b \le 1$ ——(——[//////)——]—— $\mathbb{R}$

then, $[0, 1] \bigcap (a, b) = [0, b)$ a 0 b 1

if, $0 \le a < 1 \;\wedge\; b > 1$ ——[——(//////]——)—— $\mathbb{R}$

then, $[0, 1] \bigcap (a, b) = (a, 1]$ 0 a 1 b

From the probability above the basis for induce topology τ_W is

$$\beta_W = \{[0, 1], \phi, (a, b), [0, b), (a, 1]\}$$

Notes that the elements in this family is infinite since $a, b \in \mathbb{R}$.

H = ℕ ??

The induce topology for $H = \mathbb{N}$ is discrete topology $D_{\mathbb{N}}$ since :

Let (a, b) open interval in $(\mathbb{R}, \tau_u)$; $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$; $n - 1 < a < n$ and $n < b < n+1$.

—|—(—|—)—|———|———|———|

n – 1 a n b n+1 n+2 …

This means that every singleton set (i.e., $\mathbb{N} \bigcap (a, b) = \{n\}$) from $\mathbb{N}$ is open in $\mathbb{N}$ (i.e., $\{n\} \in \tau_{\mathbb{N}}$). Therefore $\tau_{\mathbb{N}} = D$.

M = ℚ ??

We will intersect $\mathbb{Q}$ with every open interval (a, b) in $(\mathbb{R}, \tau_u)$; $a, b \in \mathbb{R}$.

————(|||||||||||||||||||||||||)———— $\mathbb{R}$

a $\mathbb{Q}$ b

$\mathbb{Q} \bigcap (a, b)$ = the rational numbers in (a, b) and the basis for induce topology $\tau_{\mathbb{Q}}$ is

$$\beta_{\mathbb{Q}} = \{\mathbb{Q} \bigcap (a, b) \;;\; a, b \in \mathbb{R}\}.$$

K = [2, 3) ??

To compute the induce topology τ_K ; K = [2, 3) is similar of compute the induce topology τ_W ; W = [0, 1] above by replace [0,1] by [2, 3) as follow :

if $a, b \le 2 \;\vee\; a, b \ge 3$ —(——) [//////////////) (——)— $\mathbb{R}$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = \phi$ a b 2 3 a b

if $a < 2 \;\wedge\; b \ge 3$ —(——[//////////////)——)— $\mathbb{R}$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = [2, 3)$ a 2 3 b

if $a \ge 2 \;\wedge\; b \le 3$ ——[(//////)——)—— $\mathbb{R}$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = (a, b)$ 2 a b 3

if $a < 2 \;\wedge\; 2 < b \le 3$ ——(——[//////)——)—— $\mathbb{R}$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = [2, b)$ a 2 b 3

if $2 \le a < 3 \;\wedge\; b > 3$ ——[——(//////)——)— $\mathbb{R}$

$\Rightarrow$ $[2, 3) \bigcap (a, b) = (a, 3)$ 2 a 3 b

From the probability above the basis for induce topology τ_K is

$$\beta_K = \{[2, 3), \phi, (a, b), [2, b), (a, 3)\}.$$

We can compute the induce topology for the intervals [c, d], [c, d), (c, d], (c, d) by similar way by taken this probality and replace W = [0, 1] or K = [2, 3) by [c, d], [c, d), (c, d], (c, d).

Theorem : Let (X, τ) be a topological space and (W, τ_W) be a subspace topology of X. If $W \in \tau$, then τ_W is subfamily of τ. i.e.,
If W open set in X, then every open set in W is open in X.

Proof : We must prove the following statement $\tau_W \subseteq \tau$ (i.e., if $V \in \tau_W \Rightarrow V \in \tau$)

Let $V \in \tau_W \Rightarrow \exists\, U \in \tau$; $V = W \cap U$ (def. of τ_W)

$\because$ $W \in \tau$ (by hypothesis) $\wedge$ $U \in \tau$

$\Rightarrow W \cap U \in \tau$ (def. of Top.)

$\Rightarrow V \in \tau$ (since $V = W \cap U$)

The following example clear this theorem :

Example : Let X = {a, b, c, d}, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and W = {a, b, c}. Find τ_W .

