EXAMPLE:

Let $\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$

Then:

1. $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$

2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$
$= 2(5) + 3(7) = 31$

3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

2.1 Integration by Substitution

THEOREM Substitution in Definite Integrals: If g' is continuous on the interval *[a,* b] and *f* is continuous on the range of *g(x) = u,* then

$$\int_{a}^{b} f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

SOL:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_{0}^{2} \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

EXAMPL

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

(a) $\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta\,d\theta = \int_{1}^{0} u \cdot (-du)$

Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$,
$-du = \csc^2\theta\,d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) =$
When $\theta = \pi/2$, $u = \cot(\pi/2) =$

$$= -\int_{1}^{0} u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b) $\int_{-\pi/4}^{\pi/4} \tan x\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\,dx$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x$, $du = -\sin x\,dx$.
When $x = -\pi/4$, $u = \sqrt{2}/2$.
When $x = \pi/4$, $u = \sqrt{2}/2$.

$$= -\ln|u|\Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0$$

Integrate, zero width interval

THEOREM:
Let f be continuous on the symmetric interval [-a, a].

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

EXAMPLE: Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$.

SOL: Since $f(x) = x^4 - 4x^2 + 6$ satisfies *f(-x) = f(x)* , it is even on the symmetric interval [- 2, 2] , so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$
$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$
$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

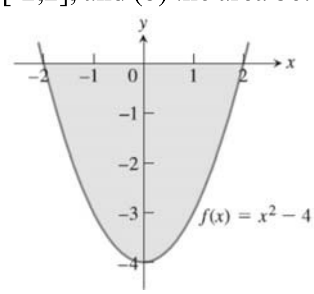
DEFINITION: If *y = f(x)* is nonnegative and integrable over a closed interval *[a, b],* then the area under the curve *y = f(x)* over *[a, b]* is the integral of *f* from *a* to b.

$$A = \int_{a}^{b} f(x)dx$$

If f(x) is negative then $A = \int_a^b |f(x)|dx$

EXAMPLE

Let $f(x) = x^2 - 4$, compute (a) the definite integral over the interval [-2,2], and (b) the area between the graph and the x-axis over [-2,2].



Solution:

(a) $\int_{-2}^{2} f(x)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3},$

(b) The area between the graph and the x-axis is $\left|-\frac{32}{3}\right| = \frac{32}{3}$

EXAMPLE: Find the area between the graph $\boldsymbol{f(x) = x^3 - 2x^2 - x}$

SOL: f(x)=0 then $\boldsymbol{(x^2 - 1)(x - 2) = 0}$ that is x=1, -1 and x=2

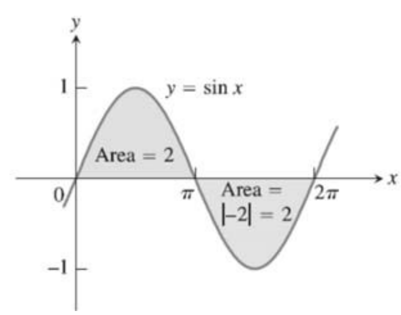
$A = \boldsymbol{A_1 + A_2} = \int_{-1}^{1} \boldsymbol{|f(x)|dx} + \int_{1}^{2} \boldsymbol{|f(x)|dx}$

$$= \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right] + \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]$$

EXAMPLE: Let the *function f(x)* = sin *x* between *x* = 0 and *x* = 2π. Compute

(a) the definite integral of *f(x)* over [0, 2π].

(b) the area between the graph of *fix)* and the x-axis over [0, 2π].



Solution

(a) The definite integral for $f(x) = sinx$ is given by

$$\int_{0}^{2\pi} \sin x\,dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

(b) To compute the area between the graph of *f(x)* and the x-axis over $[0, 2\pi]$ we should find the points in which f is intersect x-axis i.e. f(x)=0 this implies to sin x=0 i.e. $x=0$, $x=\pi$ *or* $x=2\pi$

Now subdivide $[0, 2\pi]$ into two pieces: the interval $[0, \pi]$ and the interval $[\pi, 2\pi]$.

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = -[\cos\pi - \cos 0] = -[-1-1] = 2$$

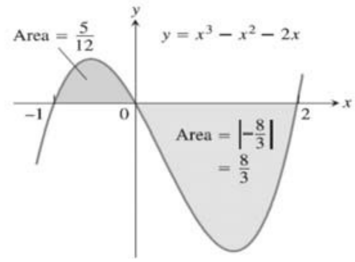
$$\int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos\pi] = -[1-(-1)] = -2$$

$$\text{Area} = |2| + |-2| = 4.$$

EXAMPLE:

Find the area of the region between the x-axis and the graph of

$$\boldsymbol{f(x) = x^3 - x^2 - 2x, \qquad -1 \le x \le 2}$$



Solution

First find the zeros of f. $f(x) = x^3 - x^2 - 2x = 0$

$$x(x^2 - x - 2) = 0$$

$$x(x+1)(x-2) = 0$$

x = 0, -1, and 2 . The zeros subdivide [-1,2] into two subintervals: [-I, 0], on which f≥ 0, and [0, 2], on which *f*≤*0*. We integrate *f* over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

EXAMPLE: Find $\boldsymbol{\int_{-1}^{2}|x-1|dx}$

Since $\boldsymbol{|x-1|} = \begin{cases} \boldsymbol{x-1} & \boldsymbol{x \ge 1} \\ \boldsymbol{-x+1} & \boldsymbol{x < 1} \end{cases}$ then $\boldsymbol{\int_{-1}^{2}|x-1|dx = \int_{-1}^{1}(-x+1)dx + \int_{1}^{2}(x-1)dx}$

.3 Indefinite Integrals and the Substitution Method

Since any two antiderivatives of f differ by a constant, the indefinite integral notation means that for any antiderivative *F* of f,