Solution :

Clear $W \in \tau$. To compute τ_W :

$\tau_W = \{W \cap U : U \in \tau\}$

$\tau_W = \{W \cap X, W \cap \phi, W \cap \{a\}, W \cap \{a, b\}, W \cap \{a, b, c\}\}$

$= \{W, \phi, \{a\}, \{a, b\}\}$

Notes that τ_W is subfamily of τ (i.e., $\tau_W \subseteq \tau$).

Remark : From definition of induce topology τ_W , notes that :

$$V \in \tau_W \Leftrightarrow \exists\, U \in \tau \ ; V = W \cap U$$

The question now what about the close set, the previous statement satisfy or not ??
The answer **yes** such that :

$$A \in (\tau_W)^c \Leftrightarrow \exists\, F \in \tau^c \ ; A = W \cap F$$

By other statement :

$$A \in \mathscr{F}_W \Leftrightarrow \exists\, F \in \mathscr{F} \ ; \ A = W \cap F$$

Such that $\mathscr{F}$ is the family of closed sets in X and $\mathscr{F}_W$ is the family of closed sets in W.

Example : Let X = {a, b, c}, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and W = {b, c}.

Then $\tau_W = \{W, \phi, \{b\}\}$, $\mathscr{F} = \{X, \phi, \{b, c\}, \{a, c\}, \{c\}\}$ and $\mathscr{F}_W = \{W^c, \phi^c, \{b\}^c\}$ such that :

$\mathscr{F}_W = \{W^c, \phi^c, \{b\}^c\} = \{\phi, W, \{c\}\}$

$= \{W \cap X, W \cap \phi, W \cap \{b, c\}, W \cap \{a, c\}, W \cap \{c\}\}$

<u>Theorem :</u> If K is a subspace from W and W is a subspace from X, then K is a subspace from X.

<u>Proof :</u> Let (X, τ) be a topological space and $W \subseteq X$, $K \subseteq W$, to prove K is a subspace from X, must prove :

(1) $K \subseteq X$

(2) if $A \in (\tau_W)_K \Rightarrow \exists\ U \in \tau$; $A = K \cap U$

Now,

(1) Since $K \subseteq W \subseteq X \Rightarrow K \subseteq X$.

(2) Let $A \in (\tau_W)_K \Rightarrow \exists\ V \in \tau_W$; $A = K \cap V$

(def. of induce top. and K is a sub space from W)

$\because\ V \in \tau_W \Rightarrow \exists\ U \in \tau$; $V = W \cap U$

(def. of induce top. and W is a sub space from X)

$\because\ A = K \cap V \Rightarrow A = K \cap (W \cap U)$ (since $V = W \cap U$)

$\Rightarrow A = (K \cap W) \cap U$ ($\cap$ associative)

$\Rightarrow A = K \cap U$ (since $K \subseteq W$ and $K = K \cap W$)

<u>Definition :</u> Restriction Function

Let $f : X \to Y$ be a function and let $A \subseteq X$. We say the function $g : A \to Y$ such that $g(a) = f(a)$ for all $a \in A$ is the **<u>restriction function</u>** on the set A and denoted by $g = f|_A$.

<u>Example :</u> Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 1$ be a function.

Notes that the domain of f is $\mathbb{R}$, take $\mathbb{N} \subseteq \mathbb{R}$ and

$f|_{\mathbb{N}} : \mathbb{N} \to \mathbb{R}$ such that $(f|_{\mathbb{N}})(x) = x + 1$.

$f|_{\mathbb{N}}$ is restriction function on the set $\mathbb{N}$.

We will use this definition to introduce the following theorem :

<u>Theorem :</u> Let $f : (X, \tau) \to (Y, \tau')$ be a continuous function and W be a subspace topology from X. Then $f|_W$ is continuous.

Proof : To prove, $f|_W : (W, \tau_W) \to (Y, \tau')$ is continuous ??

i.e., to prove if $V \in \tau' \Rightarrow (f|_W)^{-1}(V) \in \tau_W$

Let $V \in \tau' \Rightarrow f^{-1}(V) \in \tau$ (since f is continuous)

$\Rightarrow W \cap f^{-1}(V) \in \tau_W$ (By def. of τ_W)

$\Rightarrow (f|_W)^{-1}(V) \in \tau_W$ (since $W \cap f^{-1}(V) = (f|_W)^{-1}(V)$)

$\therefore f|_W$ is cont.

Remark : From previous theorem we can get on an infinite number from continuous functions thought out know one continuous function for example :

$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ such that $f(x) = x + 2$ is continuous function.

$\therefore$ every functions of follows is continues function :

$f|_{\mathbb{N}}$, $f|_{\mathbb{Q}}$, $f|_{[2, 3]}$, $f|_{(0,\infty)}$ … etc.

We can get another of an infinite number from new continuous functions by theorem blow :

Theorem : Let (X, τ) be a topological space and (W, τ_W) be a subspace of X. Then the inclusion function $i : (W, \tau_W) \to (X, \tau)$ such that $i(x) = x$ for all $x \in W$ is continuous.

Proof : To prove if $V \in \tau \Rightarrow i^{-1}(V) \in \tau_W$

Let $V \in \tau \Rightarrow i^{-1}(V) = W \cap V$ (since $W \subseteq X$ and def. of i)

$\Rightarrow i^{-1}(V) \in \tau_W$ (since $W \cap V \in \tau_W$ and by def. of τ_W)

$\therefore i$ is cont.

Let (X, τ) be a topological space and (W, τ_W) be a subspace topology of X and $A \subseteq W \subseteq X$. We can compute $\overline{A}$, A^o , A^b in (X, τ) and from the other hand we can compute $\overline{A}$, A^o , A^b in (W, τ_W). The question what relation between ($\overline{A}$ in X and $\overline{A}$ in W), (A^o in X and A^o in W) and (A^b in X and A^b in W) ?? The theorem blow answer on this questions :

Theorem : Let (X, τ) be a topological space and (W, τ_W) be a subspace topology of X and $A \subseteq W \subseteq X$, then

(1) $W \cap \overline{A} = \overline{A}$ in W ; $\overline{A}$ is closure of A in X

(2) $W \cap A^o \subseteq A^o$ in W.

(3) $W \cap A^b \supseteq A^b$ in W.

Proof :

(1) To prove, $W \bigcap \overline{A} = \overline{A}$ in W

we must prove, $W \bigcap \overline{A} \subseteq \overline{A}$ in W and $W \bigcap \overline{A} \supseteq \overline{A}$ in W

$\because A \subseteq W \subseteq X \Rightarrow \overline{A} \in \mathscr{F}$ (by previous theorem $\overline{A}$ is closed in X)

$\Rightarrow \overline{A} \bigcap W \in \mathscr{F}_W$ ($\overline{A}$ is closed in W)

Now,

$A \subseteq W \wedge A \subseteq \overline{A} \Rightarrow A \subseteq W \bigcap \overline{A}$

Notes that $W \bigcap \overline{A}$ is closed set in W and containing A, but $\overline{A}$ in W is the smallest closed set in W contain A, so we get

$\Rightarrow \overline{A}$ in $W \subseteq W \bigcap \overline{A}$ ---------------(1)

and,

$\because W \bigcap \overline{A} \in \mathscr{F}_W \Rightarrow \exists F \in \mathscr{F} : \overline{A}$ in $W = W \bigcap F$

$\Rightarrow A \subseteq F$

$\Rightarrow \overline{A} \subseteq \overline{F} = F \Rightarrow \overline{A} \subseteq F$ ($\overline{F} = F$ since F is closed)

$\Rightarrow W \bigcap \overline{A} \subseteq W \bigcap F = \overline{A}$ in W

$\Rightarrow W \bigcap \overline{A} \subseteq \overline{A}$ in W ----------------(2)

From (1) and (2) we have, $W \bigcap \overline{A} = \overline{A}$ in W.

(2) To prove, $W \bigcap A^o \subseteq A^o$ in W

$A^o \in \tau \Rightarrow W \bigcap A^o \in \tau_W$

$\Rightarrow W \bigcap A^o \subseteq A^o \subseteq A \Rightarrow W \bigcap A^o \subseteq A$

$\Rightarrow W \bigcap A^o \subseteq A^o$ in W (since $W \bigcap A^o$ open in W contain in A)

i.e., A^o in W must contain all open set in W contain in A.

(3) To prove, A^b in $W \subseteq A^b \bigcap W$.