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

THEOREM:

The Substitution Rule If u = g(x) is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

Therefore $\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$

As well as $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$ then $du = \frac{du}{dx}dx$

EXAMPLE:

Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx.$

Sol: let $u = x^3 + x$. then $du = \frac{du}{dx}dx = (3x^2 + 1)\,dx,$

so that by substitution we have :

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{(x^3 + x)^6}{6} + C \qquad \text{Substitute } x^3 + x \text{ for } u.$$

EXAMPLE:

Find the integral $\int \sqrt{2x + 1}\,dx.$

SOL: let u=2x+1 and n=1/2, $du = \frac{du}{dx}dx = 2\,dx$

because of the constant factor 2 is missing from the integral. So we write

$$\int \sqrt{2x + 1}\,dx = \frac{1}{2}\int \underbrace{\sqrt{2x + 1}}_{u} \cdot \underbrace{2\,dx}_{du}$$

$$= \frac{1}{2}\int u^{1/2}\,du \qquad \text{Let } u = 2x + 1,\ du = 2\,dx.$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{3}(2x + 1)^{3/2} + C \qquad \text{Substitute } 2x + 1 \text{ for } u.$$

EXAMPLE: Find $\int \sec^2(5t + 1) \cdot 5\,dt.$

SOL: Let $u = 5t + 1$ and $du = 5\,dx$. Then,

$$\int \sec^2(5t + 1) \cdot 5\,dt = \int \sec^2 u\,du \qquad \text{Let } u = 5t + 1,\ du = 5\,dt.$$

$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \tan(5t + 1) + C \qquad \text{Substitute } 5t + 1 \text{ for } u.$$

EXAMPLE: $\int \cos(7\theta + 3)\, d\theta$.

SOL: Let $u = 7\theta + 3$ so that $du = 7\, d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7. Then,

$$\begin{aligned}
\int \cos(7\theta + 3)\, d\theta &= \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\, d\theta && \text{Place factor } 1/7 \text{ in front of integral.}\\
&= \frac{1}{7}\int \cos u\, du && \text{Let } u = 7\theta + 3,\ du = 7\, d\theta.\\
&= \frac{1}{7}\sin u + C && \text{Integrate.}\\
&= \frac{1}{7}\sin(7\theta + 3) + C && \text{Substitute } 7\theta + 3 \text{ for } u.
\end{aligned}$$

EXAMPLE:

$$\begin{aligned}
\int x^2 \sin(x^3)\, dx &= \int \sin(x^3)\cdot x^2\, dx\\
&= \int \sin u \cdot \frac{1}{3}\, du && \text{Let } u = x^3,\ du = 3x^2\, dx,\ (1/3)\, du = x^2\, dx.\\
&= \frac{1}{3}\int \sin u\, du\\
&= \frac{1}{3}(-\cos u) + C && \text{Integrate.}\\
&= -\frac{1}{3}\cos(x^3) + C && \text{Replace } u \text{ by } x^3.
\end{aligned}$$

EXAMPLE: Evaluate $\int x\sqrt{2x + 1}\, dx$

SOL: $u = 2x + 1$ to obtain $x = (u - 1)/2$, and find that $x\sqrt{2x + 1}\, dx = \frac{1}{2}(u - 1)\cdot\frac{1}{2}\sqrt{u}\, du.$

The integration now becomes

$$\begin{aligned}
\int x\sqrt{2x + 1}\, dx &= \frac{1}{4}\int (u - 1)\sqrt{u}\, du = \frac{1}{4}\int (u - 1)u^{1/2}\, du && \text{Substitute.}\\
&= \frac{1}{4}\int (u^{3/2} - u^{1/2})\, du && \text{Multiply terms.}\\
&= \frac{1}{4}\left(\frac{2}{5}u^{5/2} \int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} . C\right. && \text{Integrate.}\\
&= \frac{1}{10}(2x + 1)^{5/2} - \frac{1}{6}(2x + 1)^{3/2} + C && \text{Replace } u \text{ by } 2x + 1. \ \blacksquare
\end{aligned}$$

$$\begin{aligned}
\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2 + 1,\ du = 2z\, dz.\\
&= \int u^{-1/3}\, du && \text{In the form } \int u^n\, du\\
&= \frac{u^{2/3}}{2/3} + C && \text{Integrate.}\\
&= \frac{3}{2}u^{2/3} + C\\
&= \frac{3}{2}(z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } z^2 + 1.
\end{aligned}$$

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u}$$ Let $u = \sqrt[3]{z^2+1}$, $u^3 = z^2 + 1$, $3u^2\,du = 2z\,dz$.

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C$$ Integrate.

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $(z^2+1)^{1/3}$.

Example: The Integrals of $\boldsymbol{sin^2 x}$ and $\boldsymbol{cos^2 x}$

(a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$$ $\sin^2 x = \frac{1-\cos 2x}{2}$

$$= \frac{1}{2}\int (1 - \cos 2x)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$ $\cos^2 x = \frac{1+\cos 2x}{2}$ ■

DEFINITION: If u is a differentiable function that is never zero, $\int \frac{1}{u}\,du = \ln|u| + C.$

In general $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

EXAMPLE

$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$ $u = x^2 - 5$, $du = 2x\,dx$, $u(0) = -5$, $u(2) = -1$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

<u>**The Integrals of tan x, cot x, sec x, and esc x**</u>

1- $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$ $u = \cos x > 0$ on $(-\pi/2, \pi/2)$, $du = -\sin x\,dx$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C.$$ Reciprocal Rule

2- $$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u}$$ $u = \sin x$, $du = \cos x\,dx$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

3- $$\int \sec x\,dx = \int \sec x\,\frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$ $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)\,dx$