Let $x \in A^b$ in $W \Rightarrow \forall V \in \tau_W, x \in V$; $V \bigcap A \neq \phi \wedge V \bigcap A^c \neq \phi$

(By def. of boundary point)

$\because V \in \tau_W \Rightarrow \exists U \in \tau ; V = W \bigcap U$ (def. of τ_W)

$\Rightarrow \forall U \in \tau, x \in U, U \bigcap A \neq \phi \wedge U \bigcap A^c \neq \phi$

$\Rightarrow x \in A^b \Rightarrow x \in A^b \bigcap W$ (since $x \in V \subseteq W$)

$\therefore A^b$ in $W \subseteq A^b \bigcap W$

Remark : The equality of properties (2) and (3) in the previous theorem is not true in general and the following example clear that :

Example : Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $W = \{a, c, d\}$.

Then $\tau_W = \{W, \phi, \{a\}, \{a, c\}\}$, $A = \{a, c\}$

$A = \{a, c\} = A^o$ in W since $A \in \tau_W$ and $A^o = \{a\}$

Since $\{a\}$ is largest open set in X containing in A $\Rightarrow$

$A^o \bigcap W = \{a\} \bigcap \{a, c, d\} = \{a\} \Rightarrow A^o$ in $W \neq A^o \bigcap W$ since $\{a, c\} \neq \{a\}$

On the other hand to compute A^b, A^b in W, we will compute A^x, A^x in W such that

A^x in W = ϕ , then

A^b in W = W – (A^o in W $\bigcup$ A^x in W) = W – {a, c} = {d}

$\therefore$ A^b in W = {d}

$A^x = \phi \Rightarrow A^b = X - (A^o \bigcup A^x) = X - \{a\} = \{b, c, d\}$

$\Rightarrow A^b \bigcap W = \{c, d\}$

$\therefore$ A^b in W $\neq A^b \bigcap W$

To check property (1) in the previous theorem we compute $\overline{A}$ and $\overline{A}$ in W as follow :

$\overline{A}$ in W = W and $\overline{A} = X \Rightarrow \overline{A} \bigcap W = X \bigcap W = W$

$\therefore$ $\overline{A}$ in W = $\overline{A} \bigcap W$

Product Space

Definition : **Cartesian Product**

Let X and Y be any two sets. The **Cartesian product** , or simply **product** of X by Y is denoted by X×Y and denoted as :

$$X\times Y = \{(x, y) ; x \in X \wedge y \in Y\}$$

Definition : **Product Space**

Let (X, τ) and (Y, τ') be two topological spaces. We say the topology has a base β ;

$$\beta = \{ U\times V ; U \in \tau \wedge V \in \tau'\}$$

Is the **Product Topology** on the set X×Y and denoted by $\tau_{X\times Y}$ and called the spaces $(X\times Y, \tau_{X\times Y})$ is the **Product Space** of X by Y.

Remark : Notes that β in general not topology since it's not satisfy the third condition of topology, but since β is a base for topology, so we can get the three condition of topology and the following example show that :

Example : Let X = {1, 2, 3}, τ = {X, ϕ, {1}}, Y = {a, b} and τ' = {Y, ϕ, {b}}. Compute $\tau_{X\times Y}$.

Solution :

$X\times Y = \{(x, y) ; x \in X \wedge y \in Y\}$

$= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$\beta = \{ U\times V ; U \in \tau \wedge V \in \tau'\}$

$= \{X\times Y, X\times\phi, X\times\{b\}, \phi\times Y, \phi\times\phi, \phi\times\{b\}, \{1\}\times Y, \{1\}\times\phi, \{1\}\times\{b\}\}$

$= \{X\times Y, \phi, X\times\{b\}, \{1\}\times Y, \{1\}\times\{b\}\}$

Since $A\times\phi = \phi$ and $\phi\times A = \phi$ for any set A.

$\therefore\ \beta = \{X\times Y, \phi, \{(1, b), (2, b), (3, b)\}, \{(1, a), (1, b)\}, \{(1, b)\}\}$

Notes that β is not topology since

$\{(1, b), (2, b), (3, b)\} \bigcup \{(1, a), (1, b)\}\} \notin \beta$

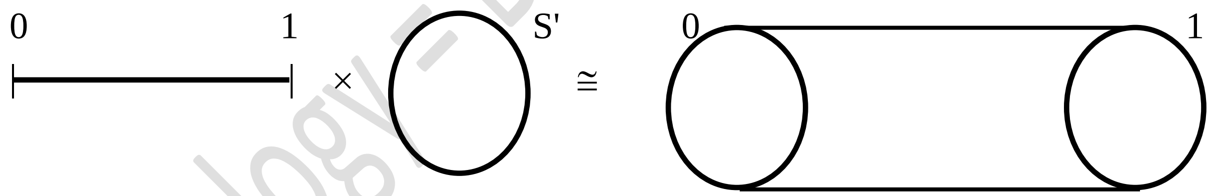
The elements of $\tau_{X\times Y}$ is elements of β and add all possible union of elements of β ;

$\therefore\ \tau_{X\times Y} = \{X\times Y, \phi, \{(1, b), (2, b), (3, b)\}, \{(1, a), (1, b)\}, \{(1, b)\}, \{(1, b), (2, b), (3, b), (1, a)\}\}$.

Remark : We can compute the product space $(X\times X, \tau_{X\times X})$ depending on (X, τ_X) only, also we can compute $(Y\times Y, \tau_{Y\times Y})$, $(Y\times X, \tau_{Y\times X})$, $(X\times Y\times Z, \tau_{X\times Y\times Z})$, … etc., there are an infinite number from product spaces which can computing from one space known or more than one space. In general $X\times Y \neq Y\times Y$.

From known product spaces which we use always is usual space $\mathbb{R}^n$; $n \in\mathbb{N}$ and the most common one is $\mathbb{R}^2$ which represent the plane and its product space follow from product $(\mathbb{R}, \tau_u)$ by self.

Example : Let $X = [0, 1]$ be a subspace of $(\mathbb{R}, \tau_u)$ and take $S' = \{(x, y) \in \mathbb{R}^2; x^2+y^2 =1\}$ be a subspace of $(\mathbb{R}^2, \tau_u)$ such that S' geometry represented as a circle in plane its center the original point (0, 0). Then $[0, 1] \times S'$ is a cylinder as follow :



Remarks : Let (X, τ) and (Y, τ') be any two topological spaces.

[1] If $\tau = I_X$ and $\tau' = I_Y$, then $\tau_{X\times Y} = I_{X\times Y}$, i.e.,

If $\tau =\{X, \phi\}$ and $\tau' = \{Y, \phi\}$, then $\tau_{X\times Y} = \{X\times Y, \phi\}$ and β is :

$$\beta = \{X\times Y, X\times\phi, \phi\times Y, \phi\times\phi\} = \{X\times Y, \phi\} = I_{X\times Y}$$

[2] If $\tau = I_X$ and $\tau' \neq I_Y$ or $\tau \neq I_X$ and $\tau' = I_Y$, then $\tau_{X\times Y} \neq I_{X\times Y}$, for example :

If $X = \{1, 2\}$, $\tau =\{X, \phi, \{1\}\}$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi\}$, then

$$\beta = \{X\times Y, \phi, \{1\}\times Y\} = \{X\times Y, \phi, \{(1, a), (1, b)\}\} = \tau_{X\times Y} \neq I_{X\times Y}$$

[3] If $\tau = D_X$ and $\tau' = D_Y$, then $\tau_{X\times Y} = D_{X\times Y}$,

Proof. To prove any topology is discrete topology it's enough to prove every singleton set is open i.e., ($\forall$ $\{(x, y)\}$ singleton set $\Rightarrow \{(x, y)\} \in \tau_{X\times Y}$)

$\{(x, y)\} = \{x\} \times \{y\}$ (def. Cartesian product of X by Y)

$\{x\} \in \tau$ (since $\tau = D_X$)

$\{y\} \in \tau'$ (since $\tau' = D_Y$)

$\{(x, y)\} \in \beta_{X\times Y}$

$\Rightarrow \{(x, y)\} \in \tau_{X\times Y}$ (def. product spaces)

[4] If $\tau \neq D_X$ or $\tau' \neq D_Y$, then $\tau_{X\times Y} \neq D_{X\times Y}$, and the following example show that :
If $X = \{1, 2\}$, $\tau =\{X, \phi, \{1\}, \{2\}\} = D_X$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi\} \neq D_Y$, then

$$\beta = \{X\times Y, \phi, \{1\}\times Y, \{2\}\times Y\}$$
$$= \{X\times Y, \phi, \{(1, a), (1, b)\}, \{(2, a), (2, b)\}\} = \beta_{X\times Y} = \tau_{X\times Y} \neq D_{X\times Y}$$

[5] If $A \subseteq X$ and $B \subseteq Y$, then $A\times B \subseteq X\times Y$ and we can compute the closure of $A\times B$ in $X\times Y$ (i.e., $\overline{A \times B}$), on the other hand there are $\overline{A}$ in X and $\overline{B}$ in Y, also we can compute $\overline{A} \times \overline{B}$ and the question what relation between $\overline{A \times B}$ and $\overline{A} \times \overline{B}$ and the answer $\overline{A \times B} = \overline{A} \times \overline{B}$.
Also, by similar way we can conclusion $(A\times B)^o = A^o\times B^o$.

[6] There are two natural projection functions from product space $X\times Y$ to codomain X and others to codomainY and denoted by P_X and P_Y and called the first project $X\times Y$ on X and called the second project $X\times Y$ on Y. We will show that the two functions are surjective , continuous and open as follows :

$P_X : X\times Y \to X$; $P_X((x, y)) = x$ and

$P_Y : X\times Y \to Y$; $P_Y((x, y)) = y$

; the first projection map the order pair (x, y) to first coordinate while the second projection map the order pair (x, y) to the second coordinate.

<u>To prove, P_X is continuous function</u>

We must prove, if $U \in \tau \Rightarrow P_X^{-1}(U) \in \tau_{X\times Y}$

Let $U \in \tau \Rightarrow P_X^{-1}(U) = U\times Y$ (By def. of P_X)

$\because U \in \tau \wedge Y \in \tau' \Rightarrow U\times Y \in \beta_{X\times Y}$

$\Rightarrow U\times Y \in \tau_{X\times Y}$ (since $\beta_{X\times Y} \subset \tau_{X\times Y}$)

$\Rightarrow P_X^{-1}(U) \in \tau_{X\times Y}$

$\therefore P_X$ is continuous functions

By similar way we prove P_Y is continuous functions

Let $V \in \tau' \Rightarrow P_Y^{-1}(V) = X\times V$ (By def. of P_Y)

$\because X \in \tau \wedge V \in \tau' \Rightarrow X\times V \in \beta_{X\times Y}$

$\Rightarrow X\times V \in \tau_{X\times Y}$ (since $\beta_{X\times Y} \subset \tau_{X\times Y}$)

$\Rightarrow P_Y^{-1}(V) \in \tau_{X\times Y}$

$\therefore P_Y$ is continuous functions

<u>To prove, P_X is open function</u>

Let $U\times V \in \beta_{X\times Y} \Rightarrow U\times V$ open set in $X\times Y$; $U \in \tau \wedge V \in \tau'$

$\Rightarrow P_X(U\times V) = U$

$\because U \in \tau \Rightarrow P_X(U\times V) \in \tau$

$\therefore$ P_X is open functions

By similar way we prove P_Y is open functions

Let $U\times V \in \beta_{X\times Y}$ $\Rightarrow$ $U\times V$ open set in $X\times Y$; $U \in \tau$ $\wedge$ $V \in \tau'$

$\Rightarrow$ $P_Y(U\times V) = V$

$\because$ $V \in \tau'$ $\Rightarrow$ $P_Y(U\times V) \in \tau$

$\therefore$ P_Y is open functions.

[7] Notes that $X\times Y \neq Y\times X$ since $(x, y) \neq (y, x)$ in general, but $X\times Y \cong Y\times X$ (i.e., $X\times Y$, $Y\times X$ are Homeomorphic), to prove this :

Define $f : X\times Y \to Y\times X$; $f((x, y)) = (y, x)$

f is 1-1 function since,

Let $f((x_1, y_1)) = f((x_2, y_2))$ $\Rightarrow (y_1, x_1) = (y_2, x_2)$

$\Rightarrow x_1 = x_2 \wedge y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$.

f is onto function since,

$\forall$ $(y, x) \in Y\times X$ $\exists$ $(x, y) \in X\times Y$; $f((x, y)) = (y, x)$.

f is continuous function since,

Let β be a base of $X\times Y$ and β' be a base of $Y\times X$

Let $V\times U \in \beta'$ $\Rightarrow$ $V \in \tau'$ $\wedge$ $U \in \tau$

$\Rightarrow V\times U \in \tau_{Y\times X}$

$\Rightarrow f^{-1}(V\times U) = U\times V$ open set in $X\times Y$

f is open function since, the image of every open set in domain is open set in codomain ;

Let $U\times V \in \beta$ $\Rightarrow$ $U\times V$ open set in $X\times Y$; $U \in \tau$ $\wedge$ $V \in \tau'$

$\Rightarrow$ $f(U\times V) = V\times U \in \tau_{Y\times X}$

$\therefore$ f is homeomorphism function.

[8] If $y_0 \in Y$, then the product space $X\times\{y_0\}$ topological equivalent the space X. i.e., $X\times\{y_0\} \cong X$; $X\times\{y_0\} = \{(x, y_0) : x \in X\}$

To prove this :

Define $f : X \to X\times\{y_0\}$; $f(x) = (x, y_0)$ $\forall$ $x \in X$

f is 1-1 function since,

Let $f(x_1) = f(x_2)$ $\Rightarrow (x_1, y_0) = (x_2, y_0)$

$\Rightarrow x_1 = x_2$ $\forall$ $x_1, x_2 \in X$

f is onto function since,

$\forall$ $(x, y_0) \in X\times\{y_0\}$ $\exists$ $x \in X$; $f(x) = (x, y_0)$

f is continuous function, since the sets in the base of the space $X\times\{y_0\}$ is $U\times\{y_0\}$; $U \in \tau$ or ϕ, then

$f^{-1}(U\times\{y_0\}) = U \in \tau$ and $f^{-1}(\phi) = \phi \in \tau$

f is open function, since if U is open in domain X, then $f(U) = U\times\{y_0\}$ and $U\times\{y_0\}$ is open in codomain $X\times\{y_0\}$.

$\therefore$ f is homeomorphism function.

[9] If $x_0 \in X$, then the product space $\{x_0\}\times Y$ topological equivalent the space Y. i.e., $\{x_0\}\times Y \cong Y$; $\{x_0\}\times Y = \{(x_0, y) : y \in Y\}$

To prove this : (By a similar way of [8])

Define $f : Y \to \{x_0\}\times Y$; $f(y) = (x_0, y)$ $\forall\ y \in Y$

f is 1-1 function since,

Let $f(y_1) = f(y_2) \Rightarrow (x_0, y_1) = (x_0, y_2)$

$\Rightarrow y_1 = y_2$ $\forall\ y_1, y_2 \in Y$

f is onto function since,

$\forall\ (x_0, y) \in \{x_0\}\times Y\ \exists\ y \in Y ; f(y) = (x_0, y)$

f is continuous function, since the sets in the base of the space $\{x_0\}\times Y$ is $\{x_0\}\times V$; $V \in \tau'$ or ϕ, then

$f^{-1}(\{x_0\}\times V) = V \in \tau'$ and $f^{-1}(\phi) = \phi \in \tau'$

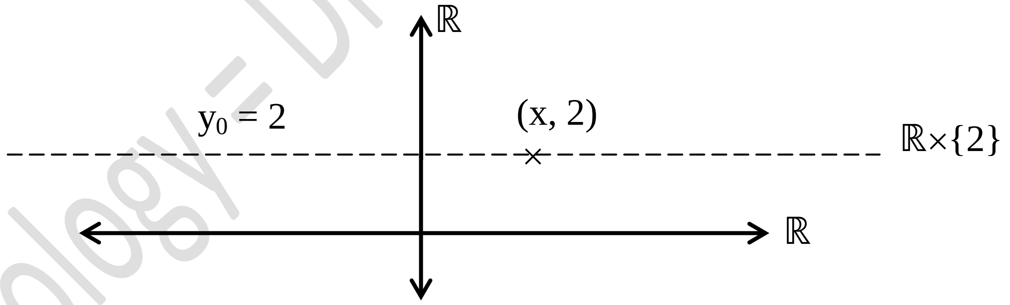
f is open function, since if V is open in domain Y, then $f(V) = \{x_0\}\times V$ and $\{x_0\}\times V$ is open in codomain $\{x_0\}\times V$.

$\therefore$ f is homeomorphism function.

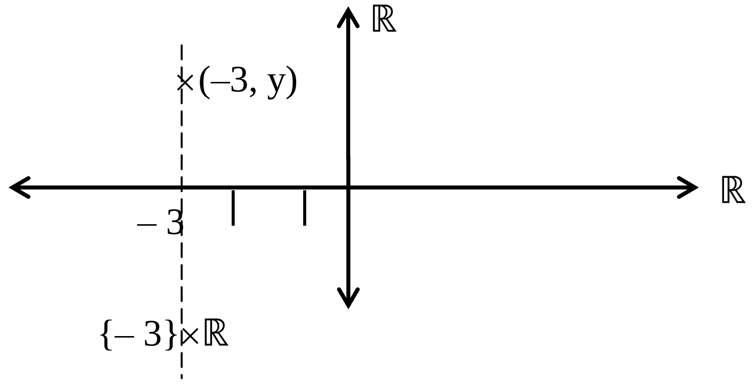
Notes that $X\times\{y_0\}$ is a sub space of the space $X\times Y$ and represented horizontal section in the space $X\times Y$ at the point y_0 . Also, $\{x_0\}\times Y$ is a sub space of the space $X\times Y$ and represented vertical section in the space $X\times Y$ at the point x_0.

<u>For example</u>, take $X = Y = \mathbb{R}$ and $\tau = \tau' = \tau_u$, then the product space $X\times Y$ is the known plane $\mathbb{R}^2$.

Let $y_0 = 2$, then $X\times\{y_0\} = \mathbb{R}\times\{2\}$ is subspace from $\mathbb{R}^2$ and represented as horizontal line segment and the following figure show this :



Let $x_0 = -3$, then $\{x_0\}\times Y = \{-3\}\times\mathbb{R}$ is subspace from $\mathbb{R}^2$ and represented as vertical line segment and the following figure show this :



Definition : Quotient Space

Let (X, τ_X) be a topological space and Y be any set. Let $f : X \to Y$ be a surjective function, then the set

$$\tau_f = \{G \subseteq Y \; ; \; f^{-1}(G) \in \tau_X\}$$

Is a topology on Y this topology called **quotient topology** on Y generated by f and (X, τ_X).

Question : The topology $\tau_f = \{G \subseteq Y \; ; \; f^{-1}(G) \in \tau_X\}$ is the largest topology on Y make the function f continuous.

Answer : Let τ be another topology on Y making f continuous.

$\Rightarrow$ $f^{-1}(G)$ is open in X $\forall$ $G \in \tau$.

$\Rightarrow$ $G \in \tau_f$ (def. of τ_f)

$\Rightarrow$ $\tau \subseteq \tau_f$

$\Rightarrow$ τ_f is the largest topology on Y making f is continuous.

Theorem : Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous surjective function, if f either open or closed, then $\tau_f = \tau_Y$.

Proof :

Clearly, $\tau_Y \subseteq \tau_f$ (by previous question)

Now, to show that $\tau_f \subseteq \tau_Y$

Let $G \in \tau_f \Rightarrow f^{-1}(G) \in \tau_X$

$\Rightarrow$ $f(f^{-1}(G)) = G$ is open in Y (since f is open)

$\Rightarrow$ $G \in \tau_Y$

$\Rightarrow$ $\tau_f \subseteq \tau_Y$

So, $\tau_f = \tau_Y$.

By similar way if f is closed.

Theorem : Let Y has the quotient space generated by the surjective function $f : X \to Y$, then $g : Y \to Z$ is continuous function if and only if gof is continuous function.

Proof :

($\Rightarrow$) The composition of continuous functions is continuous.

($\Leftarrow$) Let G be open set in Z

Since gof is cont. $\Rightarrow$ $(gof)^{-1}(G) = f^{-1}(g^{-1}(G))$ is open in X

But $g^{-1}(G) \subseteq Y \wedge f^{-1}(g^{-1}(G))$ is open in X

$\Rightarrow$ $g^{-1}(G)$ is open in Y (by definition of τ_f , $g^{-1}(G) \in \tau_f$)

$\Rightarrow$ g is continuous.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gof"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Remarks :

[1] Let X be a nonempty set. The **partition or decomposition** on X with the relation R is the family of disjoint nonempty subsets of X and their union equal X. The elements of this partition called **equivalence classes** and denoted by [x].

[2] The set of equivalence classes for X is called **quotient set** for X with the relation R and denoted by $X|R = \{[x] : x \in X\}$.

[3] The mapping $p : X \to X/R$; $p(x) = [x]$ is called **quotient mapping**.

Definition : Quotient Space

Let (X, τ) be a topological space and R be equivalence relation on X. Let $p : X \to X/R$; $p(x) = [x]$ be surjective quotient mapping from X to $X|R$, then the quotient topology on $X|R$ is the largest topology make the function f continuous and the space $(X|R, \tau_{X|R})$ is called **quotient space**.